33-652: String Theory
Assignment 6

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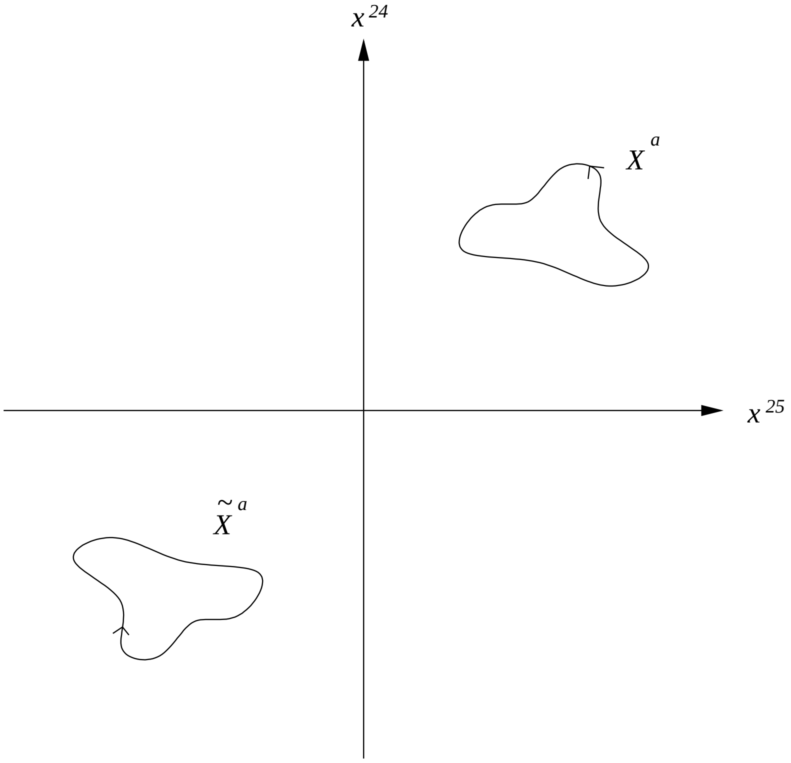
Problem 13.6: Orientifold Op-planes.

a. For an O23-plane the two normal directions x^{24}, x^{25} can be represented by a plane. A closed string at a fixed τ appears as a parametrized closed curve $X^a(\tau, \sigma)$ in this plane. Draw such an oriented closed string that lies fully in the first quadrant of the (x^{24}, x^{25}) plane. Draw also the string $\tilde{X}^a(\tau, 2\pi - \sigma)$.



b. For any operator $\mathcal{O}$ the action of Ω_p is defined by $\mathcal{O} \to \Omega_p \mathcal{O} \Omega_p^{-1}$. Use the expansion (13.24) to calculate the action of Ω_p on the operators x_0^a, p^a, α_n^a, $\bar{\alpha}_n^a$, x_0^i, p^i, α_n^i, and $\bar{\alpha}_n^i$.

The closed string coordinates are represented by

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(\alpha_n^\mu e^{in\sigma} + \bar{\alpha}_n^\mu e^{-in\sigma}). \tag{1}$$

The action of Ω_p on each of the tangential coordinates is the same as that of the twist operator in problem 13.5. Thus,

$$\Omega\, x_0^i\, \Omega^{-1} = x_0^i \tag{2}$$
$$\Omega\, p^i\, \Omega^{-1} = p^i \tag{3}$$
$$\Omega\, \alpha_n^i\, \Omega^{-1} = \bar{\alpha}_n^i \tag{4}$$
$$\Omega\, \bar{\alpha}_n^i\, \Omega^{-1} = \alpha_n^i. \tag{5}$$

For the transverse dimensions, we have

$$\Omega\, x_0^a\, \Omega^{-1} = -x_0^a \tag{6}$$
$$\Omega\, p^a\, \Omega^{-1} = -p^a \tag{7}$$
$$\Omega\, \alpha_n^a\, \Omega^{-1} = -\bar{\alpha}_n^a \tag{8}$$
$$\Omega\, \bar{\alpha}_n^a\, \Omega^{-1} = -\alpha_n^a. \tag{9}$$

For the $+$ and $-$ coordinates, Ω simply reverses the orientation. This is obviously the case for X^+. For X^- we know that the only non-trivial actions will involve products like $\alpha_n^a \alpha_{q-n}^a$ and their barred equivalents. The negative signs produced by the Ω action will cancel, and we will obtain the same result as in problem 13.5. The orientifold transformations are symmetries of closed string theory because they leave the closed string Virasoro operators invariant for exactly the same reason that they leave the X^- coordinate invariant. The closed string Virasoro operators are the operator-ingredients for the closed string Hamiltonian, so

$$\Omega\, H\, \Omega^{-1} = H \Rightarrow [\Omega, H] = 0 \tag{10}$$

c. Denote the ground states by $|p^+, p^i, p^a\rangle$. Assume that the states $|p^+, p^i, \vec{0}\rangle$ are invariant under Ω: $\Omega|p^+, p^i, \vec{0}\rangle = |p^+, p^i, \vec{0}\rangle$. Prove that

$$\Omega|p^+, p^i, p^a\rangle = |p^+, p^i, -p^a\rangle.$$

We know that p^a transforms as $\Omega\, p^a\, \Omega^{-1} = -p^a$, so Ω and p^a anticommute. Hence,

$$\begin{aligned} p^a_{\rm op}\Omega|p^+,p^i,p^a\rangle &= -\Omega p^a_{\rm op}|p^+,p^i,p^a\rangle \qquad (11)\\ &= -p^a\Omega|p^+,p^i,p^a\rangle, \qquad (12)\end{aligned}$$

so we see that

$$\Omega|p^+,p^i,p^a\rangle = |p^+,p^i,-p^a\rangle. \qquad (13)$$

d. The massless states of the oriented closed string theory are given by

$$|\Phi\rangle = \int \mathrm{d}p^+\, \mathrm{d}^{p-1}p^i\, \mathrm{d}^{d-p}p^a\ \Phi^{\pm}_{IJ}(\tau,p^+,p^i,p^a)(\alpha^I_{-1}\bar\alpha^J_{-1} \pm \alpha^J_{-1}\bar\alpha^I_{-1})|p^+,p^i,p^a\rangle.$$

Find the conditions that must be satisfied by $\Phi^\pm_{ab}$, $\Phi^\pm_{ia}$, and $\Phi^\pm_{ij}$ to guarantee Ω invariance. All of the conditions are of the form

$$\Phi^\pm_{IJ}(\tau,p^+,p^i,p^a) = \cdots \Phi^\pm_{IJ}(\tau,p^+,p^i,-p^a),$$

where the dots represent a sign factor that you must determine for each case.

We can write the action of Ω on $|\Phi\rangle$ as

$$\begin{aligned}\Omega|\Phi\rangle &= \int \mathrm{d}^D p/\mathrm{d}p^- \Phi^\pm_{IJ}(\dots)\Big(\Omega\alpha^I_{-1}\Omega^{-1}\Omega\bar\alpha^J_{-1}\Omega^{-1}\\ &\pm\ \Omega\alpha^J_{-1}\Omega^{-1}\Omega\bar\alpha^I_{-1}\Omega^{-1}\Big)\Omega|p^+,p^i,p^a\rangle. \qquad (14)\end{aligned}$$

When $(IJ) = (ab)$ the operator sum in the expression for $|\Phi\rangle$ is going to undergo the transformation $\alpha \leftrightarrow \bar\alpha$, while the action of Ω on the basis states will invert their transverse momenta. The operator sum is symmetric under the $\alpha \leftrightarrow \bar\alpha$ transformation when the $\pm$ is a $+$, and antisymmetric when the $\pm$ is a $-$. Thus we need a factor of $-$ to cancel the asymmetry and nothing to cancel the symmetry, so we need a factor of $\pm$:

$$\Phi^\pm_{ab}(p^a) = \pm\Phi^\pm_{ab}(-p^a). \qquad (15)$$

When $(IJ) = (ia)$, we pick up negative signs in the operator sum under the action of Ω, and upon reordering terms we obtain, schematically:

$$\mp\Phi^\pm_{ia}(p^a)(\alpha^i_{-1}\bar\alpha^a_{-1} \pm \alpha^a_{-a}\bar\alpha^i_{-1})|p^+,p^i,-p^a\rangle. \qquad (16)$$

So the action of Ω contributes a factor of $\mp$ to the operator terms and inverts the basis states, and we thus require, for $(IJ) = (ia)$, that

$$\Phi^{\pm}_{ia}(p^a) = \mp\Phi^{\pm}_{ia}(-p^a). \tag{17}$$

The action of Ω yields the same results when $(IJ) = (ij)$ as for when $(IJ) = (ab)$ so we need a factor of $\pm$ in this case as well:

$$\Phi^{\pm}_{ij}(p^a) = \pm\Phi^{\pm}_{ij}(-p^a). \tag{18}$$

Problem 13.7: Massive level in the open superstring.

a. Consider eight anticommuting variables b^i, with $i = 1, \ldots, 8$. Ignoring signs, how many inequivalent products of the form $b^{i_1}b^{i_2}$ can be built? How many $b^{i_1}b^{i_2}b^{i_3}$? How many $b^{i_1}b^{i_2}b^{i_3}b^{i_4}$?

Ignoring signs, the question we are trying to answer is "How many different ways can we choose N out of 8 objects?". This is the case because permutations don't matter when we are ignoring signs, and we can't choose the same object twice, or else we get zero (since the b's are Grassmannian). Thus, for $b^{i_1}b^{i_2}$ we can choose $\binom{8}{2} = 28$ different products, for $b^{i_1}b^{i_2}b^{i_3}$ we have $\binom{8}{3} = 56$, and for $b^{i_1}b^{i_2}b^{i_3}b^{i_4}$ we have $\binom{8}{4} = 70$ choices.

b. Consider the first excited level of the open superstring ($\alpha' M^2 = 1$). List the states in the NS sector and the states in the R sector. Confirm that you get the same number of states (the number is more than 64).

In the NS sector we have the following allowed states:

$$\begin{array}{rr} b^I_{-3/2}|\mathrm{NS}\rangle \otimes |p^+, \vec{p}_T\rangle & 8 \\ b^I_{-1/2}\, b^J_{-1/2}\, b^K_{-1/2}|\mathrm{NS}\rangle \otimes |p^+, \vec{p}_T\rangle & \binom{8}{3} = 56 \\ \alpha^I_{-1}\, b^J_{-1/2}|\mathrm{NS}\rangle \otimes |p^+, \vec{p}_T\rangle & 8 \times 8 = 64 \end{array}$$

yielding a total of 128 states. In the R sector we have the following allowed states:

$$\begin{array}{rr} d^I_{-1}|\mathrm{R}^{\mathrm{a}}_1\rangle \otimes |p^+, \vec{p}_T\rangle & 8 \times 8 = 64 \\ d^I_{-1}|\mathrm{R}^{\mathrm{a}}_2\rangle \otimes |p^+, \vec{p}_T\rangle & 8 \times 8 = 64 \end{array}$$

also yielding a total of 128 states. Thus we see that the NS and R sectors have the same number of $\alpha' M^2 = 1$ states.

Problem 14.1: A Dp-brane with orientifolds.

a. The Ω actions are given by

$$\begin{aligned} \Omega\, X^a(\tau,\sigma)\,\Omega^{-1} &= X^a(\tau,\pi-\sigma), && (19)\\ \Omega\, X^i(\tau,\sigma)\,\Omega^{-1} &= X^i(\tau,\pi-\sigma). && (20)\end{aligned}$$

Give the Ω action on the oscillators α_n^a and α_n^i. What is the expected Ω action on α_n^-? Does it work out?

We have the expansions

$$\begin{aligned} X^a(\tau,\sigma) &= \bar{x}^a + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^a e^{-in\tau}\sin n\sigma && (21)\\ X^i(\tau,\sigma) &= x_0^i + \sqrt{2\alpha'}\alpha_0^i\tau \\ &\quad + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_n^i e^{-in\tau} - \alpha_{-n}^i e^{in\tau})\cos n\sigma && (22)\end{aligned}$$

For the transverse oscillators, we see that a factor of $\sin(n\pi - n\sigma)$ contributes an additional minus sign to the usual $(-1)^n$ for the tangential oscillators, so we get

$$\begin{aligned} \Omega\,\alpha_n^i\,\Omega^{-1} &= (-1)^n\alpha_n^i && (23)\\ \Omega\,\alpha_n^a\,\Omega^{-1} &= (-1)^{n+1}\alpha_n^a && (24)\end{aligned}$$

For α_n^- we expect to pick up the same sign change as for the tangential oscillators. We just need to check the action of Ω on the Virasoro operators:

$$\begin{aligned} \sum_{p\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I &= \sum_{p\in\mathbb{Z}}(\alpha_p^i\alpha_{n-p}^i + \alpha_p^a\alpha_{n-p}^a) && (25)\\ &\to \sum_{p\in\mathbb{Z}}((-1)^{p+n-p}\alpha_p^i\alpha_{n-p}^i + (-1)^{p+1+n-p+1}\alpha_p^a\alpha_{n-p}^a) && (26)\\ &= (-1)^n\sum_{p\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I, && (27)\end{aligned}$$

which is what we expect.

b. Assume that the ground states $|p^+,\vec{p}\rangle$ are Ω invariant. Find the states of the theory for $N^\perp \leq 2$. As you will see, some massless sates survive. Interpret these states along the lines of the discussion below (14.37).

The possible $N^\perp \leq 2$ states that we can build are

$$\begin{array}{rc}
\alpha^a_{-1}|p^+,\vec{p}\rangle & \qquad +1 \\
\alpha^i_{-1}|p^+,\vec{p}\rangle & \qquad -1 \\
\alpha^a_{-1}\alpha^b_{-1}|p^+,\vec{p}\rangle & \qquad +1 \\
\alpha^a_{-1}\alpha^i_{-1}|p^+,\vec{p}\rangle & \qquad -1 \\
\alpha^i_{-1}\alpha^j_{-1}|p^+,\vec{p}\rangle & \qquad +1 \\
\alpha^a_{-2}|p^+,\vec{p}\rangle & \qquad -1 \\
\alpha^i_{-2}|p^+,\vec{p}\rangle & \qquad +1
\end{array}$$

If we only keep states that satisfy $\Omega = +1$ we still end up with massless states, namely those that we get by acting with α^a_{-1} on the ground states. There are $(d-p)$ of these massless scalar states. These can be viewed as excitations of the Dp-brane in its normal directions.

c. Replace the O25-plane by an Op-plane coincident with the Dp-brane at $\bar{x}^a = 0$. Let Ω_p denote the operator for which this theory keeps only the states with $\Omega_p = +1$. How should equations (19) and (20) change when Ω is replaced by Ω_p? Give the Ω_p action on the oscillators α^a_n and α^i_n.

Equations (19) and (20) change to

$$\Omega_p\, X^a(\tau,\sigma)\,\Omega_p^{-1} \;=\; -X^a(\tau,\pi-\sigma) \tag{28}$$
$$\Omega_p\, X^i(\tau,\sigma)\,\Omega_p^{-1} \;=\; X^i(\tau,\pi-\sigma). \tag{29}$$

Since Ω_p's action hasn't changed for the tangential coordinates, its action on α^i_n stays the same. However, we pick up a minus sign for the normal oscillators:

$$\Omega_p\,\alpha^a_n\,\Omega_p^{-1} = -(-1)^{n+1}\alpha^a_n = (-1)^n\alpha^a_n, \tag{30}$$

so we have

$$\Omega_p\,\alpha^I_n\,\Omega_p^{-1} = (-1)^n\alpha^I_n, \tag{31}$$

which means that we have the same allowed spectrum as for the twist-operator invariant spectrum for open strings, with a D25-brane present. This excludes the massless scalar states that we did allow for the operator Ω. We are left over with only the states for which $N^\perp$ is even. This means that we don't get the same normal excitations of the Dp-brane as we did for the case with an O25-plane.

Problem 4: Yang-Mills theory.

$U(N)$ has N^2 independent generators. The elements of $SU(N)$ must satisfy the additional constraint of $\det(M) = 1$. Since $M = \exp(i\alpha \cdot J)$ in general, and $\det(M) = \exp(i\mathrm{Tr}(\alpha \cdot J))$, we require that the generators be traceless, so this additional constraint reduces the dimensionality to $N^2 - 1$ for $SU(N)$. $SO(N)$ is just the group of rotations in N dimensions. There are $\binom{N}{2} = N(N-1)/2$ dimensions in this group because there are that many planes of rotation in N dimensions.

To generalize the QED action $-(1/4)F^{\mu\nu}F_{\mu\nu}$ to non-Abelian groups, we allow the gauge field to transform as

$$\delta A^a_\mu = gf^{abc}\alpha^b A^c_\mu + \partial_\mu \alpha^a. \tag{32}$$

The generators of the gauge group satisfy

$$[T^a, T^b] = if^{abc}T^c. \tag{33}$$

When the group is U(1), there is only one generator for the group. Since f^{abc} can be made completely antisymmetric, the fact that $[T^a, T^b] = 0$ means that the structure constants are zero, and the gauge transformation reduces to the usual transformation in QED. Define

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu. \tag{34}$$

This then transforms as

$$\begin{aligned}
F^a_{\mu\nu} \quad &\to \quad \partial_\mu(A^a_\nu + gf^{abc}\alpha^b A^c_\nu + \partial_\nu\alpha^a) - \partial_\nu(A^a_\mu + gf^{abc}\alpha^b A^c_\mu + \partial_\mu\alpha^a) \\
&\qquad +gf^{ade}(A^d_\mu + gf^{abc}\alpha^b A^c_\mu + \partial_\mu\alpha^d)(A^e_\nu + gf^{ebc}\alpha^b A^c_\nu + \partial_\nu\alpha^e) \\
&= \quad F^a_{\mu\nu} + gf^{abc}(\partial_\mu(\alpha^b A^c_\nu) - \partial_\nu(\alpha^b A^c_\mu)) + gf^{ade}A^d_\mu(gf^{ebc}\alpha^b A^c_\nu + \partial_\nu\alpha^e) \\
&\qquad +gf^{ade}(gf^{dbc}\alpha^b A^c_\mu + \partial_\mu\alpha^d)A^e_\nu + \mathcal{O}(\alpha^2) \\
&= \quad F^a_{\mu\nu} + 2gf^{abc}((\partial_\mu\alpha^b)A^c_\nu - (\partial_\nu\alpha^b)A^c_\mu) \\
&\qquad +gf^{abc}\alpha^b(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu) - g^2 f^{dce}f^{abc}\alpha^b A\mu^d A^e_\nu + \mathcal{O}(\alpha^2) \\
&= \quad F^a_{\mu\nu} + gf^{abc}\alpha^b F^c_{\mu\nu} + 2gf^{abc}((\partial_\mu\alpha^b)A^c_\nu - (\partial_\nu\alpha^b)A^c_\mu) + \mathcal{O}(\alpha^2)
\end{aligned} \tag{35}$$

to first order in α^i.