Convergence of Banach valued stochastic processes of Pettis and McShane integrable functions *†

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Abstract

It is shown that if $(X_n)_n$ is a Bochner integrable stochastic process taking values in a Banach lattice E, the convergence of $f(X_n)$ to $f(X)$ where f is in a total subset of E^* implies the a.s. convergence. For any Banach space E-valued stochastic process of Pettis integrable strongly measurable functions $(X_n)_n$, the convergence of $f(X_n)$ to $f(X)$ for each f in a total subset of E^* implies the convergence in the Pettis norm. Also convergence theorems of Mc-Shane integrable martingales are given.

1. Introduction

In [4] and [7] it is proved that if $(X_n)_n$ is a stochastic process of Bochner integrable functions taking values in a Banach space E, the convergence of $f(X_n)$ to $f(X)$ where f is in a total subset of E^*, implies the scalar convergence of X_n to X. The same result is extended to stochastic processes taking values in a Banach lattice E.

It is known that the weak Radon-Nikodym property is equivalent to the convergence in Pettis norm of a uniformly integrable martingale (see [10]). If this property does not hold, we ask for which class T of functionals f the convergence of the real valued stochastic process $f(X_n)$ to $f(X)$ implies the convergence of X_n to X in Pettis norm. In section 4 we prove that for Pettis-integrable strongly measurable martingales, T can be a total subset of E^* (Theorem 3).

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In the last section we shall deal with martingales of McShane integrable functions and the analogous of Theorem 3 is proved (see Theorem 8).

2. Preliminaries

Let E be a Banach space with norm $\|\cdot\|$, $B(E)$ its unit ball and E^* its dual. A subset T of E^* is called a *total set* over E if $f(x)=0$ for each $f\in T$ implies $x=0$.
Throughout $(\Omega,\mathcal{F},P)$ is a probability space and $(\mathcal{F}_n)_{n\in I\!N}$ a family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_m\subset\mathcal{F}_n$ if $m<n$. Moreover, without loss of generality, we will assume that $\mathcal{F}$ is the completion of $\sigma(\cup_n\mathcal{F}_n)$.
Let $\mathcal{F}_0$ be a sub-σ-algebra of $\mathcal{F}$, then a function $X:\Omega\to E$ is called *weakly $\mathcal{F}_0$ -measurable* if the function $f(X)$ is $\mathcal{F}_0$-measurable for every $f\in E^*$. A weakly $\mathcal{F}$-measurable function is called *weakly measurable.* A function $X:\Omega\to E$ is said to be *Pettis integrable* if $f(X)$ is Lebesgue integrable on Ω for each $f\in E^*$ and there exists a set function $\nu:\mathcal{F}\to E$ such that

$$f\nu(A)=\int_A fX$$

for all $f\in E^*$ and $A\in\mathcal{F}$. In this case we write $\nu(A)=P\int_A X$ and we call $\nu(\Omega)$ the *Pettis integral* of X *over* Ω and ν is the *indefinite Pettis integral of* X. The space of all E-valued Pettis integrable functions is denoted by $\mathcal{P}(E)$. The Pettis norm of a Pettis integrable functions is:

$$|X|_P=\sup\left\{\int_\Omega |f(X)| : f\in B(E^*)\right\}.$$

The pair $(X_n,\mathcal{F}_n)$ is called a *stochastic process of Pettis integrable functions* if, for each $n\in I\!N$, $X_n:\Omega\to E$ is Pettis integrable, X_n is weakly $\mathcal{F}_n$-measurable and the *Pettis conditional expectation* $E(X_n|\mathcal{F}_m)$ of X_n exists for all $n\geq m$. It should be noted that, in general, if X is only Pettis integrable, even it is strongly measurable, there is no *Pettis conditional expectation* of X with respect to a sub-σ-algebra of $\mathcal{F}$. The stochastic process $(X_n,\mathcal{F}_n)$ is called a *martingale* if $E(X_n|\mathcal{F}_m)=X_m$ for $n\geq m$.
A martingale $(X_n,\mathcal{F}_n)$ is

(*i*) *convergent* in $\mathcal{P}(E)$ if there exists a function $X\in\mathcal{P}(E)$ such that

$$\lim_{n\to\infty}|X_n-X|_P=0;$$

(*ii*) *variationally bounded* if $\sup_n|\nu_n|(\Omega)<\infty$ where $\nu_n(A)=P\int_A X_n$ and $|\nu_n|$ denotes the variation of ν_n;

(*iii*) *uniformly continuous* if $\lim_{P(A)\to 0}P\int_A X_n=0$ uniformly with respect to n;

(*iv*) *uniformly integrable* if it is variationally bounded and uniformly continuous.

3. Banach lattice valued stochastic processes

In this section we consider stochastic processes consisting of strongly measurable Bochner integrable random variables taking values in a Banach lattice (see [5], Chapter VIII). For an element $x \in E$ we denote by x^+ the least upper bound between x and 0. The Banach lattice E is said *to have the order continuous norm* or, briefly, to be *order continuous*, if for every downward directed set $\{x_\alpha\}_\alpha$ in E with $\wedge_\alpha x_\alpha = 0$, then $\lim_\alpha \|x_\alpha\| = 0$. The norm on a Banach space has the *Kadec-Klee property with respect to a set* $D \subset E^*$ if whenever $\lim_n f(x_n) = f(x)$ for every $f \in D$ and $\lim_n \|x_n\| = \|x\|$, then $\lim_n x_n = x$ strongly. If $D = E^*$ we say that the norm has the *Kadec-Klee property.* It was proved in [3] the following renorming Theorem for Banach lattices.

Theorem 1 *A Banach lattice E is order continuous if and only if there is an equivalent lattice norm on E with the Kadec-Klee property.*

It is obvious that if E is separable, the equivalent norm has the Kadec-Klee property with respect to a countable set of functionals.
A *stopping time* is a map $\tau : \Omega \to I\!N \cup \{\infty\}$ such that, for each $n \in I\!N$, $\{\tau \leq n\} = \{\omega \in \Omega : \tau(\omega) \leq n\} \in \mathcal{F}_n$. We denote by Γ be the collection of all simple stopping times (i.e. taking finitely many values and not taking the value ∞), then Γ is a set filtering to the right. A stochastic process $(X_n, \mathcal{F}_n)$ is called a *subpramart* if for each $\varepsilon > 0$ there exists $\tau_0 \in \Gamma$ such that for all τ and σ in Γ, $\tau, \sigma \geq \tau_0$ then

$$P(\{\|(X_\sigma - E(X_\tau|\mathcal{F}_\sigma))^+\| > \varepsilon\}) \leq \varepsilon.$$

If $(X_n, \mathcal{F}_n)$ is a positive supramart, then for each $f \in (E^*)^+$, where $(E^*)^+$ denotes the nonnegative cone in E^*, $(f(X_n), \mathcal{F}_n)$ and $(\|X_n\|, \mathcal{F}_n)$ are real valued positive subpramarts ([5], Lemma viii.1.12)
If E has the Radon-Nikodym property each L^1-bounded subpramart converges strongly a.s.. Without assuming this property we ask which class of functionals has the property that the scalar convergence of $f(X_n)$ to $f(X)$ for each f in the class implies the strong convergence. We are able to prove the following theorem.

Theorem 2 *([8], Theorem 3.8) Let E be an order continuous Banach lattice, which is weakly sequentially complete and let T be a total subset of E^*. Let $(X_n, \mathcal{F}_n)$ be a positive subpramart with an L^1-bounded subsequence and let X be a strongly measurable random variable. Assume that, for each $f \in T$, $f(X_n)$ converges to $f(X)$ a.s. (the null set depends on f). Then X_n converges to X strongly a.s..*

Proof. Since (X_n) and X are strongly measurable it is possible to assume that E is separable. By a decomposition theorem ([5], Lemma viii.1.17) and the fact that a subsequence of $(X_n)_n$, still denoted by $(X_n)_n$, is L^1-bounded we can also assume that

$$X_{n_k} = Y_{n_k} + Z_{n_k}$$

where Y_{n_k} and Z_{n_k} are $\mathcal{F}_{n_k}$-measurable, $(Y_{n_k})_k$ is uniformly integrable and $\lim_k Z_{n_k} = 0$, a.s.. For each $f \in (E^*)^+$, $f(X_n)_n$ is a real valued subpramart with an L^1-bounded subsequence, then it converges a.s. to a real random variable X_f. Also $f(Y_{n_k})$ converges to X_f a.s. and in L^1. In particular for each $f \in T$, $\lim_k f(Y_{n_k}) = f(X)$. So for $A \in \sigma(\cup_n \mathcal{F}_n)$

$$\lim_k \int_A f(Y_{n_k})$$

exists in $I\!R$. Hence $(\int_A Y_{n_k})_k$ is weakly Chauchy. Since the Banach lattice E is weakly sequentially complete, let for every $A \in \sigma(\cup_n \mathcal{F}_n)$

$$\mu(A) = w - \lim_k \int_A Y_{n_k}.$$

Then μ is a measure of bounded variation and it is absolutely continuous with respect to P. For each $f \in T$ we have

$$f(\mu(A)) = \lim_k \int_A f(Y_{n_k}) = \int_A f(X).$$

Let $A_n = \{\|X\| \leq n\}$, then XI_{A_n} is Bochner integrable and

$$f(\mu(A_n)) = \int_{A_n} f(X) = f \int_{A_n} X.$$

Since T is a total set it follows that

$$\mu(A_n) = \int_{A_n} X.$$

Moreover the uniform integrability of $(Y_{n_k})_k$ implies that

$$\int_{A_n} \|X\| = \|\mu\|(A_n) \leq \sup_k \int_\Omega Y_{n_k}, \tag{1}$$

and since X is strongly measurable, $P(\cup_n(\|X\| \leq n)) = 1$. Letting $n \to \infty$ in (1), we get that X is Bochner integrable and for each $A \in \sigma(\cup_n \mathcal{F}_n)$

$$\mu(A) = \int_A X.$$

It follows that

$$\int_A f(X) = f(\mu(A)) = \lim_k \int_A f(Y_{n_k}) = \int_A X_f,$$

for each $f \in (E^*)^+$ and $A \in \cup_n \mathcal{F}_n$. Hence $f(X) = X_f$ a.s. and for each $f \in (E^*)^+$, $f(X_n)$ converges to $f(X)$ a.s.. Let $\|\cdot\|$ denote the Kadec-Klee norm equivalent to $\|\cdot\|$, as in Theorem 1, and let $D \in (E^*)^+$ be a countable norming subset. Applying ([5], Lemma viii.1.15) to the sequence $\{(f(X_n), \mathcal{F}_n), n \in \mathbb{N}, f \in D\}$ it follows that $\lim_n \|X_n\| = \|X\|$, a.s.. Now invoking again Theorem 1 we get the strong convergence of X_n to X and the assert follows. $\square$

Considering that if a Banach space E does not contain c_0, it is order continuous and weakly sequentially complete, the following corollary holds.

Corollary 1 *Let E be a Banach lattice not containing c_0 as an isomorphic copy and let T be a total subset of E^*. Let $(X_n, \mathcal{F}_n)$ be a positive subpramart with an L^1-bounded subsequence and let X be a strongly measurable random variable. Assume that, for each $f \in T$, $f(X_n)$ converges to $f(X)$ a.s. (the null set depends on f). Then X_n converges to X strongly a.s..*

4. Convergence of Pettis integrable stochastic processes

In this section we consider Pettis integrable stochastic processes.

Theorem 3 *Let $(X_n, \mathcal{F}_n)$ be an uniformly integrable martingale of Pettis integrable strongly measurable functions, X a weakly measurable function. Let T be a total subset of X^*, and assume that $f(X_n)$ converges to $f(X)$ a.s. for each $f \in T$ (the null set depends on f). Then $X \in \mathcal{P}(E)$ and X_n converges to X in the Pettis norm.*

Proof. By Pettis measurability Theorem we can assume that E is separable, then since T is closed and *weak**-dense, the assert follows from [9] Theorem 1. $\square$

Remark 1 *Since in Theorem 3 we can suppose E separable, the weak measurability of X can be replaced by the measurability of the functions $f(X)$ for all $f \in T$ (see [2]).*

We will extend Theorem 3 to more general stochastic processes $(X_n, \mathcal{F}_n)$.

Definition 1 *A stochastic process $(X_n, \mathcal{F}_n)$ of Bochner integrable functions is said to be L^1-bounded if $\sup_n \int_\Omega \|X_n\| < \infty$.*

Definition 2 *A stochastic process $(X_n, \mathcal{F}_n)$ of strongly measurable functions is said to be a* game which becomes fairer with time *(briefly a P-*martingale*), if for each $\varepsilon > 0$*

$$\lim_n \sup_{m \geq n} P(\|E(X_m|\mathcal{F}_n) - X_n\| > \varepsilon) = 0.$$

If for each $\varepsilon > 0$

$$\lim_n \sup_{m \geq n} P(\sup_{n \leq q \leq m} \|E(X_m|\mathcal{F}_q) - X_q\| > \varepsilon) = 0$$

the sequence $(X_n, \mathcal{F}_n)$ *is called a* mil.

Definition 3 *A stochastic process* $(X_n, \mathcal{F}_n)$ *of Pettis integrable functions is* σ-bounded *if there exists an increasing sequence* $(B_n)_n$, $B_n \in \mathcal{F}_n$, *such that* $\lim_n P(B_n) = 1$ *and the sequence* (X_n) *restricted to each* B_m, $m = 1, 2, \ldots$, *is* L^1*-bounded.*

For more details and the proofs of the following Theorems see [9].

Theorem 4 *Let* $(X_n, \mathcal{F}_n)$ *be a* σ*-bounded* P*-martingale of Pettis integrable functions and* X *a weakly measurable function. Let* T *be a total subset of* E^*, *and assume that* $f(X_n)$ *converges to* $f(X)$ *a.s. for each* $f \in T$ *(the null set depends on* f*). Then* X_n *converges to* X *in probability (i.e. for every* $\varepsilon > 0$ *we have* $\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0$*).*

Theorem 5 *Let* $(X_n, \mathcal{F}_n)$ *be a* σ*-bounded mil of Pettis integrable strongly measurable functions and* X *a weakly measurable function. Moreover let* T *be a total subset of* E^*, *and assume that* $f(X_n)$ *converges to* $f(X)$ *a.s. for each* $f \in T$ *(the null set depends on* f*). Then* X_n *converges to* X *a.s. in the strong topology.*

As we noted in Remark 1 the hypothesis of weak measurability of X in Theorems 4 and 5 can be substituted by the measurability of the functions $f(X)$ for all $f \in T$.
Assuming a weaker strong measurability condition on the martingale $(X_n, \mathcal{F}_n)$, in Theorem 3 we obtain:

Theorem 6 *Let* $(X_n, \mathcal{F}_n)$ *be an uniformly integrable martingale of Pettis integrable functions such that the indefinite integrals of all* X_n *have norm relatively compact range and let* X *be a weakly measurable function. Assume that there exists an increasing sequence of measurable sets* $(B_m)_m$, $B_m \in \mathcal{F}_m$, *such that* $\lim_m P(B_m) = 1$ *and that the function* X_n *restricted to each* B_m *is strongly measurable,* $n = 1, 2, \ldots$. *Assume, moreover, that for each* $f \in T$, *where* T *is a total set,* $f(X_n)$ *converges to* $f(X)$ *a.s. (the null set depends on* f*). Then* $X \in \mathcal{P}(E)$ *and* X_n *converges to* X *in the Pettis norm.*

Theorem 3 and Theorem 6 hold also for amarts, changing the proofs as in [13] Theorem 2.

5. Martingale of McShane integrable functions

In this section we consider stochastic processes of McShane integrable functions.

Let $(\Omega, \mathcal{A}, \mathcal{F}, P)$ be a probability space which is a quasi-Radon, outer regular and compact probability space. A *Mc-Shane partition* of Ω is a set $\{(S_i, \omega_i), i = 1, \dots, p\}$ where $(S_i)_i$ is a disjoint family of measurable sets of finite measure, $P(\Omega \setminus \cup_{i=1}^{p} S_i) = 0$ and $\omega_i \in \Omega$ for each $i = 1, \dots, p$. A *gauge* on Ω is a function $\Delta : \Omega \to \mathcal{A}$ such that $\omega \in \Delta(\omega)$ for each $\omega \in \Omega$. A Mc-Shane partition $\{(S_i, \omega_i), i = 1, \dots, p\}$ is *subordinate to a gauge* Δ if $S_i \subset \Delta(\omega_i)$ for $i = 1, \dots, p$. A function $f : \Omega \to E$ is *McShane integrable* (briefly M-integrable), with *Mc-Shane integral* $z \in E$ if for each $\varepsilon > 0$ there exists a gauge $\Delta : \Omega \to \mathcal{A}$, such that

$$\left\| \sum_{i=1}^{p} P(S_i) f(\omega_i) - z \right\| < \varepsilon$$

for each McShane partition $\{(S_i, \omega_i) : i = 1, \dots, p\}$ subordinate to Δ.

It is known that if $f : \Omega \to E$ is M-integrable, then $\nu_f(\Omega) = \left\{(M)\int_A f : A \in \mathcal{F}\right\}$ is totally bounded (see [1], Theorem B and [6], Corollary 3E), hence it is norm relatively compact. Denote by $M(E)$ the set of all M-integrable functions endowed with the seminorm

$$|X|_M = \sup \left\{ \int_{\Omega} |f(X)| : f \in B(E^*) \right\},$$

which is equivalent to the seminorm ([11])

$$\sup \left\{ \left\| M \int_A X \right\| : A \in \mathcal{F} \right\}.$$

If $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, X is McShane integrable and Y is McShane integrable on $(\Omega, \mathcal{A}, \mathcal{G}, P)$, then Y is called the *McShane conditional expectation* of X with respect to $\mathcal{G}$ if

- (i) Y is weakly $\mathcal{G}$-measurable;
- (ii) for every $A \in \mathcal{G}$, $M\int_A Y = M\int_A X$.

The symbol $Y = E_M(X|\mathcal{G})$ will denote the McShane conditional expectation of X with respect to $\mathcal{G}$.

We say that $(X_n, \mathcal{F}_n)$ is a *stochastic process of M-integrable functions*, if for each $n \in I\!N$, X_n is M-integrable, X_n is weakly measurable with respect to $\mathcal{F}_n$ and the McShane conditional expectation $E_M(X_n|\mathcal{F}_m)$ of X_n exists for all $n \geq m$. Also we observe that the conditional expectation of a M-integrable function does not always exist, indeed the same is true for strongly measurable Pettis integrable functions and a strongly measurable Pettis integrable function is McShane integrable.

As in case of a stochastic process of Pettis integrable functions, we say that $(X_n, \mathcal{F}_n)$ is a martingale if X_n is a M-integrable function for each n, and if for all $n \geq m$ $E_M(X_n|\mathcal{F}_m) = X_m$ or equivalently for all $A \in \mathcal{F}_m$

$$M\int_A X_m = M\int_A X_n.$$

If X is M-integrable and $E_M(X|\mathcal{F}_n)$ exists for all n, then $X_n = E_M(X|\mathcal{F}_n)$ is called a closed martingale. Since a M-integrable function is Pettis integrable and $\nu_f(\Omega) = \{(M)\int_A f : A \in \mathcal{F}\}$ is norm relatively compact, there exists a sequence of simple functions $f_n : \Omega \to E$, converging to f in $|\cdot|_M$, i.e. $\lim |f_n - f|_M = 0$. The following proposition is an extension of Lemma 1.4 of [12] to a martingale of McShane integrable functions. The proof follows with suitable changes.

Proposition 1 *Let $(X_n, \mathcal{F}_n)$ be a martingale of M-integrable functions. Then the following are equivalent:*

(i) *there exists a M-integrable function X such that X_n is $|\cdot|_M$ convergent to X;*

(ii) *there exists a M-integrable function X such that $E_M(X|\mathcal{F}_n) = X_n$ for each $n \in I\!N$;*

(iii) *there exists a M-integrable function X such that for each $A \in \cup_n \mathcal{F}_n$*

$$\lim_n M\int_A X_n = M\int_A X.$$

The condition $(ii) \Rightarrow (i)$ in the previous Proposition says that a closed martingale is $|\cdot|_M$ convergent. We have the following:

Proposition 2 *Let $(X_n, \mathcal{F}_n)$ be a martingale of M-integrable functions. Then, for all $A \in \cup_n \mathcal{F}_n$, the set function $\mu(A) = \lim_n M\int_A X_n$ is absolutely continuous and has norm relatively compact range if and only if the martingale $(X_n, \mathcal{F}_n)$ is $|\cdot|_M$ Chauchy.*

PROOF. First we prove the necessary part.
Since μ has norm relatively compact range, by Hoffman-Jorgensen Theorem for each $\varepsilon > 0$ there exists a function $H_\varepsilon : \Omega \to E$ such that $H_\varepsilon = \sum_{i=1}^k x_i I_{A_i}$, with $A_i \in \cup_n \mathcal{F}_n$ and $x_i \in E$, so that

$$\sup\left\{\left\|\mu(A) - \int_A H_\varepsilon\right\| : A \in \cup_n \mathcal{F}_n\right\} < \varepsilon.$$

Take $\varepsilon > 0$ and let $H = H_{\varepsilon/4}$, there exists m_0 for which $A_i \in \mathcal{F}_{m_0}$, for $i = 1, \ldots, k$. Since $\mu(A) = \lim_n M\int_A X_n$ there is m_0 such that $\|\mu(A) - M\int_A X_n\| < \frac{\varepsilon}{4}$ for $n > m_0$. Let $n, m \geq m_0$.

We have

$$
\begin{aligned}
& \sup\left\{\left\|M\textstyle\int_A(X_n-X_m)\right\| : A\in\cup_n\mathcal{F}_n\right\}\\
\leq\ & \sup\left\{\left\|M\textstyle\int_A(X_n-H)\right\| : A\in\cup_n\mathcal{F}_n\right\}+\sup\left\{\left\|M\textstyle\int_A(H-X_m)\right\| : A\in\cup_n\mathcal{F}_n\right\}\\
\leq\ & \sup\left\{\left\|M\textstyle\int_A X_n-\mu(A)\right\| : A\in\cup_n\mathcal{F}_n\right\}+\sup\left\{\left\|\mu(A)-M\textstyle\int_A H\right\| : A\in\cup_n\mathcal{F}_n\right\}\\
+\ & \sup\left\{\left\|M\textstyle\int_A X_m-\mu(A)\right\| : A\in\cup_n\mathcal{F}_n\right\}+\sup\left\{\left\|\mu(A)-M\textstyle\int_A H\right\| : A\in\cup_n\mathcal{F}_n\right\}\\
<\ & \tfrac{\varepsilon}{4}+\tfrac{\varepsilon}{4}+\tfrac{\varepsilon}{4}+\tfrac{\varepsilon}{4}=\varepsilon.
\end{aligned}
$$

Then $|X_n-X_m|_M<\varepsilon$ for $n,m\geq m_0$.
Conversely choose $\varepsilon>0$ and find m_0 such that if $n,m\geq m_0$ then $|X_n-X_m|_M<\varepsilon$. If $\mu_n(A)=M\int_A X_n$ for $A\in\cup\mathcal{F}_n$ then

$$\|\mu_n(A)-\mu_m(A)\|_M\leq|X_n-X_m|_M<\varepsilon.$$

So the sequence of measures μ_n is Chauchy, therefore $\lim_n\mu_n(A)=\mu(A)$ exists. The functions X_n are M-integrable, then μ_n has a norm relatively compact range and since the convergence is uniform in $A\in\cup_n\mathcal{F}_n$, it follows that μ is absolutely continuous and has a norm relatively compact range. □

Proposition 1 and Proposition 2 hold also for M-integrable martingales indexed by a directed set.
We will prove now two convergence theorems for a M-integrable martingale.

Theorem 7 *Let $(X_n,\mathcal{F}_n)$ be an uniformly integrable martingale of M-integrable functions and suppose that there exists a weakly measurable function $X:\Omega\to E$ such that $f(X_n)$ converges to $f(X)$ a.s.. Then X_n is $|\cdot|_M$ convergent to X.*

Proof. Since $(X_n)_n$ is uniformly integrable the set function $\nu:\cup_n\mathcal{F}_n\to E$ defined as

$$\nu(A)=\lim_n M\int_A X_n$$

is an absolutely continuous measure of bounded variation and it can be extended to the whole $\mathcal{F}$ to an absolutely continuous measure of bounded variation. Moreover for each $\omega\notin N$ with $P(N)=0$, $f(X_n(\omega))$ converges to $f(X(\omega))$ for each $f\in E^*$. Hence it follows from [6] Theorem 4A that X is M-integrable and $\nu(\Omega)=M\int_\Omega X$. Then for each $A\in\cup_n\mathcal{F}_n$

$$\lim_n M\int_A X_n=M\int_A X$$

and the assert follows from Proposition 1. □

Definition 4 *A function $X : \Omega \to E$ is called* weakly asymptotically measurable *with respect to an increasing family $(\mathcal{F}_n)_n$ of sub-σ-algebras of $\mathcal{F}$ if there exists an integer N such that for all $n > N$ and for all $f \in E^*$ $f(X)$ is $\mathcal{F}_n$-measurable.*

Theorem 8 *Let $(X_n, \mathcal{F}_n)$ be an uniformly integrable martingale of M-integrable functions and let T be a weak*-sequentially dense subset of E^*. Assume that there exists a weakly measurable function $X : \Omega \to E$ such that X is weakly asymptotically measurable with respect to $(\mathcal{F}_n)$ and such that, for each $f \in T$, $f(X_n)$ converges to $f(X)$ a.s. (the null set depends on f). Then X_n is $|\cdot|_M$ convergent to X.*

Proof. Since each McShane integrable function is Pettis integrable it follows by [9] Theorem 1 that X is Pettis integrable, (X_n) converges to X in the Pettis norm and

$$\mu(A) = \lim_n M\int_A X_n = P\int_A X$$

for all $A \in \cup_n \mathcal{F}_n$. We want to prove that X is M-integrable. Since X is weakly asymptotically measurable there exists $N \in \mathbb{N}$ such that X is weakly $\mathcal{F}_N$-measurable, then

$$E(X|\mathcal{F}_N) = X \tag{2}$$

and also

$$E(X|\mathcal{F}_N) = X_N. \tag{3}$$

Then (2) and (3) implies that $X = X_N$ a.s. and X is M-integrable. Therefore the assert follows from Proposition 1. □

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