I want to challenge myself solving an exercise in discrete mathematics. I want to find the coeffiecients of the genarating function $\sqrt{1-x}$, but without deriving the function, but using induction instead; I will only be inspired by derivatives.
If I fail. At least I will have a story about learning math in the university.
Why would I do such a thing? Because it is still in my head.

Background

I wa sitting in the discrete math tutor class, learning about the product of generating function, when suddenly a student asked a question about the range in which the function converges. That was a hell of an obstacle to surmount. Well, the tutor had tried to explain that it doesn't matter because what one should care about, and the student had argued that this is Calculus. WTF? Calculus is a prerequisite, and teachers should teach it. Finally, our tutor showed that:

$$(1-x)(1+x+x^2+x^3+...)=1-x+x-x^2+x^2-x^3\pm\ldots$$

Still not good enough! Well, the lesson needs to continue.

Generating functions are limit of polynomial, and polynomial converge for every real x. But all we have to care about is the derivatives at one point: x=0.

Maybe, that little exercise from our calculus class can help:

Prove that if f(x) and g(x) are derivable at least n times, and h(x)=f(x)g(x), then

$$h^{(n)}(x)=\sum_{k=0}^{n}\binom{n}{k}f^{(k)}(x)g^{(n-k)}(x)$$

Now, if we divide both sides by n!, we'll get that

$$\frac{h^{(n)}(x)}{n!}=\sum_{k=0}^{n}\binom{n}{k}\frac{f^{(k)}(x)g^{(n-k)}(x)}{n!}=\frac{\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}}{n!}f^{(k)}(x)g^{(n-k)}(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x)}{k!}\frac{g^{(n-k)}(x)}{(n-k)!}$$

Now, Let's Find The Generating Function $\sqrt{(1-x)}$

Or a function f(x) such that $f(x)^2=1-x$

Let's find a sequence of numbers a_n for every integer $n\geq 0$, such that:

$$\sum_{k=0}^{n}a_k a_n-k=\begin{Bmatrix}1\,\text{for}\,n=0\\-1\,\text{for}\,n=1\\0\,\text{otherwise}\end{Bmatrix}$$

Lets define for each n:

$$b_n=n!a_n$$

Now,

$a_0^2=1\rightarrow a_0=1$ because we choose the positive root

$$a_0 a_1 + a_1 a_0 = -1 \rightarrow 2a_1 = -1 \rightarrow a_1 = -\frac{1}{2}$$

$$\frac{b_0 b_2}{2!} \cdot b_1^2 + \frac{b_2}{2!} \cdot b_0 \rightarrow b_0 b_2 = -b_1^2 = -\frac{1}{4} \rightarrow b_2 = -\frac{1}{4} = \frac{-1}{2} \cdot \frac{1}{2}$$

$$\frac{b_0 b_3}{3!} + \frac{b_1 b_2}{2!} = 0 \rightarrow \frac{b_3}{3!} = \frac{-b_1 b_2}{2!} = \frac{1}{2} \cdot \left(-\frac{1}{4 \cdot 2!}\right) \rightarrow b_3 = \frac{-1}{4} \cdot \frac{3}{2} = \frac{-1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2}$$

OK, now let's see if for every n>0

$$b_n = -\prod_{k=1}^{n} \frac{2k-3}{2} = -\frac{2(2n-2)!}{4^n (n-1)!}$$

This is correct for n=1, now let's assume it is correct from n and prove for n+1

Let us prove that for any integer $n \geq 0$, $b_{n+1} = \frac{2n-1}{2} b_n$

if n=0, then $b_{n+1} = b_1 = -\frac{1}{2} = \frac{2 \cdot 0 - 1}{2}$

Now, if n>1 $0 = \sum_{k=0}^{n+1} a_k a_{n+1-k}$

Thus, $2 a_0 a_{n+1} = -\sum_{k=1}^{n} a_k a_{n+1-k}$

$a_0 = 1$

Thus, $2 a_{n+1} = -\sum_{k=1}^{n} a_k a_{n+1-k}$

Because $b_n = \frac{a_n}{n!}$:

$$\frac{2 b_{n+1}}{(n+1)!} = -\sum_{k=1}^{n} \frac{b_k}{k!} \frac{b_{n+1-k}}{(n+1-k)!}$$

Let us multiply both sides by (n + 1)! :

$$2 b_{n+1} = -\sum_{k=1}^{n} \frac{(n+1)!}{k!(n+1-k)!} b_k n_{n+1-k} = -\sum_{k=1}^{n} \binom{n+1}{k} b_k b_{n+1-k} =$$

$$= -\binom{n+1}{1} b_1 b_n - \sum_{k=2}^{n} \binom{n+1}{k} b_k b_{n+1-k} = -(n+1) b_n - \sum_{k=2}^{n} [\binom{n}{k-1} + \binom{n}{k}] b_k b_{n+1-k} =$$

$$= \frac{(n+1)}{2} b_n - [\sum_{k=2}^{n} \binom{n}{k-1} b_k b_{n+1-k}] - \sum_{k=2}^{n} \binom{n}{k} b_k b_{n-1-k} =$$

$$= \frac{(n+1) b_n}{2} - [\sum_{k=1}^{n-1} \binom{n}{k} b_{k+1} b_{n-k}] - \sum_{k=2}^{n} \binom{n}{k} b_k b_{n+1-k} =$$

$$= \frac{(n+1) b_n}{2} - [\sum_{k=1}^{n-1} \binom{n}{k} \frac{(2k-1)}{2} b_k b_{n-k}] - \sum_{k=2}^{n} \binom{n}{k} \frac{(2n-2k-1)}{2} b_k b_{n-k}$$

Now, if n=1, summing from 1 to n-1 or from 2 to n will give the result 0, thus:

$$2 b_2 = (1+1) \frac{b_1}{2} = -\frac{1}{2} \rightarrow b_2 = -\frac{1}{4} = \frac{(2*1-1)}{2} * (-\frac{1}{2})$$

Now, let's prove for $n \geq 2$

$$2 b_{n+1} = \frac{(n+1) b_n}{2} + \binom{n}{n} \frac{2n-1}{2} b_n b_0 + \binom{n}{1} \frac{2n-3}{2} b_1 b_{n-1} - (n-1) \sum_{k=1}^{n} \binom{n}{k} b_k b_{n-k} =$$

$$= \frac{(n+1) b_n}{2} + \frac{(2n-1) b_n}{2} - \frac{(2n-3) n b_{n-1}}{4} - (n-1) n! \sum_{k=1}^{n} \frac{b_k}{k!} \frac{b_{n-k}}{(n-k)!} =$$

$$= \frac{3 n b_n}{2} - \frac{n b_n}{2} - (n-1) n! \sum_{k=1}^{n} a_k a_{n-k} = 2 n b_n + (n-1) n! a_0 a_n =$$

$$= n b_n + (n-1) b_n = (2n-1) b_n$$

Let us divide both sides by 2 and get that $b_{n+1} = \frac{(2n+1) b_n}{2}$

OK. now let’s prove our formula:

Assuming that $b_n = \frac{-2(2n-2)!}{4^n (n-1)!}$,

Let's prove for $n+1$

$$b_{n+1} = \frac{(2n-1)b_n}{2} = \frac{-2(2n-2)!}{4^n (n-1)!} \cdot \frac{(2n-1)}{2} = \frac{-2(2n-2)!(2n-1)2n}{4^n (n-1)! \cdot 2 \cdot 2n} = \frac{-2*(2n)!}{4^{n+1} n!} =$$

$$= \frac{-2 \cdot [2(n+1)-2]!}{4^{n+1}[(n+1)-1]!}$$

Wait!

$$a_n = \frac{b_n}{n!} = \frac{-2(2n-2)!}{4^n (n-1)! n!} = \frac{-2}{4^n n} \binom{2n-2}{n-1}$$

So, isn't using what you've learned in calculus better?