Catalan Numbers

There are at least two problems that can be solved using the Catalan Numbers

1. In how many ways can you form a balanced parenthesis string of length 2n for a given natural number n?

2. In how many ways can you divide a convex ploygon with n sides into triangles by diagonals with no diagonals crossing each other?

Solution

1. Balance Parentheses Problem

Let’s solve problem #1 and use the solution to solve problem #2.

Let c_n , Catalan number for n, be the number of ways to form a string of length 2n. For convenience, c_0 will be 1.

For n=1, there is only one such string: “()”.

For n+1:

The closer of the first symbol is at 2k for k=1,2,3,…,n+1

Between these two positions, there is a blanced string of length 2(k – 1).

If k<n+1, the first balanced substring is followed by another string from postion 2k+1 to 2n+2, the latter is of lengthe 2(n + 1– k)

Thus,

$$c_{n+1}=\sum_{k=1}^{n+1} c_{k-1}\, c_{n-1-k}=\sum_{k=0}^{n} c_k c_{n-k}$$

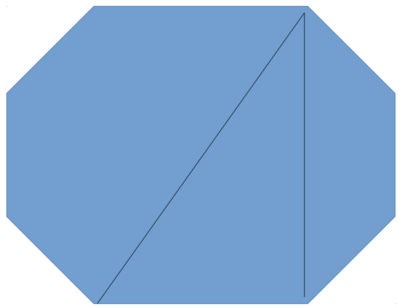
2. Dividing Polygons

Let a_n be the number of ways to split a polygon of n sides into triangle by diagonals without diagonals crossing each other. There are no polygons with less than 3 sides, but for convenience a_2 will be defined as 1.

For n+1:

Every side of the polygon is a side of a triangle. Vertices 1 and n+1 should be connected to a vertex k, for k=2,3,4,5,...n+1

As you can see from the following image, the trainle splits the polygon into 3 areas: a polygon with k sides, the triangle, and a traingle of vertices k, k+1,…,n+1 (n+k-2 sides)



Thus, $a_{n+1}=\sum_{k=2}^{n} a_k a_{n+2-k} = \sum_{k=0}^{n-2} a_{k+2} a_{n-k}$

Now, if we assign $a_n = c_{n-2}$ for every n, we will get that

$a_{k+2}=c_k$ and

$a_{n-k}=c_{(n-2)-k}$

Thus,

$$c_{n-1}=\sum_{k=0}^{n-2} c_k c_{n-2-k}$$

Or, for every non-negative integer n:

$$c_n=\sum_{k=0}^{n} c_k c_{n-k}$$

The same Catalan number.

Calculating The Formula Using Generating Function

Let S be the generating function for Catalan numbers

Then

$$S-c_0=\sum_{n=0}^{\infty} c_{n+1\text{¿}} x^{n+1} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_k c_{n-k} x^{n+1} = xS^2$$

The generating function can be found by solving the quadratic equation

$$xS^2 - s + 1 = 0$$
$$Thus, S_{1,2} = \frac{1 \pm \sqrt{1-4x}}{2x}$$

We need a solution for which the numerator is 0 when x=0, so we choose

$$S = \frac{1-\sqrt{1+4x}}{2x}$$

Now,

$$c_0 = \lim_{x \to 0} \frac{1-\sqrt{1-4x}}{2x} = \lim_{x \to 0} \frac{\left(1-\sqrt{1-4x}\right)\left(1+\sqrt{1-4x}\right)}{2x\left(1+\sqrt{1-4x}\right)} =$$
$$= \lim_{x \to 0¿} \frac{1-(1-4x)}{2x\left(1+\sqrt{1-4x}\right)} = \lim_{x \to 0} \frac{4x}{2x \cdot 2} = 1$$

Let's derive the numerator and devide by 2, to find the formula:

$$\frac{d}{dx} - \sqrt{1-4x} = \frac{4*1}{2\sqrt{1-4x}}$$

Assigning x=0 and dividing by 2, we get:

$$c_0 = \frac{1}{2} \cdot \frac{4}{2} = 1$$

Let's take second derivative, divide by 2 and then by 2!

Assigning x=0, we get

$$c_1 = \frac{1}{2 \cdot 2!} \cdot \frac{4}{1} = 1$$

Let's take now the 3rd derivative, and divide by 2 and then by 3!

$$\frac{1}{2 \cdot 3!} \frac{d^3}{dx^3} - \sqrt{1-4x} = \frac{1}{2 \cdot 3!} \frac{d}{dx} \frac{4}{\sqrt{(1-4x)^3}} = \frac{1}{12} \frac{4 \cdot \frac{3}{2} \cdot 4}{\sqrt{(1-4x)^5}} = \frac{1}{12} \frac{24}{\sqrt{(1-4x)^5}}$$

Assigning x=0, we get that

$$c_2 = 2$$

OK, do you see by what we multpily our result each time?

We multiply by 4 because of the coefficent of x under the root sign. We multiply by $\frac{2n-1}{2}$ because of the exponent. To get the nth coefficient we divide our result by (n+1)!

From this we can get a result, such as

$$c_n=\frac{1}{(n+1)!}\cdot\prod_{k=1}^{n}4k-2=\frac{1}{(n+1)!}\cdot 2^n\cdot\prod_{k=1}^{n}(2k-1)=\frac{1}{(n+1)!}\cdot 2^n\frac{\prod_{k=1}^{2n}k}{\prod_{k=1}^{n}2k}=\frac{\frac{1}{(n+1)!}\cdot 2^n\cdot(2n)!}{2^n\cdot n!}=$$

$$=\frac{(2n)!}{n!(n+1)!}=\frac{1}{n+1}\binom{2n}{n}$$

$$c_0=\frac{1}{1}\binom{0}{0}=1$$

$$c_1=\frac{1}{2}\cdot\binom{2}{1}=1$$

$$c_2=\frac{1}{3}\cdot\binom{4}{2}=2$$

Now, it’s time to prove by induction:

Let’s prove that for every natural n,

$$\frac{d^n}{dx^n}-\sqrt{1-4x}=\frac{((2n-2)!)}{(n-1)!\sqrt{(1-4x)^{2n-1}}}$$

For n=1:

$$\frac{d}{dx}-\sqrt{1-4x}=\frac{-(-4)\cdot(\frac{1}{2})}{\sqrt{1-4x}}=\frac{2\cdot 0!}{\sqrt{(1-4x)^{(2\cdot 1-1)}}}$$

Good!

Now, assuming that it is correct for n, let’s prove for n+1:

$$\frac{d^{n+1}}{dx^{n+1}}-\sqrt{1-4x}=\frac{d}{dx}\frac{2\cdot(2n-2)!}{(n-1)!\sqrt{(1-4x)^{(2n-1)}}}=\frac{4\cdot\frac{(2n-1)}{2}\cdot 2\cdot(2n-2)!}{(n-1)!\sqrt{(1-4x)^{(2n+1)}}}=$$

$$\text{cdot}\frac{4\cdot(2n-1)\cdot 2n\cdot(2n-2)}{2n\cdot(n-1)!\sqrt{(1-4x)^{(2n+1)}}}=\frac{2\cdot(2n)!}{n!\sqrt{(1-4x)^{(2n+1)}}}=\frac{2\cdot[2(n+1-1)]!}{(n+1-1)!\sqrt{(1-4x)^{2(n+1)-1}}}$$

QED

Now, let us devide this eerivative by 2(n + 1)!, assign x=0 and get

$$c_n=\frac{1}{2*(n+1)!}\frac{2*(2n)!}{n!}=\frac{(2n)!}{n!(n+1)!}=\frac{1}{n+1}\binom{2n}{n}$$