Pell’s Equations

Definition

Pell's equations are diophantine equations of form

$$x^2 - dy^2 = 1$$

Where x, y are integers and d is a constant natural number but not a perfect square. The number of solution is infinity or $\aleph_0$

Examples

$$1 = 1^2$$
$$1+2+3+4+5+6+7+8 = 36 = 6^2$$

How many more natural values of n satisfy that $1+2+\ldots+n$ is a perfect square? Solution:

$$\sum_{j=1}^{n} j = k^2 \rightarrow$$
$$\rightarrow \frac{n(n+1)}{2} = k^2 \rightarrow$$
$$\rightarrow n(n+1) = 2k^2 \rightarrow$$
$$\rightarrow n^2 + n = 2k^2 \rightarrow$$
$$\rightarrow 4n^2 + 4n = 8k^2 \rightarrow$$
$$\rightarrow 4n^2 + nn + 1 = 8k^2 + 1 \rightarrow$$
$$\rightarrow (2n+1)^2 = 8k^2 + 1 \rightarrow$$
$$\rightarrow (2n+1)^2 - 8k^2 = 1$$

That’s Pell’s equation with
x=2n+1, y=k and d=8

The answer is infinity, and to show that there are more solutions, take a natural n and raise (x^2-dy^2) to the power of n.

A close form of solutions will follow

Another example: Pythagorean triplets

A triplet (a,b,c) is called a Pythagorean triplet if $a^2+b^2=c^2$

Let us find Pythagorean triplets of natural numbers, such that:

b-a = 1

One solution is (3,4,5).

Another is (20,21,29) because $20^2+21^2=400+441=841=29^2$

To find others:

$$a^2+(a+1)^2=c^2 \rightarrow$$
$$\rightarrow a^2+a^2+2a+1=c^2 \rightarrow$$
$$\rightarrow 2a^2+2a+1=c^2 \rightarrow$$
$$4a^2+4a+2=2c^2 \rightarrow$$
$$\rightarrow (4a^2+4a+1)+1=2c^2 \rightarrow$$
$$\rightarrow (2a+1)^2+1=2c^2 \rightarrow$$
$$\rightarrow (2a+1)^2-2c^2=-1$$

Not Pell’s equation, but the rest of the solutions can be found using the equation:

$x^2-2y^2=1$

A Closed Form

Pell's equations are based on Bahmagupta's identity:
$(x^2+ny^2)(a^2+nb^2)=$
$=a^2x^2+na^2y^2+nb^2x^2+n^2b^2y^2=$
$=(ax)^2+na^2y^2+n(bx)^2+n^2(by)^2=$
$=(ax)^2+(nby)^2+n(bx)^2+n(ay)^2=$
$=(ax)2-naxby+nby)^2+n(bx)2+2naxby+n(ay)^2=$
$=(ax-nby)^2+n(bx+ay)^2$

This, if both (x_1,y_1) and (x_2,y_2) are solutions Of Pell's equation $x^2-dy^2=1$, so is
$(x_1x_2+dy_1y_2,\ x_1y_2+x_2y_1)$

To find a closed form, let us write the equation as:

$$(x+\sqrt{d}\,y)(x-\sqrt{d}\,y)=1$$

Now, let us define a fundamental solution (x_1,y_1) as ssolution in natural numbers, such that if (x,y) is a solutions in natural numbers, then

$$x+\sqrt{d}\,y\geq x_1+\sqrt{d}\,y_1$$

To find the n^{th} solution (x_n,y_n)

$$x_n+\sqrt{d}\,y_n=(x_1+\sqrt{d}\,y_1)^n$$

And using the binomial expansion:

$$x_n-\sqrt{d}\,y_n=(x_1-\sqrt{d}\,y_1)^n$$

By adding the above equations:

$$x_n=\frac{(x_1+\sqrt{d}\,y_1)^b+(x_1-\sqrt{d}\,y_1)^n}{2}$$

By subtracting

$$y_n=\frac{(x_1+\sqrt{d}\,y_1)^n-(x_1-\sqrt{d}\,y_1)^n}{2\sqrt{d}}$$

With simple induction, we can prove that all solutions in natural numbers where found.
What's left to be done is proove that a fundamental solution exists.
Before that, please allow me to refer back to slide 6, and solve the equations from my examples using LibreOffice Calc. The full solution set, which includes negative numbers is:

$$x+\sqrt{d}\,y=\pm(x_1+\sqrt{d}\,y_1)^n \quad n\in\mathbb{Z}$$

First: for which values of n is 1+2+...+n a perfect square?

Equation:

$(2n+1)^2-8k^2=1$
First n to display: 1
Fist k to display: 1
First x: 3
First y: 1
Fundamental solution: (3,1)

Following is a list of some results:

x	k=y	n
3	1	1
17	6	8
99	35	49
577	204	288
3363	1189	1681
19601	6930	9800
114243	40391	57121
665857	235416	332928

Second: Pythagorean Triplets
Equation: $(2a+1)^2-2c^2=-1$

First triplet: (3,4,5)
First x: 7
First y: 5
Fundamental soultion: (3,2)

Following is a list of some results:

a	b	c=y	x
3	4	5	7
20	21	29	41
119	120	169	239
696	697	985	1393
4059	4060	5741	8119
23660	23661	33461	47321
137903	137904	195025	275807
803760	803761	1136689	1607521

A Fundamental Solution Exists

To prove that afundamental solution exists, it is sufficient to prove the existence of a non-trivial solution Because solutions are pairs of natural numbers and can be sorted by ascending order.

Let us start with lemma 1:
If d is a natural number which is not a perfect square, there exist infinity pairs of (x,y), such that:

$$|x-\sqrt{d}\,y|<\frac{1}{y}$$

Proof, let us use the fact that the square root of d is irrational because d is not a perfect square.

Let N be a natural number, so we can divide interval [0,1) into N sub-intervals:

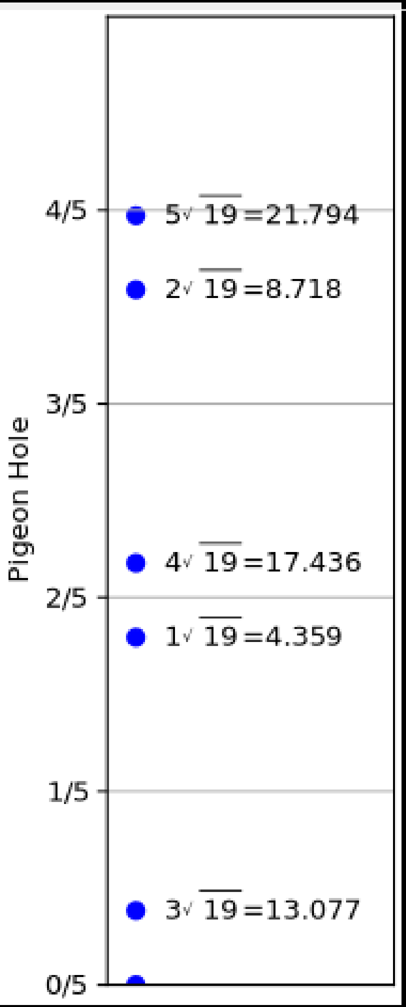
$$\left[0,\frac{1}{N}\right),\left[\frac{1}{N},\frac{2}{N}\right),\ldots,\left[\frac{N-1}{N},\frac{N}{N}\right)$$

And take the N+1 numbers:

$$0,1\sqrt{d}.2\sqrt{d},\ldots,N\sqrt{d}$$

So, according to the pigeon-hole principle, there is a sub interval that contains at least two of the numbers' fractional parts (the parts after the decimal points)

In the following figure with N=5 and d=19. You can see two sub-intervals with 2 fractional parts inside them.



To put it formally, let us define:

$$f_k = \lfloor k\sqrt{d} \rfloor, \quad r_k = k\sqrt{d} - f_k \quad k \in \{0, 1, \ldots, N$$

There exist :

k, l $\in$ {0,1,…N}

n $\in$ {0,1,…N-1}

Such that k>l and:

$$\frac{n}{N} \le f_k < \frac{n+1}{N}, \quad \frac{n}{N} < f_k \le \frac{n+1}{N} \rightarrow$$

$$\rightarrow |f_k - f_l| < \frac{1}{N}$$

Take

$y=k-l$, $x=f_k-f_l$

Then,

$$|x - \sqrt{d}\,y| = |f_k - f_l - (k-l)\sqrt{d}| = |f_k - f_l - k\sqrt{d} + l\sqrt{d}| =$$

$$= |(f_k - \sqrt{d}\,k) - (f_l - \sqrt{d}\,l)| = |r_l - r_k| < \frac{1}{N} \le \frac{1}{y}$$

Let us prove that for such pairs,

$$|x^2 - dy^2| < 1 + 2\sqrt{d}$$

Proof:

$$|x + \sqrt{d}\,y| = |x - \sqrt{d}\,y + 2\sqrt{d}\,y| \le |x - \sqrt{d}\,y| + 2\sqrt{d}\,y \le \frac{1}{y} + 2\sqrt{d}\,y \le 1 + 2\sqrt{d}\,y$$

Multiply both sides by $|x - \sqrt{d}\,y|$, and get:

$$|x^2 - dy^2| \le |x^2 - dy^2|(1 + 2\sqrt{d}\,y) \le \frac{1}{y} + 2\sqrt{d} \le 1 + 2\sqrt{d}$$

Add the fact that x^2-dy^2 is an integer, thus we can replace "≤" by "<".
So there are infinity integers x^2-dy^2 bounded. And in a bounded sequence, we can find a converging sequence.

And that means that there exist a non-zero integer M, and infinity pairs of (x,y) such that:

$x^2-dy^2=M$

x and y leave remainders when divided by |M|, and the remainders are bounded. thus we can find two pairs:

(x_1,y_1) and (x_2,y_2) such that

$x_1 \equiv x_2 \mod |M|$

And

$y_1 \equiv y_2 \mod |M|$

Thus, there exist two non-zero integers k, m such that:

$x_2=x_1+kM$

$y_2=y_2+mM$

Let us write M as $M=x^2-dy^2$ and plug it into the equations above:

$$x_2=x_1+k(x_1^2-dy_1^2)$$

and

$$y_2=y_1+m(x_1^2-dy_1^2)$$

Let’s add:

$$x_2+\sqrt{d}\,y_2=x_1+\sqrt{d}\,y_1+k(x_1^2-dy_1^2)+\sqrt{d}\,m(x_1^2-dy_1^2)=$$
$$=(x_1+\sqrt{d}\,y_1)[1+(k+\sqrt{d}\,m)(x_1-\sqrt{d}\,y_1)]=$$
$$=(x_1+\sqrt{d}\,y_1)[(1+kx_1-dmy_1)+\sqrt{d}(mx_1-ky_1)]$$

Let’s subtract:

$$x_2-\sqrt{d}\,y_2=x_1-\sqrt{d}\,y_1+k(x_1^2-dy_1^2)-\sqrt{d}\,m(x_1^2-dy_1^2)=$$
$$=(x_1-\sqrt{d}\,y_1)[1+(k-\sqrt{d}\,m)(x_1+\sqrt{d}\,y_1)]=$$
$$=(x_1-\sqrt{d}\,y_1)[(1+kx_1-dmy_1)-\sqrt{d}(mx_1-ky_1)]$$

Now, let us define:

$x=1+kx_1-dmy_1$

$y=mx_1-ky_1$

The equations become

$$x_2+\sqrt{d}\,y_2=(x_1+\sqrt{d}\,y_1)(x+\sqrt{d}\,y)$$
$$x_2-\sqrt{d}\,y_2=(x_1-\sqrt{d}\,y_1)(x+\sqrt{d}\,y)$$

Let’s multiply:

$(x_2)^2-d(y_2)^2=[(x_1)^2-d(y_1)^2](x^2-dy^2)$

$M=M(x^2-dy^2)$

And because $M\neq 0$:

$x^2-dy^2=1$