A NONABSOLUTELY CONVERGENT INTEGRAL DEFINED BY PARTITIONS OF THE UNITY *

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Abstract.
In a compact metric measure space with measure μ, we define an integral by partitions of the unity such that a $\mu-$integrable function is also integrable and a function which is integrable but it is not $\mu-$integrable is constructed in a suitable non Euclidean space.
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Introduction

In [5], a type of integral defined by partitions of the unity (PU-integral) is defined on an abstract compact metric measure space and it is proved that a PU-integrable function is $\mu-$integrable and that the $\mu-$integral is equivalent to the PU-integral. Moreover an example of a non Euclidean space on which is defined this type of integral is given. The PU-integral is obtained by approssimations of type Riemann sums. The advantage to use a such integral is that it does not use the geometry of the space so it can be defined in any abstract space.

In this paper X denotes a compact metric space, $\mathcal{M}$ a σ-algebra of subsets of X such that each open set is in $\mathcal{M}$, μ a non-atomic, finite, complete Radon measure on $\mathcal{M}$ such that:

α) each ball $U(x,r)$ centered at x with radius r has a positive measure,

β) for every x in X there is a number $h(x) \in \Re$ such that $\mu(U[x,2r]) \le h(x) \cdot \mu(U[x,r])$ for all $r > 0$ (where $U[x,r]$) is the closed ball),

γ) $\mu(\partial U(x,r)) = 0$ where $\partial U(x,r)$ is the boundary of $U(x,r)$.

Definition 1 A partition of unity (PU-partition) in X is, by definition, a finite collection $P = \{(\theta_i, x_i)\}_{i=1}^p$ where $x_i \in X$ and θ_i are non negative, μ-measurable and μ-integrable real functions on X such that $\sum_{i=1}^p \theta_i(x) = 1$ a.e. in X.

The PU-partition is a PU*-partition if $x_i \in S_{\theta_i} = \{x \in X : \theta_i(x) \neq 0\}$.

Definition 2 If δ is a positive function on X, a PU-partition $P = \{(\theta_i, x_i)\}_{i=1}^p$ is said to be δ-fine if $S_{\theta_i} \subset U(x_i, \delta(x_i)) \quad (i = 1,2,..,p)$.

Definition 3 A real function f on a compact set $A \subseteq X$ is said to be (PU)-integrable on A if there exists a real number I with the property that, for every

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given $\epsilon > 0$, there is a positive function δ such that $| \sum_{i=1}^{p} f(x_i) \cdot \int_A \theta_i d\mu - I | < \epsilon$ for each δ-fine (PU)-partition $P = \{(\theta_i, x_i)\}_{i=1}^{p}$ in A.

The number I is called the (PU)-integral of f on A and we write $I = (PU) \int_A f$.

For (PU)*-partitions, we have the (PU)*-integral and set $I = (PU)^* \int_A f$.

Main results

Proposition 1 If δ is a positive function on a compact set $A \subseteq X$ then there is a δ-fine PU*-partition in A.

Proof . See the Proof of Proposition 1.1 in [5].

Denoting by $\mathcal{PU}^*(A)$ the family of all the PU*-integrable real functions on A, the following Proposition is an immediate consequence of the Definition 4.

Proposition 2 If $A \subseteq X$ is compact, then:

1) $\mathcal{PU}^*(A)$ is a linear space and the map $f \to (PU)^* \int_A f$ is a non negative linear functional on $\mathcal{PU}^*(A)$;

2) if $k \in \Re$ and $f(x) = k$ for each $x \in A$ then $f \in \mathcal{PU}^*(A)$ and $(PU)^* \int_A f = k\mu(A)$.

3) if f , $g \in \mathcal{PU}^*(\mathcal{A})$ and $f \leq g$ then $(PU)^* \int_A f \leq (PU)^* \int_A g$.

If $P = \{(\theta_i, x_i)\}_{i=1}^{n}$ is a partition in A, set $\sigma(f, P) = \sum_{i=1}^{n} f(x_i) \int_A \theta_i d\mu$.

Proposition 3 If A is a compact subset of X and if $f \in \mathcal{PU}^*(X)$ then $f \in \mathcal{PU}^*(A)$.

Proof . See Proposition 1.3 in [5].

Proposition 4 If f is a real function on a compact set $A \subseteq X$, then $f \in \mathcal{PU}^*(A)$ if and only if for each $\epsilon > 0$ there is a positive function δ on A such that $|\sigma(f, P) - \sigma(f, Q)| < \epsilon$ for every $P = \{(\theta_i, x_i)\}_{i=1}^{n}$ and $Q = \{(\theta'_i, x'_i)\}_{i=1}^{p}$ $\delta-$fine PU*-partitions in A.

Proof . See proposition 1.2 in [5].

Proposition 5 If f is $\mu-$measurable and $\mu-$integrable on X, then $f \in \mathcal{PU}^*(X)$ and $(PU)^* \int_X f = \int_X f d\mu$.

Proof. It follows by the equivalence between the PU-integral and the $\mu-$integral (see [5]) and because a PU*-partition is also a PU-partition.

Proposition 6 A PU*-integrable function is $\mu-$measurable.

Proof It is analogue to that used in [5] Propositions 2.3 and 2.4.

Proposition 7 If f, g are two real functions on X and $f = g$ a.e. in X then g is (PU)*-integrable if and only if f is (PU)*-integrable and the two integral coincide.

Proof If f is (PU)*-measurable then by Proposition 6 it is $\mu-$measurable and by completeness of measure also g is $\mu-$measurable, then $f-g=0$ a.e. and it is $\mu-$measurable, $\mu-$integrable and (PU)*-integrable with $(PU)^*\int_X(f-g)=0$. So $g=f-(f-g)$ is (PU)*-integrable.

Lemma 1 If f is a real $\mu-$integrable function on X, A, $B\in\mathcal{M}$, with $A\subset B$, and if $c\in\Re$ and $\int_A fd\mu\le c\le\int_B fd\mu$ then there exists a $\mu-$measurable set C such that $A\subset C\subset B$ and $\int_C fd\mu=c$.

Proof Consider the $\sigma-$algebra $\mathcal{D}=\{D\in\mathcal{M}: D\subset B-A\}$ and the signed measure $\alpha: D\to\int_D fd\mu$ for $D\in\mathcal{D}$.

By Liapounoff theorem (see [7]), the set $\{\alpha(D): D\in\mathcal{D}\}$ is a compact interval. So

$$\alpha(\emptyset)=0<c-\int_A fd\mu<\int_{B-A}fd\mu$$

and exists $D_1\in\mathcal{D}$ such that

$$\int_{D_1}fd\mu=c-\int_A fd\mu$$

$$c=\int_{A\cup D_1}fd\mu,\qquad A\subset A\cup D_1\subset B.$$

Proposition 8 If f is a $\mu-$measurable and PU*-integrable function on X, then for each $\epsilon>0$ there is a $\mu-$measurable set E such that $\mu(X-E)<\epsilon$, f is $\mu-$integrable on E and $\int_E fd\mu=(PU)^*\int_X f$.

Proof Suppose that f be not $\mu-$integrable; set

$$E_n=\{x\in X: n-1\le f(x)<n\},\quad F_n=\{x\in X: -n\le f(x)<-n+1\}\ n=1,2,3,...,$$

then

$$X=\bigcup_{n=1}^{\infty}(E_n\cup F_n)=\bigcup_{n=1}^{\infty}(\bigcup_{i=1}^{n}(E_i\cup F_i))=\bigcup_{n=1}^{\infty}H_n,$$

where $H_n=\bigcup_{i=1}^n(E_i\cup F_i)$ is an increasing sequence of measurable sets.

By a property of the measure, it results $\lim_{n\to\infty}\mu(H_n)=\mu(X)$ and for each $\epsilon>0$ there is $\bar{n}\in N$ such that for $n_0>\bar{n}$ it is

$$\mu(X)-\mu(H_{n_0})=\mu(X-H_{n_0})<\epsilon\qquad(*)$$

f is bounded on H_{n_0} so it is $\mu-$integrable on H_{n_0}.

Suppose that $\int_{H_{n_0}}fd\mu<(PU^*)\int_X f$; since f is not $\mu-$integrable, then the series $\sum_n\int_{E_n}fd\mu$ and $\sum_n\int_{F_n}fd\mu$ are divergent to $+\infty$ and to $-\infty$ respectively. In fact, if $\sum_n\int_{E_n}fd\mu=+\infty$ and $\sum_n\int_{F_n}fd\mu>-\infty$, consider the functions

$$f_1(x)=f(x)\quad\text{if}\quad x\in\bigcup_n E_n\quad\text{and}\quad f_1(x)=0\quad\text{elsewhere},$$

$$f_2(x) = f(x) \quad \text{if} \quad x \in \bigcup_n F_n \quad \text{and} \quad f_2(x) = 0 \quad \text{elsewhere},$$

then $f_2(x)$ is $\mu-$integrable and hence (PU)*-integrable and $f_1(x) = f(x) - f_2(x)$ is (PU)*-integrable, but it is also $\mu-$integrable with integral $+\infty$ and this is impossible. So for $\epsilon > 0$ there exists $K > n_0$ such that

$$\int_{H_{n_0}} f d\mu + \int_{E_{n_0+1}} f d\mu + ... + \int_{E_{n_0+k}} f d\mu > (PU)^* \int_X f$$

and set $H = H_{n_0} \cup E_{n_0+1} \cup .... \cup E_{n_0+k}$, it results

$$\int_{H_{n_0}} f d\mu < (PU)^* \int_X f < \int_H f d\mu.$$

By Lemma 1 there exists a $\mu-$measurable set E with $H_{n_0} \subset E \subset H$ such that $\int_E f d\mu = (PU)^* \int_X f$ and by relation (*) we have:

$$\mu(X - E) \leq \mu(X - H_{n_0}) < \epsilon.$$

Lemma 2 If f is $\mu-$measurable and there exists finite $\int_X f d\mu$, given $\epsilon > 0$ there is a positive function δ on X such that

$$\sum_i |(f(x_i) \int_X \theta_i d\mu - \int_X f \theta_i d\mu)| < \epsilon$$

for each $\delta-$fine (PU)*-partition $P = \{(\theta_i, x_i)\}$ in X.

Proof See Proposition 3.1 in [5]

Proposition 9 A $\mu-$measurable function f is (PU)*-integrable on X if and only if given $\epsilon > 0$ there is a positive function δ on X and a $\mu-$measurable set E such that $\mu(E^C) < \epsilon$, f is $\mu-$integrable on E and $|\sum_i f \chi_{E^C}(x_i) \int_X \theta_i d\mu| < \epsilon$ for each $\delta-$fine (PU)*-partition $P = \{(\theta_i, x_i)\}$. Moreover $\int_E f d\mu = (PU)^* \int_X f$.
We have set $E^C = X - E$.

Proof If f is (PU)*-integrable, by previous Proposition, let $\epsilon > 0$ there is $E \in \mathcal{M}$ such that $\mu(E^C) < \epsilon$, f is $\mu-$integrable on E and $\int_E f d\mu = (PU)^* \int_X f$; so $f \chi_E$ is $\mu-$integrable and hence (PU)*-integrable and

$$(PU)^* \int_X f \chi_E = \int_X f \chi_E d\mu = \int_E f d\mu = (PU)^* \int_X f.$$

By the (PU)*-integrability of f and $f \chi_E$, at correspondence of $\epsilon > 0$ there is a positive function δ on X such that for each $\delta-$fine (PU)*-partition $\{(\theta_i, x_i)\}$, it results

$$|\sum_i f(x_i) \int_X \theta_i d\mu - (PU)^* \int_X f| < \frac{\epsilon}{2}$$

and

$$\sum_i f(x_i)\chi_E \int_X \theta_i d\mu - (PU)^* \int_X f| < \frac{\epsilon}{2}.$$

So we have

$$|\sum_i f(x_i)\chi_{E^C} \int_X \theta_i d\mu| = |\sum_i f(x_i) \int_X \theta_i d\mu - \sum_i f(x_i)\chi_E \int_X \theta_i d\mu| \leq$$

$$\leq |\sum_i f(x_i) \int_X \theta_i d\mu - (PU)^* \int_X f| + |\sum_i f\chi_E(x_i) \int_X \theta_i d\mu - (PU)^* \int_X f| < \epsilon.$$

Conversely, for $\epsilon > 0$ let E be a $\mu-$measurable and $\mu-$integrable set with $\mu(E^C) < \epsilon$ and let δ be a positive function on X such that $|\sum_i f\chi_E^C(x_i) \int_X \theta_i d\mu| < \frac{\epsilon}{2}$ for each $\delta-$fine (PU)*-partition P in X.

By the $\mu-$integrability of f on E , then also the function $f\chi_E$ is $\mu-$integrable and, by lemma 2, there is a positive function δ_1 on X such that

$$|\sum_i f\chi_E(x_i) \int_X \theta_i d\mu - \int_X f\chi_E d\mu| < \frac{\epsilon}{2}.$$

If $\bar{\delta}(x) = \min(\delta(x), \delta_1(x))$ for each $x \in X$, then for each $\bar{\delta}-$fine (PU)*-partition P consider:

$$|\sum_i f(x_i) \int_X \theta_i d\mu - \int_E f d\mu| \leq |\sum_i f\chi_E(x_i) \int_X \theta_i d\mu - \int_E f d\mu| +$$

$$+ |\sum_i f\chi_E^C(x_i) \int_X \theta_i d\mu| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So f is (PU)*-integrable and $(PU)^* \int_X f = \int_E f d\mu$.

Proposition 10 If f and $|f|$ are (PU)*-integrable then f is $\mu-$integrable.

Proof If f and $|f|$ are (PU)*-integrable, consider the bounded sequence $f_n = |f| \wedge n$ for each $n \in N$ it converges increasing to $|f|$ and it is $\mu-$integrable and

$$\int_X |f| d\mu = \lim_n \int_X f_n d\mu = \lim_n (PU)^* \int_X f_n \leq (PU)^* \int_X |f| < +\infty.$$

So $|f|$ and f are $\mu-$integrable.

Proposition 11 If $(f_n)_n$ is an increasing sequence of (PU)*-integrable functions converging to f pointwise and $\lim_n (PU)^* \int_X f_n < \infty$ then f is (PU)*-integrable and $(PU)^* \int_X f = \lim_n (PU)^* \int_X f_n$.

Proof Consider the increasing sequence $(f_n - f_1)_n$ converging to $f - f_1$; since the functions $(f_n - f_1)_n$ are non negative, then by Proposition 10, they are $\mu-$integrable and

$$\lim_n \int_X (f_n - f_1) d\mu = \lim_n (PU)^* \int_X (f_n - f_1) =$$

$$= \lim_n (PU)^* \int_X f_n - (PU)^* \int_X f_1 < +\infty.$$

So by the monotone theorem for the $\mu-$integrable functions, the function $(f - f_1)$ is $\mu-$integrable and hence (PU)*-integrable. Therefore $f = (f - f_1) + f_1$ is (PU)*-integrable.

Proposition 12 If $(f_n)_n$ is a sequence of (PU)* integrable functions converging to f and such that there are two functions h and g (PU)*-integrable with $h \leq f_n \leq g$ for each $n \in N$ then f is (PU)*-integrable and $(PU)^* \int_X f = \lim_n (PU)^* \int_X f_n$.

Proof Consider the sequence $(f_n - h)_n$; it is non negative and (PU)*-integrable, so it is $\mu-$integrable and results:

$$0 \leq (f_n - h) \leq (g - h).$$

Since the function $g-h$ is non negative and (PU)*-integrable, it is $\mu-$integrable and by the dominate convergent theorem, the sequence of functions $(f_n - h)$ converges to $f - h$ which is a $\mu-$integrable function and hence (PU)*-integrable. By the equality $f = (f - h) + h$ it follows the (PU)*-integrability of f.

Proposition 13 If f is $\mu-$measurable and exists finite $\int_X f d\mu$ but $\int_X |f| d\mu = +\infty$ then f is (PU)*-integrable and $\int_X f d\mu = (PU)^* \int_X f$.

Proof If $\epsilon > 0$, by lemma 2, there is positive function δ on X such that if $P = \{(\theta_i, x_i)\}$ is a (PU)*-partition in X, then we have:

$$\epsilon > |\sum_i (f(x_i) \int_X \theta_i d\mu - \int_X f\theta_i d\mu)| = |\sum_i f(x_i) \int_X \theta_i d\mu - \sum_i f\theta_i d\mu| =$$

$$= |\sum_i (f(x_i) \int_X \theta_i d\mu - \int_X f d\mu)|.$$

An example of a function which is PU*-integrable but it is not PU-integrable.

Consider the space $X = \{0,1\}^N$. Let $\bar{\alpha} = \alpha_1 \alpha_2 ... \alpha_k$ be a finite string of 0 and 1; consider the set $A_{\bar{\alpha}} = [\bar{\alpha}]_k = \{\gamma \in X : \gamma = \bar{\alpha}\beta$, for some $\beta \in X\}$, it is a clopen set (i.e. an open and closed set) with respect to the topology induced by the metric ρ so defined:

if $\alpha, \beta \in X$ $\rho(\alpha, \beta) = \frac{1}{2^n}$ if $\alpha \neq \beta$ and $\alpha_1 = \beta_1, ..., \alpha_n = \beta_n, \alpha_{n+1} \neq \beta_{n+1}$ $\rho(\alpha, \alpha) = 0$.

With respect to this metric ρ, $X = \{0,1\}^N$ is a complete, separable and compact metric space (see [3]). Define on the family $\{A_{\bar{\alpha}}\}$ the following set function m:

$$m(A_{\bar{\alpha}}) = \frac{1}{2^k}$$

and let m^* be the outer measure induced by m on the family of all the subsets of X. If $\mathcal{M}$ is the $\sigma-$algebra of all the subsets of X m^*-measurable in the Caratheodory sense, then the space $(X, \mathcal{M}, m^*)$ satisfies the conditions $\alpha), \beta), \gamma)$ (see [3] and [5]).

Define on X the following real function

$$f(\alpha) = \begin{cases} a_1 & if\ \alpha_1 = 0 \\ a_2 & if\ \alpha_1 = 1, \alpha_2 = 0 \\ a_n & if\ \alpha_1, \alpha_2, ...\alpha_{n-1} = 1, \alpha_n = 0 \\ ... \\ . \end{cases}$$

$$f(1111...111..) = 0$$

where $\alpha = (\alpha_1, \alpha_2, ...) \in \{0,1\}^N$ and $a_n = (-1)^n \frac{2^n}{n}$.

Then, by Proposition 13, we have:

$$\int_X f dm = \sum_{n=1}^{\infty} a_n \frac{1}{2^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} = (PU)^* \int_X f,$$

so f is PU*-integrable but $|f|$ is not μ-integrable so it is not also (PU)-integrable.

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