# 33-652: String Theory
# Assignment 7

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**Problem 1:** *Differential forms*

A $p$-form is a totally antisymmetric tensor, so it has $\binom{D}{p}$ independent components, or

$$\frac{D!}{p!(D-p)!} \tag{1}$$

independent components. Since any $p$-form has the representation

$$\omega = \frac{1}{p!} A_{a_1 a_2 \ldots a_p} dx^{a_1} \wedge dx^{a_2} \wedge \cdots \wedge dx^{a_p}, \tag{2}$$

it can be represented as a linear combination of wedge products of $p$ basis one-forms. Examine the form

$$dx^{a_1} \wedge dx^{a_2} \wedge \cdots \wedge dx^{a_p}.$$

In order for this form to be different from zero, each of the $\{a_i\}$ must take on a different value, since

$$dx^{a_i} \wedge dx^{a_i} = 0 \tag{3}$$

for all $a_i$. Since each of the $a_i$'s can run over the index set $\{1, \ldots, D\}$, we see that if we had more than $D$ forms wedging to create our basis form, then at least two of them would have to have the same index. We can then push one of them through to the other, and by (3) we see that the form we have constructed vanishes. Hence the maximum possible value for $p$ is $D$. Denoting the space of $p$-forms on a manifold $M$ by $\Omega_p(M)$, we have the exterior derivative $d$ as a mapping $d : \Omega_p(M) \to \Omega_{p+1}(M)$, whose action on a $p$-form $\omega_p$ is

$$d\omega_p = \frac{1}{p!} \partial_{a_{p+1}} A_{a_1 a_2 \ldots a_p} dx^{a_{p+1}} \wedge dx^{a_2} \wedge \cdots \wedge dx^{a_p}. \tag{4}$$

If we act on this new $p+1$-form with $d$ a second time, we obtain

$$d^2\omega_p = \frac{1}{p!} \partial_{a_{p+2}} \partial_{a_{p+1}} A_{a_1 a_2 \ldots a_p} dx^{a_{p+2}} \wedge dx^{a_{p+1}} \wedge dx^{a_1} \wedge dx^{a_2} \wedge \cdots \wedge dx^{a_p}. \tag{5}$$

Since the partial derivatives commute, we can commute them and relabel indices to arrive at

$$d^2\omega = -d^2\omega, \tag{6}$$

which proves that $d^2 = 0$ when acting on any $p$-form. Considering Maxwell's equations, we can write the gauge field $A$ as

$$A = A_\mu dx^\mu. \tag{7}$$

We then have

$$\begin{aligned} dA &= \partial_\nu A_\mu dx^\nu \wedge dx^\mu \\ &= \partial_\nu A_\mu (dx^\nu \otimes dx^\mu - dx^\mu \otimes dx^\nu) \\ &= (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \otimes dx^\nu \\ &= F_{\mu\nu} dx^\mu \otimes dx^\nu \\ &= F, \end{aligned} \tag{8}$$

where $F$ is the field strength two-form. Now let's examine the components of $dA$:

$$
\begin{aligned}
dA &= \partial_\mu A_\nu dx^\mu \wedge dx^\nu \\
&= (\partial_t A_x - \partial_x \Phi) dt \wedge dx \\
&+ (\partial_t A_y - \partial_y \Phi) dt \wedge dy \\
&+ (\partial_t A_z - \partial_z \Phi) dt \wedge dz \\
&+ (\partial_x A_y - \partial_y A_x) dx \wedge dy \\
&+ (\partial_z A_x - \partial_x A_z) dz \wedge dx \\
&+ (\partial_y A_z - \partial_z A_y) dy \wedge dz \\
&= E_x dt \wedge dx + E_y dt \wedge dy + E_z dt \wedge dz \\
&+ B_z dx \wedge dy + B_y dz \wedge dx + B_x dy \wedge dz. \qquad (9)
\end{aligned}
$$

Taking the exterior derivative of the above expression, we have

$$
\begin{aligned}
d^2 A &= \partial_y E_x dy \wedge dt \wedge dx + \partial_z E_x dz \wedge dt \wedge dx \\
&+ \partial_x E_y dx \wedge dt \wedge dy + \partial_z E_y dz \wedge dt \wedge dy \\
&+ \partial_x E_z dx \wedge dt \wedge dz + \partial_y E_z dy \wedge dt \wedge dz \\
&+ \partial_t B_x dt \wedge dy \wedge dz + \partial_x B_x dx \wedge dy \wedge dz \\
&+ \partial_t B_y dt \wedge dz \wedge dx + \partial_y B_y dy \wedge dz \wedge dx \\
&+ \partial_t B_z dt \wedge dx \wedge dy + \partial_z B_z dz \wedge dx \wedge dy \\
&= (\partial_z E_y - \partial_y E_z + \partial_t B_x) dt \wedge dy \wedge dz \\
&+ (\partial_x E_z - \partial_z E_x + \partial_t B_y) dt \wedge dz \wedge dx \\
&+ (\partial_y E_x - \partial_x E_y + \partial_t B_z) dt \wedge dx \wedge dy \\
&+ (\partial_x B_x + \partial_y B_y + \partial_z B_z) dx \wedge dy \wedge dz \\
&= 0. \qquad (10)
\end{aligned}
$$

This is a fancy way of writing

$$
\begin{aligned}
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0, \qquad (11) \\
\nabla \cdot \mathbf{B} &= 0. \qquad (12)
\end{aligned}
$$

Now consider the Hodge dual of a form, defined as

$$\star dx^{a_1} \wedge \cdots \wedge dx^{a_p} \equiv \frac{1}{(D-p)!} \epsilon^{a_1 \cdots a_p}{}_{a_{p+1} \cdots a_D} dx^{a_{p+1}} \wedge \cdots \wedge dx^{a_D}. \qquad (13)$$

To construct the vector curl in three dimensional Euclidean space, we examine the Hodge dual of the exterior derivative of a one-form. Let a one-form $\omega$ defined as

$$\omega = \omega_a dx^a \qquad (14)$$

be given. Its exterior derivative is given by

$$d\omega = \left( \frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z} \right) dy \wedge dz + \left( \frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x} \right) dz \wedge dx + \left( \frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} \right) dx \wedge dy. \qquad (15)$$

Now consider the Hodge dual of the form $dx^a \wedge dx^b$, given by

$$\star dx^a \wedge dx^b = \epsilon^{abc} dx^c. \qquad (16)$$

Applying this to equation (15), we see that

$$\star d\omega = \left(\frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z}\right) dx + \left(\frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x}\right) dy + \left(\frac{\partial \omega_x}{\partial x} - \frac{\partial \omega_x}{\partial y}\right) dz. \tag{17}$$

Now, for each point $p$ on a manifold $M$, we can consider the metric to be a map between the tangent space $T_pM$ and the dual space $T_p^*M$ of that manifold. Denote this operation by $\sharp$, so that $\sharp\, dx^\alpha = \partial/\partial x^\alpha$ and $\sharp\, \partial/\partial x^\alpha = dx^\alpha$. From the above equations we see that the curl of a vector field $V$ is given (in a really retarded way) by

$$\nabla \times V = \sharp \star d \sharp V. \tag{18}$$

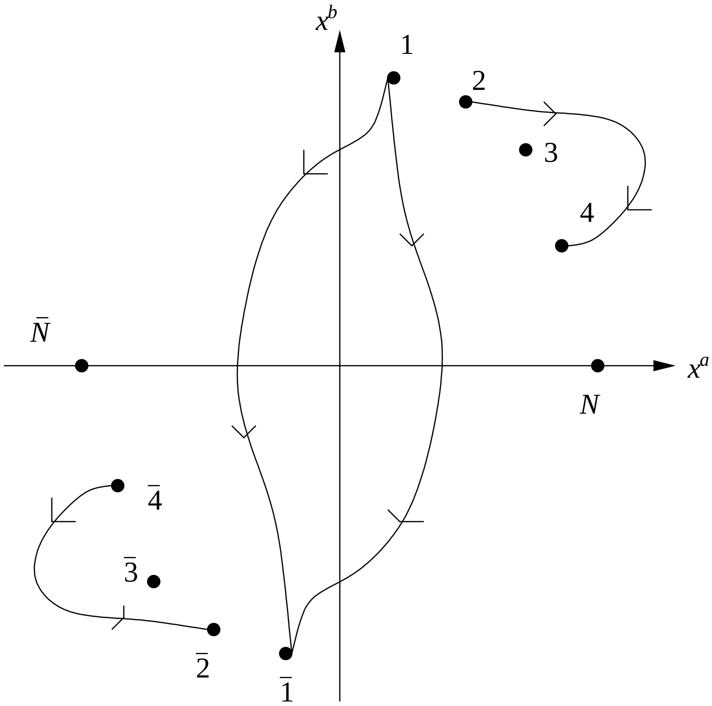
**Problem 14.4:** *Separated Dp-branes and an Op-plane*

**a.** Show the two strings obtained by the orientifold symmetry. Since the arguments $p^+, \vec{p}$ of the ground states are always present, let us omit them for brevity. The ground states are of four types:

$$|[ij]\rangle, \quad |[i\bar{j}]\rangle, \quad |[\bar{i}j]\rangle, \quad |[\bar{i}\bar{j}]\rangle. \tag{19}$$

Each class contains $N^2$ ground states since $i$ and $j$ run from 1 to $N$ and $\bar{i}$ and $\bar{j}$ run from $\bar{1}$ to $\bar{N}$. Define an expected action of $\Omega_p$ on the ground states in (19). Show that your choice satisfies $\Omega_p^2 = 1$ acting on the ground states.

The two strings and their orientifolding partners are depicted as follows:




We expect the following action of $\Omega_p$ on the ground states in (19):

$$\Omega_p|[ij]\rangle = |[\bar{j}\bar{i}]\rangle, \quad \Omega_p|[i\bar{j}]\rangle = |[\bar{j}i]\rangle, \quad \Omega_p|[\bar{i}j]\rangle = |[j\bar{i}]\rangle, \quad \Omega_p|[\bar{i}\bar{j}]\rangle = |[ji]\rangle. \tag{20}$$

It is obvious that $\Omega_p^2 = 1$ when acting on any of the ground states.

**b.** What are the possible interactions between strings in the four types of sectors built on the states (19)?

The following interactions between strings can take place:

$$\begin{aligned} &[ij] * [jk] = [ik] \quad [ij] * [j\bar{k}] = [i\bar{k}] \quad [i\bar{j}] * [\bar{j}k] = [ik] \quad [i\bar{j}] * [\bar{j}\bar{k}] = [i\bar{k}] \\ &[\bar{i}j] * [jk] = [\bar{i}k] \quad [\bar{i}j] * [j\bar{k}] = [\bar{i}\bar{k}] \quad [\bar{i}\bar{j}] * [\bar{j}k] = [\bar{i}k] \quad [\bar{i}\bar{j}] * [\bar{j}\bar{k}] = [\bar{i}\bar{k}]. \end{aligned} \tag{21}$$

**c.** String products have a component along the ground state, as in

$$|[i\bar{j}]\rangle * |[\bar{j}k]\rangle = |[ik]\rangle + \cdots . \tag{22}$$

Write the other possible ground state products. Test the consistency of your definition of $\Omega_p$ by acting with $\Omega_p$ on both sides of the equations giving ground state products. $\Omega_p$ acts on a product via

$$\Omega_p(|A\rangle * |B\rangle) = (\Omega_P|B\rangle) * (\Omega_p|A\rangle). \tag{23}$$

The other possible ground state products are

$$\begin{aligned}
|[ij]\rangle * |[jk]\rangle &= |[ik]\rangle + \cdots \\
|[ij]\rangle * |[j\bar{k}]\rangle &= |[i\bar{k}]\rangle + \cdots \\
|[i\bar{j}]\rangle * |[\bar{j}\bar{k}]\rangle &= |[i\bar{k}]\rangle + \cdots \\
|[\bar{i}j]\rangle * |[jk]\rangle &= |[\bar{i}k]\rangle + \cdots \\
|[\bar{i}j]\rangle * |[j\bar{k}]\rangle &= |[\bar{i}\bar{k}]\rangle + \cdots \\
|[\bar{i}\bar{j}]\rangle * |[\bar{j}k]\rangle &= |[\bar{i}k]\rangle + \cdots \\
|[\bar{i}\bar{j}]\rangle * |[\bar{j}\bar{k}]\rangle &= |[\bar{i}\bar{k}]\rangle + \cdots .
\end{aligned} \tag{24}$$

We act on the above equations using (23) to get

$$\begin{aligned}
|[\bar{k}\bar{j}]\rangle * |[\bar{j}\bar{i}]\rangle &= |[\bar{k}\bar{i}]\rangle + \cdots \\
|[k\bar{j}]\rangle * |[\bar{j}\bar{i}]\rangle &= |[k\bar{i}]\rangle + \cdots \\
|[kj]\rangle * |[j\bar{i}]\rangle &= |[k\bar{i}]\rangle + \cdots \\
|[\bar{k}\bar{j}]\rangle * |[\bar{j}i]\rangle &= |[\bar{k}i]\rangle + \cdots \\
|[k\bar{j}]\rangle * |[\bar{j}i]\rangle &= |[ki]\rangle + \cdots \\
|[\bar{k}j]\rangle * |[ji]\rangle &= |[\bar{k}i]\rangle + \cdots \\
|[kj]\rangle * |[ji]\rangle &= |[ki]\rangle + \cdots ,
\end{aligned} \tag{25}$$

which are consistent.

**d.** Find $\Omega_p$ action on $\alpha_n^i$ by using $\Omega_p\, X^i(\tau,\sigma)\, \Omega_p^{-1} = X^i(\tau, \pi-\sigma)$. Find $\Omega_p$ action on $\alpha_n^a$ using

$$\Omega_p\, \dot{X}^a(\tau,\sigma)\, \Omega_p^{-1} = -\dot{X}^a(\tau, \pi-\sigma). \tag{26}$$

Verify that for any arbitrary product $R$ of oscillators of both types, $\Omega_p\, R\, \Omega_p^{-1} = (-1)^{N_\perp} R$, where $N_\perp$ is the total number of $R$.

We use the method of the previous homework problem on orientifolds and D-branes to get that

$$\Omega_p\, \alpha_n^i\, \Omega_p^{-1} = (-1)^n \alpha_n^i, \tag{27}$$

where the $(-1)^n$ comes from a factor of $\cos(n\pi)$. To obtain the action of $\Omega_p$ on the transverse oscillators, we use the expansion for $\dot{X}^a$

$$\dot{X}^a = -i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^a e^{-in\tau} \sin n\sigma, \tag{28}$$

which turns out to be the same for a string stretched between two parallel D-branes as for one D-brane. We now apply relation (26) to get

$$\Omega_p\, \alpha_n^a\, \Omega_p^{-1} = (-1)^n \alpha_n^a. \tag{29}$$

Thus each $\alpha_n^{(i,a)}$ contributes a $(-1)^n$, so we see that for a product of oscillators of both kinds $R$ with number $N^\perp$, the action of $\Omega_p$ is given by $(-1)^{N_\perp}$.

**e.** Describe the orientifold spectrum in terms of the spectrum of a single D$p$-brane.

We can build a general state from any product $R$ of oscillators in the following manner:

$$|\Psi_R\rangle = \sum_{ij} \left( r_{ij} R|[ij]\rangle + r_{i\bar{j}} R|[i\bar{j}]\rangle + r_{\bar{i}j} R|[\bar{i}j]\rangle + r_{\bar{i}\bar{j}} R|[\bar{i}\bar{j}]\rangle \right) . \tag{30}$$

Acting on both sides with $\Omega_p$, we obtain

$$\Omega_p|\Psi_R\rangle = \sum_{ij} (-1)^{N_\perp} \left( r_{ij} R|[\bar{j}\bar{i}]\rangle + r_{i\bar{j}} R|[j\bar{i}]\rangle + r_{\bar{i}j} R|[\bar{j}i]\rangle + r_{\bar{i}\bar{j}} R|[ji]\rangle \right) . \tag{31}$$

If we require that our theory be $\Omega_p$ invariant, then we obtain the conditions

$$\begin{aligned} r_{ij} &= (-1)^{N_\perp} r_{\bar{j}\bar{i}} \\ r_{i\bar{j}} &= (-1)^{N_\perp} r_{j\bar{i}} \\ r_{\bar{i}j} &= (-1)^{N_\perp} r_{\bar{j}i} . \end{aligned} \tag{32}$$

The first of these relations states that the matrix $r_{\bar{i}\bar{j}}$ is completely determined once we fix the $N^2$ parameters in $r_{ij}$ (one is the $\pm$ transpose of the other). The other two conditions state that the $r_{i\bar{j}}$ and $r_{\bar{i}j}$ matrices are either both symmetric or both antisymmetric. For the case of $N_\perp$ odd, we see that the antisymmetric condition holds. This tells us that there are in total $N^2 + 2 \times (N/2)(N-1) = 2N(N-1)$ independent parameters that can be used to build states. When $N_\perp$ is even, the symmetric condition holds and we then have $N^2 + 2 \times (N/2)(N+1) = 2N(N+1)$ independent parameters. This was what we wanted to show.

**Problem 15.3:** *Properties of the string charge $\vec{Q}$*

**a.** Consider a string at some fixed time $t_0$ and a region $\mathcal{R}$ of space that contains a portion of this string: the string enters the region $\mathcal{R}$ at a point $\vec{x}_i$ and leaves $\mathcal{R}$ at a point $\vec{x}_f$ (assume there is no compactification of space). Calculate the string charge $\vec{Q} = \int_{\mathcal{R}} \mathrm{d}^d x \vec{j}^0$ contained in $\mathcal{R}$ at time $t_0$. Use your result to show that the total string charge associated with a closed string is zero.

We know that $\vec{j}^0$ is given by

$$\vec{j}^0 = \frac{1}{2} \int \mathrm{d}\sigma \, \delta\left(\vec{x} - \vec{X}(t,\sigma)\right) \frac{\partial \vec{X}(t,\sigma)}{\partial \sigma}. \tag{33}$$

Thus, $\vec{Q}$ is given by

$$\begin{aligned} \vec{Q} &= \int_{\mathcal{R}} \mathrm{d}^d x \, \frac{1}{2} \int \mathrm{d}\sigma \, \delta\left(\vec{x} - \vec{X}(t_0,\sigma)\right) \frac{\partial \vec{X}(t_0,\sigma)}{\partial \sigma} \\ &= \frac{1}{2} \int \mathrm{d}\sigma \int_{\mathcal{R}} \mathrm{d}^d x \, \delta\left(\vec{x} - \vec{X}(t_0,\sigma)\right) \frac{\partial \vec{X}(t_0,\sigma)}{\partial \sigma} \\ &= \frac{1}{2} \int \mathrm{d}\sigma \, \frac{\partial \vec{X}(t_0,\sigma)}{\partial \sigma} \left\{ \begin{array}{ll} 1, & \vec{X}(t_0,\sigma) \in \mathcal{R} \\ 0, & \vec{X}(t_0,\sigma) \notin \mathcal{R} \end{array} \right\} \\ &= \frac{1}{2} \left( \vec{X}(t_0, \vec{x}_f) - \vec{X}(t_0, \vec{x}_i) \right). \end{aligned} \tag{34}$$

Thus we see that the charge associated with a closed string is $\vec{0}$ because $\vec{x}_i$ and $\vec{x}_f$ are identical when the region encloses the entire string.

**b.** A more abstract proof that $\vec{Q}$ is zero for any *localized* configuration of closed strings requires showing that $\nabla \cdot \vec{j}^0 = 0$ implies $\int \mathrm{d}^d x \, \vec{j}^0 = 0$.

First we note that the following identity holds for any two functions $\phi$ and $\psi$:

$$\nabla \cdot (\phi\psi\vec{j}^0) = \phi\nabla\psi \cdot \vec{j}^0 + \psi\nabla\phi \cdot \vec{j}^0 + \phi\psi\nabla \cdot \vec{j}^0. \tag{35}$$

Integrating $\nabla \cdot (\phi\psi\vec{j}^0)$ over all space and using the divergence theorem, we have

$$\int_{\text{space}} \mathrm{d}^d x \, \nabla \cdot (\phi\psi\vec{j}^0) = \oint_{\partial(\text{space})} \phi\psi\vec{j}^0 \cdot \mathrm{d}\vec{\sigma} = 0, \tag{36}$$

where we used the fact that $\vec{j}^0$ is localized in space. We can then use our previously-derived identity, dropping off the term containing $\nabla \cdot \vec{j}^0$, which vanishes, and using $\phi = 1$ and $\psi = x^i$:

$$\int \mathrm{d}^d x \, \frac{\partial x^i}{\partial x^j} \, \hat{e}^j \cdot \vec{j}^0 = \int \mathrm{d}^d x \, \delta^i_j \, j^{0,j} = \int \mathrm{d}^d x \, j^{0,i} = 0, \tag{37}$$

which holds for all $i$, so we've proven what we wanted to prove.

**c.** Assume now that one space coordinate $x$ is curled up into a circle of radius $R$, and consider a closed string wrapped around this circle. Calculate the string charge $\vec{Q}$. Explain why the answer is not zero, and compare with the result obtained in (15.72).

The string charge is given by

$$\frac{1}{2} \left( \vec{X}(t_0, \vec{x}_f) - \vec{X}(t_0, \vec{x}_i) \right). \tag{38}$$

Now this evaluates to $\frac{1}{2} 2\pi R m = m\pi R$, where $m$ is the number of times the string is wrapped around the compactified dimension. This is nonzero because there can be two different string coordinates associated with identified points in space if some of the dimensions are compactified.

**Problem 17.3:** *Charge carried by winding strings*

The zero mode expansion of a string with winding number $\ell$ is given by

$$X^m(\tau,\sigma) = x^m(\tau), \qquad X(\tau,\sigma) = \ell R\sigma, \tag{39}$$

where the index $m$ is used to denote all string coordinates except for $X^{25} \equiv X$. Consider the coupling term of the Kalb-Ramond field to the string:

$$S = -\int \mathrm{d}\tau \mathrm{d}\sigma \, \frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \sigma} B_{\mu\nu}(X). \tag{40}$$

Calculate the terms in $S$ that couple $B_{m,25} = -B_{25,m}$ to the string trajectory $x^a(\tau)$.

The term which couples $B_{m,25}$ to the string trajectory is given by

$$\left(\partial_\tau X^m \partial_\sigma X - \partial_\tau X \partial_\sigma X^m\right) B_{m,25}, \tag{41}$$

which simplifies to

$$\ell R \frac{\mathrm{d}x^m}{\mathrm{d}\tau} B_{m,25}. \tag{42}$$

Note that this is strikingly similar to the term which couples a Maxwell field to the trajectory of a particle,

$$S_{\mathrm{EM}} = \frac{e}{c}\int_{\mathcal{P}} \mathrm{d}\tau \, A_\mu\left(x(\tau)\right) \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}(\tau). \tag{43}$$

Making the identification $(e/c)A_\mu \leftrightarrow (\ell R)B_{m,25}$ (up to a sign), we see that the electric charge carried by the string is proportional to its winding number $\ell$.