ON THE DUAL SPACE OF THE HENSTOCK-KURZWEIL INTEGRABLE FUNCTIONS IN N DIMENSIONS*

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ABSTRACT

The dual space of the class of Henstock-Kurzweil integrable functions is well known in the one-dimensional case and corresponds to the space of multipliers which, in turn, coincides with the class of functions of bounded essential variation. Comparable results in higher dimensions have been elusive. For cases in which the partitions defining the Henstock-Kurzweil integrals are defined on n-cells (parallelepipeds) with their sides parallel to the coordinate axes (i.e., Cartesian products of compact intervals), we prove that a function is a multiplier of the class of n-dimensional ($n \geq 2$) Henstock-Kurzweil integrable functions if and only if it is of strongly bounded essential variation as defined by Kurzweil. This result was proved earlier by T.Y. Lee, T. S. Chew, and P.Y. Lee in the two-dimensional case by a different method. The sufficiency part of our proof makes use of a generalization of a method used earlier by P.Y. Lee in the one-dimensional case.

**2000 Mathematics Subject Classification.* Primary 26A39; Secondary 46E99
Key Words and Phrases: Nonabsolute integration, Henstock-Kurzweil integrability, dual spaces, multipliers.

1. INTRODUCTION

In recent years there has been a great deal of interest in theories of integration more general than the well known Lebesgue theory. These theories are nonabsolute in the sense that an integral $\int f$ may exist even when the integral $\int |f|$ does not, a property that Lebesgue integrals do not have. The best known of these theories bear the names of Denjoy, Perron, Henstock, and Kurzweil (for summaries, see [CD, F, G, H, Ku1, L1, M, Mc, P1, S1]).

In the present work we will be concerned with the Henstock-Kurzweil approach, which is a generalization of Riemann's original constructive method. This approach has been thoroughly investigated in the one-dimensional case, but less extensively studied in higher dimensions. We will restrict ourselves to the discussion of the case of functions defined on particular types of compact sets in R^n (n-cells, see below).

In order to simplify our presentation and to make otherwise long complicated formulas comprehensible, we will use vector notation where possible. We will also use it in some nonconventional cases, where the intent should be obvious. In other cases, we will define our meaning. For example, if $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$ are sequences of real numbers with $a_i < b_i$, $i = 1, 2, \cdots, n$; we denote the corresponding n-cell by

$$I_n \equiv [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]. \tag{1.1}$$

We denote this by $I_n = [\mathbf{a}, \mathbf{b}]$. Let $\mathbf{x} = (x_1 \cdots x_n)$, and for $q \leq n$, let $\mathbf{x}_q = x_1, \cdots, x_q$ and $\mathbf{x}_{n-q} = x_{q+1}, \cdots, x_n$. Below, we record most of the nonconventional cases:

$$I' = i_1, \cdots, i_n, \quad I'_q = i_1, \cdots, i_q, \quad K' = k_1, \cdots, k_n, \quad K'_q = k_1, \cdots, k_q,$$

$$[\mathbf{a} : \mathbf{x}_{K'}^{[1,n]}] = a_1, x_{k_1}^{(1)}; \cdots; a_n, x_{k_n}^{(n)},$$

$$\xi_{I'}^{[1,n]} = \xi_{i_1}^{(1)}, \cdots, \xi_{i_n}^{(n)}, \quad [\xi_{K'}^{[1,n]} : \xi_{K'+1}^{[1,n]}] = \xi_{k_1}^{(1)}, \xi_{k_1+1}^{(1)}, \cdots, \xi_{k_n}^{(n)}, \xi_{k_n+1}^{(n)}.$$

One says that a real-valued function f defined on I_n is integrable in the Henstock-Kurzweil sense or is Henstock-Kurzweil integrable, and denotes its integral $\int_{I_n} f(\mathbf{x})\, d\mathbf{x}$ if, for each $\varepsilon > 0$, there exists a positive function $\delta : I_n \to R_+$ (generally called a gauge or gage) such that if $P = \{J_i \,|\, i = 1, 2, \cdots, m\}$ is a partition of I_n into a set of nonoverlapping n-cells $\{J_i\}$ with union $I_n = \bigcup_{i=1}^{m} J_i$ such that if $\mathbf{x}^{(i)} \in$

$J_i \subset B(\mathbf{x}^{(i)}, \delta(\mathbf{x}^{(i)}))$, where $B(\mathbf{a}, r)$ denotes the open ball of radius r and center $\mathbf{a}$ in R^n, then

$$\left| \sum_{i=1}^{m} f(\mathbf{x}^{(i)}) \mu(J_i) - \int\limits_{I_n} f(\mathbf{x}) d\mathbf{x} \right| < \varepsilon,$$

where $\mu(J_i)$ denotes the Lebesgue measure of the n-cell (J_i). Such a partition is called a δ-fine partition. Henstock-Kurzweil integrals can be described in terms of primitive functions, thereby providing a formulation of the fundamental theorem of calculus [CD, F, G, Ku1, L1, M, Mc, P1, S1]. In the one-dimensional case for example, if f is Henstock-Kurzweil integrable on the interval $[a, b]$, then there exists a (primitive) function $F(x)$, $x \in [a, b]$, such that

$$\int\limits_{a}^{b} f(x) dx = F(b) - F(a), \tag{1.2}$$

and the derivative of F exists and is equal to f a.e. on $[a, b]$. Generalizations of this result to higher dimensions are known [CD, H, Ku1, Ku2, M, MO, O2, Mc, P1] and it will be convenient to describe one of these by using a formalism analogous to that discussed by Tolstov [To]. For a real-valued function ϕ defined on I_n, consider the first difference:

$$\Delta_i(\phi(\mathbf{x}); [a_i, b_i]) \equiv \phi(\mathbf{x}_{i-1}, b_i, \mathbf{x}_{n-i-1}) - \phi(\mathbf{x}_{i-1}, a_i, \mathbf{x}_{n-i-1}). \tag{1.3}$$

Similarly, second differences are defined as (for $i \neq j$):

$$\begin{aligned} \Delta_{x_i x_j}(\phi; [a_i, b_i] \times [a_j, b_j]) &\equiv \Delta_{x_j}(\Delta_{x_i}(\phi; [a_i, b_i]); [a_j, b_j]) \\ &= \phi(\cdots, b_i, \cdots, b_j, \cdots) - \phi(\cdots, a_i, \cdots, b_j, \cdots) \\ &\quad - \phi(\cdots, b_i, \cdots, a_j, \cdots) + \phi(\cdots, a_i, \cdots, a_j, \cdots). \end{aligned}$$

One easily shows that

$$\Delta_{x_i x_j}(\phi; [a_i, b_i] \times [a_j, b_j]) = \Delta_{x_j x_i}(\phi; [a_i, b_i] \times [a_j, b_j]), \tag{1.4}$$

and the following convenient formula for n^{th} differences has been given by Tolstov ([To], p. 70):

$$\Delta_{\mathbf{x}_{I'}}(\phi; I_n) = \phi(\mathbf{b}) + \sum_{q=1}^{n-1} \sum_{k_1=1}^{n} \cdots \sum_{k_q=1}^{n} (-1)^q \phi_{K'_q} + (-1)^n \phi(\mathbf{a}), \tag{1.5}$$

where the summation indices $k_1, \cdots, k_q$ satisfy the constraints $k_1 < k_2 < \cdots < k_q$ and the functions $\phi_{K'_q}$ are obtained from ϕ by evaluating that function at the lower limits $a_1, \cdots, a_q$ at the $x_{k_1} = a_{k_1}, \cdots, x_{k_q} = a_{k_q}$, and at the upper limits $x_m = b_m$ for the remaining indices. The expressions (1.5) will be very useful to us later in this paper.

It will be convenient for our subsequent considerations to use some notation that is slightly different from that employed by Tolstov. Thus, we will denote a difference by the symbol Δ with a subscript that indicates the order of the difference with particular differences being indicated by a specification of the intervals upon which the function is evaluated and, for purposes of brevity, we will omit the square brackets in the indication of the intervals. The latter simplification should not cause any confusion because we only consider closed intervals. Thus, we will indicate differences such as (1.3) by $\Delta_1(\phi(\mathbf{x}_{i-1}, \cdot, \mathbf{x}_{n-i-1}); a_i, b_i)$ and (1.5) by $\Delta_n(\phi; I_n)$. The symmetry properties (1.4) and higher order versions thereof show that differences are uniquely defined by a specification of their order and of the intervals upon which they act.

The n-dimensional ($n \geq 2$) generalizations of (1.2) are

$$\int\limits_{I_n} f(\mathbf{x}) d\mathbf{x} = \Delta_n(F; I_n), \tag{1.6}$$

where F denotes the n-dimensional primitive function of f on the n-cell I_n. (See eq. (3.13) in the proof of Theorem 3.1 for the justification of the above equation.) We will denote by HK_n the collection of Henstock-Kurzweil integrable functions in n dimensions, where $n \geq 1$.

The main object of this paper is to investigate the dual spaces of the class of HK_n functions for $n = 2, 3, \cdots$. As was first done by Alexiewicz [A] in the one-dimensional case, the HK_n functions on I_n (1.1) can be topologized by using the Alexiewicz norm:

$$\|f\|_A \equiv \sup_{\mathbf{x} \in I_n} |F(\mathbf{x})|,$$

where F denotes the indefinite integral of the HK_n integrable function f:

$$F(\mathbf{x}) \equiv \int\limits_{a_1}^{x_1} \cdots \int\limits_{a_n}^{x_n} f(\mathbf{y})\, d\mathbf{y}.$$

It is known that the space of HK_n functions is not complete relative to the topology defined by the Alexiewicz norm, but it can be completed to a space of Schwartz distributions [MO,Ku2]. It is known in dimensions $n = 1$ [Th] and $n = 2$ [O2] that the space of HK_n functions, topologized by the Alexiewicz norm, is a barreled space but is not a Banach space. In one dimension it is known that there exists a one-to-one correspondence between the continuous linear functionals (relative to the topology defined by the Alexiewicz norm) and the functions of bounded essential variation [O1, S2, MO, L1, CD], and that these functions are multipliers of HK_1 where, in general, a function g is said to be a multiplier of a space of integrable functions G if $hg \in G$ whenever $h \in G$. In particular ([L1], Sec.12), if T denotes a continuous linear functional on $HK_1([a, b])$ for a finite closed interval $[a, b]$, then

$$T(f) = \int_a^b f(x)g(x)dx$$

for all $f \in HK_1([a, b])$ and some g of bounded variation on $[a, b]$. Moreover, $\|g\|_B \leq 2\|T\|_A$, where $\|g\|_B \equiv \inf V(g_1)$, in which $V(g)$ denotes the total variation of g on $[a, b]$ and the infimum is taken over all functions g_1 of bounded variation on $[a, b]$ which are equal to g a.e. on $[a, b]$. In this one-dimensional situation, a function has bounded essential variation if and only if its first (Schwartz) distributional derivative is a signed finite Borel measure.

In dimensions $n \geq 2$, however, the situation is different because, as we will discuss in more detail below, there are many distinct classes of functions of bounded variation, each of which coincides in the one-dimensional case with the class of functions of bounded variation usually considered. Several different classes of functions of bounded variation in two dimensions were listed and their interrelationships discussed by Clarkson and Adams [CA]. We list below a few of these that we will need to use in our subsequent discussion.

DEFINITION 1.1. A function $f(x, y)$ is said to be of *bounded variation in the Vitali-Lebesgue-Frechet-de la Vallee Poussin sense* on the rectangle $I_2 \equiv [a_1, b_1] \times [a_2, b_2]$ if

$$\sum_{i=0}^{q-1} \sum_{j=0}^{p-1} \left| \Delta_2(f; [x_i^{(1)}, x_{i+1}^{(1)}] \times [x_j^{(2)}, x_{j+1}^{(2)}]) \right|$$

is finite for all partitions of I_2 into nonoverlapping nondegenerate closed rectangles: $a_i = x_0^{(i)} < x_1^{(i)} < \cdots < x_{m_i}^{(i)} = b_i$ for $i = 1, 2$ with $m_1 = q$ and $m_2 = p$.

DEFINITION 1.2. A function $f(x, y)$ is said to be of *bounded variation on I_2 in the Hardy-Krause sense* if it is of bounded variation in the sense of Definition 1.1 and, in addition, $f(x', y)$ is of bounded variation in y for at least one x' and $f(x, y')$ is of bounded variation in x for at least one y'.

DEFINITION 1.3. A function $f(x, y)$ is said to be of *bounded variation on I_2 in the Tonelli sense* if the one-dimensional total variation of $f(x, y)$ (considered as a function of y) is finite a.e. on (a_1, b_1) and its Lebesgue integral over (a_1, b_1) exists (and is finite), with a similar condition on the one-dimensional total variation of $f(x, y)$ considered as a function of x.

Kurzweil ([Ku2], see also [Ku1], p. 95) defined a real-valued function g on I_2 to be of *strongly bounded variation* if it is of bounded variation in the sense of Definition 1.1 and if , for every $x \in [a_1, b_1], g(x, \cdot)$ is of bounded variation on $[a_2, b_2]$ and, for every $y \in [a_2, b_2], g(\cdot, y)$ is of bounded variation on $[a_1, b_1]$. Thus, it is seen that the condition of strongly bounded variation is a special case of the condition of bounded variation in the sense of Definition 1.2. In [Ku2] Kurzweil proved (in the n-dimensional case) that functions of strongly bounded essential variation are multipliers for the class of Perron integrable functions and that these functions correspond to continuous linear functionals of the space of Perron integrable functions. Since the class of Perron integrable functions is equivalent to the class of HK functions, similar results hold for the dual space of the HK functions. Thus, Kurzweil's result gives a sufficient condition for a function g to be a multiplier of the space HK_n. In an important paper [LCL], Lee, Chew, and Lee proved, in the two-dimensional case, that this condition is necessary, as well as sufficient, for a function to be a multiplier of the space HK. These authors claim that their proof can be extended to all higher dimensions, and our main goal in the present paper is to give a different proof of this characterization of the multipliers of the HK_n spaces when $n \geq 2$. The sufficiency part of our proof makes use of a generalization of a method used by P. Y. Lee ([L1], pp. 72-73) in the one-dimensional case and yields an explicit expression for the integrals by means of integration-by-parts formulae in terms of Riemann-Stieltjes integrals. The existence of these higher-dimensional Riemann-Stieltjes integrals can be proven by using a generalization of Clarkson's argument [C] for the two-dimensional case. We note that it is sufficient to consider Riemann-Stieltjes integrals rather than the more general Perron-Stieltjes or Ward integrals because the primitives of Henstock-Kurzweil integrals are continuous.

It is known in the two-dimensional case [MO, O2, LCL] that $T(f)$

is a continuous linear functional on the space $HK_2(I_2)$ (relative to the topology defined by the Alexiewicz norm) if and only if there exists a finite signed Borel measure μ on $I_2 = [a_1, b_1] \times [a_2, b_2]$ such that $T(f) = \int_{I_2} f(x, y) d\mu(x, y)$ for all $f \in HK_2(I_2)$. In this connection, Krickeberg [Kr] proved that a real-valued function on R^n has first distributional derivatives that are measures if and only if it is of bounded variation in the sense of Tonelli. Ostaszewski [MO,O2] proved that distribution functions of finite Borel measures are of strongly bounded essential variation and are multipliers of $HK_2(I_2)$, but was not able to prove that these multipliers are associated with the most general form of continuous linear functional on the space of HK_2 functions. Such a proof was provided later by Lee, Chew, and Lee [LCL]. Multiplier problems have also been investigated for other types of Riemann-type integrals [L2, P1, P2, MP, BS, B].

Our objective in this paper is to give a new proof of the result of Lee, Chew, and Lee on the characterization of the class of multipliers of the space of HK_n functions for $n = 2, 3, \cdots$. We first prove, in Theorem 2.1 in Section 2, that a necessary condition that a function be a multiplier of HK_n is that it be of bounded essential variation in the n-dimensional version of the class of functions of bounded variation defined in Definition 1.1. This theorem was proved before we were aware of the work of Lee, Chew, and Lee. After the present work was presented at the conference by the second author, Lee Tuo Yeong informed him of the results in the papers [LCL] and [L2]. It then became clear to us how to modify the proof of Theorem 2.1 so that we obtain a proof of the necessity that functions of strongly bounded essential variation be multipliers for the HK_n functions.This is discussed in Section 4. The proof of the sufficiency of this condition in Theorem 3.2 makes use of a generalization to n dimensions ($n \geq 2$) of a method used by P.Y. Lee in the one-dimensional case ([L1], Theorem 12.1 and Corollary 12.2, pp. 72-73).

2. Necessary condition for existence of multipliers

In this section we obtain a necessary condition in order that a function g be a multiplier for HK_n . The idea of the proof is to reduce from an arbitrary integral dimension ($n \geq 2$) to the one-dimensional case and then use similar arguments to those used in earlier proofs for that case ([CD], Theorem 50, p.45; [L1], Theorem 12.9, pp. 78-79; [S2]). As in the Introduction, we restrict ourselves to functions defined on compact n-cells in R^n.

THEOREM 2.1. *Let f and g denote real-valued functions defined on an n-cell I_n (1.1). If $fg \in HK_n(I_n)$ for all $f \in HK_n(I_n)$, then g is equal a. e. to a function of n-dimensional ($n \geq 1$) bounded variation in the sense of Vitali et al.*

Before proceeding to the proof, we discuss n-dimensional variations in the sense of Vitale et al and their connection with the n^{th} order differences (1.5). Given the definition of n-cells I_n as in (1.1), consider partitions of these sets of the form

$$\begin{aligned} a_1 = x_0^{(1)} < x_1^{(1)} < \cdots < x_{m_1}^{(1)} = b_1, \\ a_2 = x_0^{(2)} < x_1^{(2)} < \cdots < x_{m_2}^{(2)} = b_2, \\ \vdots \\ a_n = x_0^{(n)} < x_1^{(n)} < \cdots < x_{m_n}^{(n)} = b_n. \end{aligned} \tag{2.1}$$

Then, generalizing Definition 1.1 to $n \geq 3$ dimensions, we define n-dimensional variations in the sense of Vitali et al of real-valued functions ϕ on I_n by:

$$V(\phi; I_n) \equiv ess \sup_P \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} \left| \Delta_n(\phi; [\mathbf{x}_{I'-1}^{[1,n]} : \mathbf{x}_{I'}^{[1,n]}]) \right|, \tag{2.2}$$

where the n^{th} differences Δ_n are defined by (1.5) and the essential supremum is taken over all partitions (2.1).

For the second differences we have

$$\begin{aligned} \Delta_2(\phi; x_{i_1-1}^{(1)}, x_{i_1}^{(1)}; x_{i_2-1}^{(2)}, x_{i_2}^{(2)}) = \phi(x_{i_1}^{(1)}, x_{i_2}^{(2)}) \\ -\phi(x_{i_1-1}^{(1)}, x_{i_2}^{(2)}) - \phi(x_{i_1}^{(1)}, x_{i_2-1}^{(2)}) + \phi(x_{i_1-1}^{(1)}, x_{i_2-1}^{(2)}), \end{aligned}$$

so that

$$\begin{aligned} \left| \Delta_2(\phi; x_{i_1-1}^{(1)}, x_{i_1}^{(1)}; x_{i_2-1}^{(2)}, x_{i_2}^{(2)}) \right| \leq \omega(\phi(\cdot, x_{i_2}^{(2)}); x_{i_1-1}^{(1)}, x_{i_1}^{(1)}) \\ + \omega(\phi(\cdot, x_{i_2-1}^{(2)}); x_{i_1-1}^{(1)}, x_{i_1}^{(1)}), \end{aligned} \tag{2.3}$$

where ω denotes one-dimensional essential oscillations:

$$\begin{aligned} \omega(h; a, b) &\equiv ess \sup_{x,y \in [a,b]} [h(x) - h(y)] \\ &= ess \sup_{x,y \in [a,b]} h(x) - \operatorname{ess\,inf}_{x,y \in [a,b]} h(x). \end{aligned} \tag{2.4}$$

It easily follows from (1.5) that, for example,

$$\begin{aligned} &\Delta_n(\phi; [\mathbf{x}_{I'-1}^{[1,n]} : \mathbf{x}_{I'}^{[1,n]}]) \\ &\quad = \Delta_{n-1}(\phi(\cdots, x_{i_n}^{(n)}); x_{i_1-1}^{(1)}, x_{i_1}^{(1)}; \cdots; x_{i_{n-1}-1}^{(n-1)}, x_{i_{n-1}}^{(n-1)}) \\ &\quad -\Delta_{n-1}(\phi(\cdots, x_{i_n-1}^{(n)}); x_{i_1-1}^{(1)}, x_{i_1}^{(1)}; \cdots; x_{i_{n-1}-1}^{(n-1)}, x_{i_{n-1}}^{(n-1)}), \end{aligned} \tag{2.5}$$

and the expressions on the right-hand side of (2.5) may be replaced by pairs of $(n-1)$-differences in which the function ϕ is evaluated at the endpoints of other "component" intervals of the n-cell I_n. Then, from (2.3), (2.5), and induction, we obtain:

$$\begin{aligned} &\left|\Delta_n(\phi;[\mathbf{x}^{[1,n]}_{I'-1}:\mathbf{x}^{[1,n]}_{I'}])\right| \\ &\qquad \leq \sum_{c_1=1}^{2}\cdots\sum_{c_{n-1}=1}^{2}\omega(\phi(x^{(1)}_{c_1 i_1},\cdots,x^{(n-1)}_{c_{n-1}i_{n-1}},\cdot);x^{(n)}_{i_n-1},x^{(n)}_{i_n}) \end{aligned} \tag{2.6}$$

where, in (2.6), for each c_j $(j=1,2,\cdots,n-1)$ the values $c_j=1$ or $c_j=2$ are to be chosen corresponding to whether the function ϕ is evaluated at the left or right endpoints, respectively, of the intervals $[x^{(j)}_{i_j-1},x^{(j)}_{i_j}],(j=1,2,\cdots,n-1)$.

PROOF OF THEOREM 2.1. For $n=1$ the definition of functions of bounded variation in the sense of Vitali et al reduces to the usual concept of functions of bounded variation, so the theorem is true in this case. See the references cited in the first paragraph of this section.

For $n\geq 2$, following an argument of P. Y. Lee for $n=1$, since fg is HK_n integrable for all f that are absolutely HK_n integrable (and hence Lebesgue integrable); g must be essentially bounded. To begin the proof for the $n\geq 2$ cases, we use an argument analogous to one used in the $n=1$ case in the papers cited above. Suppose that g is not equal a. e. to a function of bounded variation in the sense of Vitali et al in n dimensions $(n\geq 2)$. Thus, for any function g_1 equal to g a.e., g_1 is not of bounded variation in the stated sense. Then there exists a point $\gamma\in I_n$ such that g_1 is not of bounded variation in the stated sense on any open n-cell which contains γ. Since the variation of g_1 in I_n is unbounded, the variation of g_1 in every open subset ϑ of I_n completely enclosing γ is also unbounded. Otherwise, one would be able to enclose every point of ϑ in a sub-n-cell of I_n in which g_1 has bounded variation. But it would then follow, by an application of the Heine-Borel theorem, that there exists a finite number of n-cells which cover ϑ on each of which g_1 has bounded variation. Then g_1 would be of bounded variation on ϑ, a contradiction. In particular, g_1 is not of bounded variation in the stated sense on every n-cell with one vertex at γ.

We next use a generalization of the one-dimensional argument of Čelidze and Džvaršeĭšvili ([CD], Theorem 50, p. 45). We call $\mathbf{y}\in I_n\equiv[\mathbf{a}\,,\mathbf{b}\,]$ a *singular point* of g if, for all sub-n-cells $\tilde{I}_n\equiv[\tilde{\mathbf{a}}\,,\tilde{\mathbf{b}}\,]\subset I_n$, $\mathbf{y}\in\tilde{I}_n$ and $V(g;\tilde{I}_n)=+\infty$, where V is defined by (2.2). From the assumption that g is not of bounded essential variation, we infer that the set of such singular points is not empty. We may assume, without loss of generality, that the point $\mathbf{a} \in I_n$ is a singular point.

By the definitions (2.2) and inequalities (2.6), there exists a sub-n-cell of each I_n, $J_1^{(n)} \equiv [\alpha^{(1)}, \beta^{(1)}] \subset I_n$ and partitions of $J_1^{(n)}$:

$$\begin{array}{c} \alpha_1^{(1)} = x_1^{(1)} < \cdots < x_{q_{1,1}}^{(1)} = \beta_1^{(1)}, \\ \alpha_2^{(1)} = x_1^{(2)} < \cdots < x_{q_{2,1}}^{(2)} = \beta_2^{(1)}, \\ \vdots \\ \alpha_n^{(1)} = x_1^{(n)} < \cdots < x_{q_{n,1}}^{(n)} = \beta_n^{(1)}, \end{array}$$

such that

$$\sum_{i_1=1}^{q_{1,1}} \cdots \sum_{i_n=1}^{q_{n,1}} \sum_{c_1=1}^{2} \cdots \sum_{c_{n-1}=1}^{2} \omega(g(x_{c_1 i_1}^{(1)}, \cdots, x_{c_{n-1} i_{n-1}}^{(n-1)}, \cdot); x_{i_n-1}^{(n)}, x_{i_n}^{(n)}) \\ \geq V(g; J_1^{(n)}) \geq 1.$$

Further, on each sub-n-cell $K_1^{(n)} \equiv [x_{i_1-1}^{(1)}, x_{i_1}^{(1)}] \times \cdots \times [x_{i_n-1}^{(n)}, x_{i_n}^{(n)}] \subset J_1^{(n)}$, there exist disjoint measurable sets $L_{i_1,\cdots,i_n}^{(1)}, H_{i_1,\cdots,i_n}^{(1)} \subset K_1^{(n)}$ with $\mu(L_{i_1,\cdots,i_n}^{(1)}) = \mu(H_{i_1,\cdots,i_n}^{(1)}) \equiv \eta_1 > 0$ (where μ denotes Lebesgue measure on R^n) such that

$$g(\mathbf{z}) \geq \frac{3}{2} m_{I'}^* + \frac{1}{2} m_{I'}, \text{ if } \mathbf{z} \in L_{I'}^{(1)}, \tag{2.7a}$$

and

$$g(\mathbf{z}) \leq \frac{1}{2} m_{I'}^* + \frac{3}{2} m_{I'}, \text{ if } \mathbf{z} \in H_{I'}^{(1)}, \tag{2.7b}$$

where

$$m_{I'}^* \equiv \sum_{c_1=1}^{2} \cdots \sum_{c_{n-1}=1}^{2} ess \sup_{y \in [x_{i_n-1}^{(n)}, x_{i_n}^{(n)}]} g(x_{c_1 i_1}^{(1)}, \cdots, x_{c_{n-1} i_{n-1}}^{(n-1)}; y), \tag{2.8}$$

$$m_{I'} \equiv \sum_{c_1=1}^{2} \cdots \sum_{c_{n-1}=1}^{2} ess \inf_{y \in [x_{i_n-1}^{(n)}, x_{i_n}^{(n)}]} g(x_{c_1 i_1}^{(1)}, \cdots, x_{c_{n-1} i_{n-1}}^{(n-1)}; y), \tag{2.9}$$

and, in defining (2.8) and (2.9), we have used the fact that two sets of the forms $\{x_{c_1 i_1}^{(1)}, \cdots, x_{c_{n-1} i_{n-1}}^{(n-1)}\} \times [x_{i_n-1}^{(n)}, x_{i_n}^{(n)}]$ and $\{x_{c_1' i_1}^{(1)}, \cdots, x_{c_{n-1}' i_{n-1}}^{(n-1)}\} \times [x_{i_n-1}^{(n)}, x_{i_n}^{(n)}]$ are disjoint if $c_i \neq c_i'$ for at least one value of $i \in \{1, 2, \cdots, n-1\}$.

By repetition of the above considerations, we obtain by induction a sequence of disjoint n-cells $J_k^{(n)} \equiv [\vec{\alpha}^{(k)}, \vec{\beta}^{(k)}] \subset I_n$ with $\beta_i^{(k)} < \alpha_i^{(k-1)}$ for $i = 1, 2, \cdots, n$ and $k = 2, 3, \cdots$ such that $\lim\limits_{k\to+\infty} \alpha_i^{(k)} = \lim\limits_{k\to+\infty} \beta_i^{(k)} = a_i$ for each $i = 1, 2, \cdots, n$. Moreover, on each n-cell $J_k^{(n)}$ there exists a partition

$$\begin{aligned} \alpha_1^{(k)} &= x_{q_{1,k-1}}^{(1)} < \cdots < x_{q_{1,k}}^{(1)} = \beta_1^{(k)}, \\ \alpha_2^{(k)} &= x_{q_{2,k-1}}^{(2)} < \cdots < x_{q_{2,k}}^{(2)} = \beta_2^{(k)}, \\ &\vdots \\ \alpha_n^{(k)} &= x_{q_{n,k-1}}^{(n)} < \cdots < x_{q_{n,k}}^{(n)} = \beta_n^{(k)}, \end{aligned} \tag{2.10}$$

with

$$\sum_{q_{1,k-1}+1}^{q_{1,k}} \cdots \sum_{q_{n,k-1}+1}^{q_{n,k}} \sum_{c_1=1}^{2} \cdots \sum_{c_{n-1}=1}^{2} \omega(g(x_{c_1 i_1}^{(1)}, \cdots, x_{c_{n-1} i_{n-1}}^{(n-1)}, \cdot); x_{i_n-1}^{(n)}, x_{i_n}^{(n)}) \geq V(g; J_k^{(n)}) \geq k. \tag{2.11}$$

On each set $K_k^{(n)} \equiv \{x_{c_1 i_1}^{(1)}, \cdots, x_{c_{n-1} i_{n-1}}^{(n-1)}\} \times [x_{i_n-1}^{(n)}, x_{i_n}^{(n)}] \subset J_k^{(n)}$, there exist disjoint measurable sets $L_{I'}^{(k)}, H_{I'}^{(k)}$ such that $\eta_{k-1} > \mu(L_{I'}^{(k)}) = \mu(H_{I'}^{(k)}) \equiv \eta_k > 0$ and such that (2.7) holds with the sets $L_{I'}^{(1)}$ and $H_{I'}^{(1)}$ replaced by $L_{I'}^{(k)}$ and $H_{I'}^{(k)}$, respectively. In addition, $\lim\limits_{k\to+\infty} \eta_k = 0$.

We now define a real-valued function f on I_n by

$$f(z) = \begin{cases} p_k & \text{if } z \in L_{I'}^{(k)}, \\ -p_k & \text{if } z \in H_{I'}^{(k)}, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$\frac{1}{p_k} \equiv \eta_k \sqrt{V(g; J_k^{(n)})} \tag{2.12}$$

and $q_{j,k-1} + 1 \leq i_j \leq q_{j,k}$ for $j = 1, 2, \cdots, n$. Then

$$\int_{I_n} f dz = \int_{J_k^{(n)}} f dz = \sum_{i_1=q_{1,k-1}+1}^{q_{1,k}} \cdots \sum_{i_n=q_{n,k-1}+1}^{q_{n,k}} [\int_{L_{I'}^{(k)}} f dz + \int_{H_{I'}^{(k)}} f dz] = 0$$

so $f \in HK_n(I_n)$. Similarly, by (2.7), we obtain

$$\begin{aligned}
\int_{I_n} fgdz &= \sum_{i_1=q_{1,k-1}+1}^{q_{1,k}} \cdots \sum_{i_n=q_{n,k-1}+1}^{q_{n,k}} p_k[\int_{L_{I'}^{(k)}} gdz - \int_{H_{I'}^{(k)}} gdz] \\
&\geq \sum_{i_1=q_{1,k-1}+1}^{q_{1,k}} \cdots \sum_{i_n=q_{n,k-1}+1}^{q_{n,k}} p_k\eta_k[m_{I'}^* - m_{I'}] \\
&= \sum_{i_1=q_{1,k-1}+1}^{q_{1,k}} \cdots \sum_{i_n=q_{n,k-1}+1}^{q_{n,k}} \sum_{c_1=1}^{2} \cdots \\
&\quad \sum_{c_{n-1}=1}^{2} p_k\eta_k\omega(g(x_{c_1i_1}^{(1)}, \cdots, x_{c_{n-1}i_{n-1}}^{(n-1)}, \cdot); x_{i_n-1}^{(n)}, x_{i_n}^{(n)})
\end{aligned}$$

using (2.8), (2.9), and (2.4). Then, from the inequalities (2.11) and the definition of p_k (2.12), we have $\int_{I_n} fgdz \geq \sqrt{V(g; J_k^{(n)})}$. Since $V(g; J_k^{(n)}) \to +\infty$ as $k \to +\infty$, it follows that $fg \notin HK_n(I_n)$. This contradiction completes the proof of Theorem 2.1.

An example of a function in two dimensions that is of bounded variation in the sense of Vitali et al but not in the sense of Hardy-Krause was given by Clarkson and Adams [CA], which we change in a trivial manner to conform to the notation in the present paper. Thus, we define a real-valued function h on $I_2 = [a_1, b_1] \times [a_2, b_2]$ by:

$$h(x, y) = (x - a_1)\sin(\frac{1}{x - a_1}) \text{ for } x \neq a_1; \quad = 0 \text{ for } \quad x = a_1.$$

One can easily modify this definition to give examples of functions in n dimensions ($n \geq 2$) that belong to the n-dimensional class of Vitali et al, but are not of strongly bounded variation. Let us define, for example, h on I_n by:

$$h(\mathbf{x}) = (x_1 - a_1)\sin(\frac{1}{x_1 - a_1}) \text{ for } x \neq a_1; \quad = 0 \text{ for } \quad x = a_1.$$

Other examples of this type can easily be constructed.

3. Sufficient Conditions for Existence of Multipliers

In this section we give a new proof of the result of Kurzweil [Ku2], in the n-dimensional case with $n \geq 2$, that functions of strongly bounded essential variation are multipliers for the class of Henstock-Kurzweil integrable functions. (Actually, Kurzweil considered the equivalent class of Perron integrable functions.) Our proof yields integration-by-parts formulae analogous to the one-dimensional result of P. Y.

Lee ([L1], Theorem 12.1 and Corollary 12.2, pp. 72-73) which the proof generalizes to the n-dimensional case, and which agrees with the explicit result stated by Kurzweil in the $n = 2$ case.

We begin with a lemma on the existence of certain types of Riemann-Stieltjes integrals, which generalizes the corresponding result of Clarkson [C] when $n = 2$. Consider partitions of I_n of the form (2.1) and let $\xi_i^{(j)}$ denote real numbers which satisfy the inequalities:

$$x_{i-1}^{(j)} \leq \xi_i^{(j)} \leq x_i^{(j)} \quad (i = 1, 2, \cdots, m_j; j = 1, 2, \cdots, n) \tag{3.1}$$

with

$$\xi_1^{(j)} = x_0^{(j)} = a_j, \quad \xi_{m_j}^{(j)} = x_{m_j}^{(j)} = b_j. \tag{3.2}$$

If the sums

$$\sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} h(\xi_I^{[1,n]}) \Delta_n(\phi(\mathbf{x}_{K'}^{[1,n]})), \tag{3.3}$$

where Δ_n denotes the n^{th} difference of the function ϕ defined by (1.5), tend to a finite limit as the norm of the subdivisions approaches zero; then the integral with respect to ϕ is said to exist. It will be denoted by

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} h(\mathbf{x}) d_{\mathbf{x}} \phi(\mathbf{x}), \tag{3.4}$$

and will be called a *Riemann-Stieltjes integral.* We have:

LEMMA 3.1. *A necessary and sufficient condition that the integrals* (3.4) *exist for each real-valued continuous function h defined on I_n is that ϕ belongs to the n-dimensional class of functions of bounded variation in the sense of Vitali et al (Definition* 1.1*).*

REMARK. For $n = 2$ Clarkson [C] considered, in addition to the convergence of (3.3) to (3.4) which he called the *restricted integral*, the convergence of the sums $\sum_{i=1}^{m_1} \sum_{j=1}^{m_2} h(\xi_{ij}^{(1)}, \xi_{ij}^{(2)}) \Delta_2(\phi(x_i^{(1)}, x_j^{(2)}))$, wherein the numbers $\{\xi_i^{(j)} : i = 1, \cdots, m_j; j = 1, 2\}$ in (3.3) with $n = 2$ have been replaced by the numbers $\{\xi_{ik}^{(j)} : i = 1, \cdots, m_1; k = 1, \cdots, m_2; j = 1, 2\}$. The limit of these latter sums, when they exist, was called the *unrestricted integral.* We will not need these integrals, and so will not consider their n-dimensional generalizations.

The proof of Lemma 3.1 follows from a straightforward extension of Clarkson's argument in the $n = 2$ case.

We now have:

THEOREM 3.2. *Let f and g denote real - valued functions defined on I_n. Suppose that $f \in HK_n(I_n)$ $(n \geq 1)$ and let g be of strongly bounded essential variation. Then $fg \in HK_n(I_n)$ and we have the following integration-by-parts formulae in terms of Riemann-Stieltjes integrals:*

$$\begin{aligned} &\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(\mathbf{x})g(\mathbf{x})d\mathbf{x} = \Delta_n(Fg; \mathbf{a}\,, \mathbf{b}\,) + (-1)^n B_n^{(0)} \\ &+ \sum_{q=1}^{n-1} (-1)^q \int_{a_1}^{b_1} \cdots \int_{a_q}^{b_q} \Delta_{n-q}(F(\mathbf{x}_q; \cdots) d_{\mathbf{x}_q} g(\mathbf{x}_q; \cdots); \mathbf{a_{n-q}}, \mathbf{b_{n-q}}), \end{aligned} \tag{3.5}$$

where F denotes the n-dimensional primitive of f,

$$B_n^{(0)} \equiv \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} F(\mathbf{x})\, d_{\mathbf{x}} g(\mathbf{x}), \tag{3.6}$$

and the $(n-q)$-differences in (3.5) *are defined by* (1.5) *with the function ϕ replaced by the product of the functions F and g with the differentials $d_{x_1} \cdots d_{x_q}$ inserted between them.*

REMARK. For $n = 1$, (3.5) reduces to the result

$$\int_a^b fg dx = \Delta_1(Fg; a, b) - \int_a^b F(x) dg(x), \tag{3.7}$$

discussed by P. Y. Lee ([L1], Theorem 12.1 and Corollary 12.2, pp.72-73), while for $n = 2$ it becomes

$$\begin{aligned} &\int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x_1, x_2) g(x_1, x_2) dx_1 dx_2 = \Delta_2(Fg; a_1, b_1; a_2, b_2) + B_2^{(0)} \\ &- \int_{a_1}^{b_1} \Delta_1(F(x_1, \cdot) d_{x_1} g(x_1, \cdot); a_2, b_2) - \int_{a_2}^{b_2} \Delta_1(F(\cdot, x_2) d_{x_2} g(\cdot, x_2); a_1, b_1), \end{aligned}$$

in agreement with the result given by Kurzweil [Ku2].

Proof of THEOREM 3.2. For $n = 1$, g is of bounded variation in the usual sense, $fg \in HK_1$, and the integral of fg satisfies (3.7) to which (3.5) reduces in this case.

For the case $n \geq 2$, in terms of the δ-fine partitions (2.1), (3.1), (3.2), define

$$J_n \equiv \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} f(\xi_{I'}^{[1,n]}) A_{I'} g(\xi_{I'}^{[1,n]}), \tag{3.8}$$

where $A_{I'} \equiv \prod_{j=1}^{n} (x_{i_j}^{(j)} - x_{i_j-1}^{(j)})$. We write

$$J_n = T_n + (J_n - T_n), \tag{3.9}$$

$$T_n \equiv \sum_{i_1=1}^{m_1-1} \cdots \sum_{i_n=1}^{m_n-1} f(\xi_{I'}^{[1,n]}) A_{I'} g(\xi_{I'}^{[1,n]}),$$

and an easy calculation shows that the terms in $J_n - T_n$ involve products of at most $n-1$ summations. Using the finite additivity of the differences Δ_n (see, e.g., [To], p. 68), we decompose T_n in the form

$$T_n = T_n^{(1)} + T_n^{(2)} + T_n^{(3)}, \tag{3.10}$$

where

$$\begin{aligned} T_n^{(1)} \equiv & \sum_{k_1=1}^{m_1-1} \cdots \sum_{k_n=1}^{m_n-1} \\ & \left\{ \sum_{i_1=1}^{k_1} \cdots \sum_{i_n=1}^{k_n} f(\xi_I^{[1,n]}) A_I - \Delta_n(F; [\mathbf{a} : \mathbf{x}_{K'}]) \right\} \\ & \qquad \times (-1)^n \Delta_n(g; [\xi_{K'}^{[1,n]} : \xi_{K'+1}^{[1,n]}]), \end{aligned}$$

$$T_n^{(2)} \equiv \sum_{k_1=1}^{m_1-1} \cdots \sum_{k_n=1}^{m_n-1} \Delta_n(F; [\mathbf{a} : \mathbf{x}_{K'}])[(-1)^n \Delta_n(g; [\xi_{K'}^{[1,n]} : \xi_{K'+1}^{[1,n]}])], \tag{3.11}$$

and

$$T_n^{(3)} \equiv \sum_{i_1=1}^{m_1-1} \cdots \sum_{i_n=1}^{m_n-1} f(\xi_{I'}^{[1,n]}) A_{I'} [g(\xi_{I'}^{[1,n]}) - \Delta_n(g; [\xi_{I'}^{[1,n]} : \mathbf{b}])]. \tag{3.12}$$

First consider ${T_n}^{(1)}$. Since $f \in HK_n(I_n)$, for every $\varepsilon > 0$ there exists $\delta(\xi_{I'}^{[1,n]}) > 0$ such that for any δ-fine partition (2.1), (3.1), (3.2) (i. e., there exists a positive function δ such that each n-cell $\prod_{j=1}^{n} [x_{i_j-1}^{(j)}, x_{i_j}^{(j)}] \subset I_n$ is contained in the open ball in I_n with center $(\xi_{I'}^{[1,n]})$ and radius δ) one has

$$\left| \sum_{i_1=1}^{k_1} \cdots \sum_{i_n=1}^{k_n} f(\xi_{I'}^{[1,n]}) A_{I'} - \Delta_n(F; [\mathbf{a} : \mathbf{x}_{K'}]) \right| < \varepsilon, \tag{3.13}$$

where F denotes the n-dimensional primitive of f. (This is the justification for writing eq.(1.6).) From (3.13) and the definition (2.2) of n-dimensional variations in the sense of Vitali et al, it follows that

$$\left|T_n^{(1)}\right| < \varepsilon V(g; I_n). \tag{3.14}$$

To evaluate $T_n^{(2)}$, we have by (1.5):

$$\begin{aligned} \Delta_n(F; [\mathbf{a} : \mathbf{x}_{K'}]) &= F(\mathbf{x}_{K'}^{[1,n]}) \\ &+ \sum_{q=1}^{n-1} \sum_{k_1=1}^{n} \cdots \sum_{k_q=1}^{n} (-1)^q F_{K'_q} + (-1)^n F(\mathbf{a}), \end{aligned} \tag{3.15}$$

where $k_1 < \cdots < k_q$ and (see the discussion following eq. (1.5)) the functions $F_{K'_q}$ are obtained from F by evaluation at the lower limits a_i for the indices K'_q and evaluation at the upper limits $x_{k_i}^{(i)}$ for the remaining indices. By Lemma 3.1 the contributions to $T_n^{(2)}$ obtained by replacing $\Delta_n(F; \cdots)$ in (3.11) by the first term in (3.15) converge to the Riemann-Stieltjes integrals (3.6). Similarly, for the contributions to $T_n^{(2)}$ obtained from the terms in (3.15) involving the functions $F_{K'_q}$, we use repeatedly the decompositions (2.5) of $\Delta_n(F; \cdots)$ into differences of lower order to prove, again using Lemma 3.1, that these contributions to $T_n^{(2)}$ converge to the following Riemann-Stieltjes integrals:

$$\begin{aligned} &\sum_{k_1=1}^{n} \cdots \sum_{k_q=1}^{n} (-1)^q \int_{a_{q+1}}^{b_{q+1}} \cdots \int_{a_n}^{b_n} \left\{F(\mathbf{a}_{K'_q}; \mathbf{x}_{n-q})\right\} \times \\ &\qquad \left\{d_{\mathbf{x}_{n-q}} \Delta_q(g(\cdots; \mathbf{x}_{n-q}); [\mathbf{a}_{K'_q} : \mathbf{b}_{K'_q}])\right\} \\ &\equiv \sum_{j_1=1}^{n} \cdots \sum_{j_q=1}^{n} (-1)^q B_n^{(j_1, \cdots, j_q)}, \end{aligned}$$

and, as usual, we have the constraints $j_1 < \cdots < j_q$ on the summation indices. Finally, for the contributions to $T_n^{(2)}$ arising from the remaining term in (3.15), we obtain

$$(-1)^n F(\mathbf{a}) \sum_{k_1=1}^{m_1-1} \cdots \sum_{k_n=1}^{m_n-1} \Delta_n(g; \xi_I^{[1,n]}) = F(\mathbf{a}) \Delta_n(g; [\mathbf{a} : \mathbf{b}])$$

using the additivity of the Δ_n and the identifications (3.2). Thus, upon collecting the above results, we obtain:

$$T_n^{(2)} = \tilde{T}_n^{(2)} + F(\mathbf{a}) \Delta_n(g; [\mathbf{a} : \mathbf{b}]) \tag{3.16}$$

with, for every $\varepsilon > 0$, $\left|\tilde{T}_n^{(2)} - B_n\right| < \varepsilon$, where

$$B_n \equiv (-1)^n B_n^{(0)} + \sum_{q=1}^{n-1} \sum_{j_1=1}^{n} \cdots \sum_{j_q=1}^{n} (-1)^q B_n^{(j_1,\cdots,j_q)}, \tag{3.17}$$

where $j_1 < \cdots < j_q$.

We now consider $T_n^{(3)}$ as given by (3.12). Again using formula (1.5), we have:

$$\Delta_n(g; [\xi_{I'}^{[1,n]} : \mathbf{b}]) = g(\mathbf{b}) + \sum_{q=1}^{n-1} \sum_{j_1=1}^{n} \cdots \sum_{j_q=1}^{n} (-1)^q g_{j_1,\cdots,j_q} + (-1)^n g(\xi_{I'}^{[1,n]}), \tag{3.18}$$

where the subscripts $j_1, \cdots, j_q$ are subject to the constraints $j_1 < \cdots < j_q$ and the functions $g_{J'_q}$ are derived from g in a similar manner as described in the discussions following (1.5) and (3.15). Substitution of (3.18) into (3.12) gives:

$$T_n^{(3)} = (-1)^{n-1} \sum_{i_1=1}^{m_1-1} \cdots \sum_{i_n=1}^{m_n-1} f(\xi_I^{[1,n]}) A_{I'}[g(\mathbf{b}) + \sum_{q=1}^{n-1} \sum_{i_1=1}^{n} \cdots \sum_{i_q=1}^{n} (-1)^q g_{I'_q}] \tag{3.19}$$

where, as in (3.18), $i_1 < \cdots < i_q$.

We now transform the summations $\sum_{i_j=1}^{m_j-1}$ $(j = 1, 2, \cdots, n)$ in (3.19) to summations of the form $\sum_{i=1}^{m}$ and call the resulting expressions $T_n^{(4)}$:

$$T_n^{(4)} = (-1)^{n-1} \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} f(\xi_{I'}^{[1,n]}) A_{I'}[g(\mathbf{b}) + \sum_{q=1}^{n-1} \sum_{i_1=1}^{n} \cdots \sum_{i_q=1}^{n} (-1)^q g_{I'_q}], \tag{3.20}$$

with $i_1 < \cdots < i_q$. The difference between $T_n^{(3)}$ and $T_n^{(4)}$ involves terms in which the number of summations is $\leq n-1$ and one easily shows by straightforward calculations that these terms cancel with some of the terms in $J_n - T_n$. We note that the summations for given values of q in (3.20) are special cases of the summations (3.8) because, in the functions $g_{J'_q}$, some of the arguments are the variables $\xi_{i_j}^{(j)}$ and some are upper limits. Thus, the terms in (3.20) can be evaluated by using the same procedure that we are using to evaluate J_n; viz., by using analogues of the decompositions (3.10), (3.16), and the transformation from (3.19) to (3.20).

For the terms in (3.19) with a given value of q in the set $\{1, 2, \cdots, n-$

$1\}$ we have, by analogy with the decompositions (3.10):

$$\begin{aligned} T_{n,q}^{(4)} &\equiv (-1)^{n+q-1} \sum_{\alpha_1=1}^{n} \cdots \sum_{\alpha_q=1}^{n} \\ &\sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} f(\xi_{I'}^{[1,n]}) A_{I'} g(\xi_{\alpha_1}^{(1)}, \cdots, \xi_{\alpha_q}^{(q)}; \mathbf{b}_{n-q}) \\ &= T_{n,q}^{(4,1)} + T_{n,q}^{(4,2)} + T_{n,q}^{(4,3)}, \end{aligned} \tag{3.21}$$

where $\alpha_1 < \cdots < \alpha_q$ and

$$\begin{aligned} T_{n,q}^{(4,1)} \equiv (-1)^{n+q-1} &\sum_{\alpha_1=1}^{n} \cdots \sum_{\alpha_q=1}^{n} \sum_{k_1=1}^{m_1-1} \cdots \sum_{k_q=1}^{m_q-1} \\ &\Big\{ \sum_{i_1=1}^{k_1} \cdots \sum_{i_q=1}^{k_q} \sum_{i_{q+1}=1}^{m_{q+1}} \cdots \sum_{i_n=1}^{m_n} f(\xi_{I'}^{[1,n]}) A_{I'} \\ &-\Delta_n(F; [\mathbf{a}_q : x_{K_q'}^{[1,q]}]; [\mathbf{a}_{n-q} : \mathbf{b}_{n-q}]) \Big\} \\ &\times (-1)^q \Delta_q(g(\cdots; \mathbf{b}_{n-q}); [\xi_{K_q'}^{[1,q]} : \xi_{K_q'+1}^{[1,q]}]), \end{aligned}$$

$$\begin{aligned} T_{n,q}^{(4,2)} \equiv (-1)^{n+q-1} &\sum_{\alpha_1=1}^{n} \cdots \sum_{\alpha_q=1}^{n} \sum_{k_1=1}^{m_1-1} \cdots \sum_{k_q=1}^{m_q-1} \\ &\Big\{ \Delta_n(F; [\mathbf{a}_q : x_{K_q'}^{[1,q]}]; [\mathbf{a}_{n-q} : \mathbf{b}_{n-q}]) \Big\} \times \\ &\qquad \Big\{ (-1)^q \Delta_q(g(\cdots; \mathbf{b}_{n-q}); [\xi_{K_q'}^{[1,q]} : \xi_{K_q'+1}^{[1,q]}]) \Big\}, \end{aligned}$$

$$\begin{aligned} T_{n,q}^{(4,3)} \equiv (-1)^{n+q-1} &\sum_{k_1=1}^{n} \cdots \sum_{k_q=1}^{n} \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} \Big\{ f(\xi_{I'}^{[1,n]}) A_{I'} \Big\} \\ &\times \Big[g(\xi_{K_q'}^{[1,q]}; \mathbf{b}_{n-q}) + (-1)^{q-1} \Delta_q(g(\cdots; \mathbf{b}_{n-q}); [\xi_{K_q'}^{[1,q]}, \mathbf{b}_q]) \Big], \end{aligned}$$

where we again have the constraints $k_1 < \cdots < k_q$ on the summation indices.

Each expression $T_{n,q}^{(4,1)}$ can be estimated in a similar manner as $T_n^{(1)}$ in (3.14), and a decomposition analogous to (3.16) for $T_n^{(2)}$ can be obtained for $T_{n,q}^{(4,2)}$ with associated terms $\tilde{T}_{n,q}^{(4,2)}$ converging to appropriate Riemann-Stieltjes integrals analogous to (3.17). For $T_{n,q}^{(4,3)}$, we apply formula (1.5) to Δ_q and obtain expressions analogous to (3.19):

$$\begin{aligned} T_{n,q}^{(4,3)} = (-1)^n &\sum_{i_1=1}^{m_1-1} \cdots \sum_{i_n=1}^{m_n-1} \sum_{k_1=1}^{n} \cdots \sum_{k_q=1}^{n} \Big[f(\xi_{I'}^{[1,n]}) A_{I'} \Big] \times \Big[g(\mathbf{b}) + \\ &\sum_{q'=1}^{q-1} \sum_{\beta_1=1}^{q} \cdots \sum_{\beta_{q'}=1}^{q} (-1)^{q'} g_{\beta_1,\cdots,\beta_{q'}}([\xi_{K_{q'}'}, \mathbf{b}_{K_{q'}'}]; \mathbf{b}_{n-q'-1}) \Big], \end{aligned} \tag{3.22}$$

with the constraints $k_1 < \cdots < k_q$ and $\beta_1 < \cdots < \beta_{q'}$ on the summation indices.

We now transform the summations $\sum\limits_{i_j=1}^{m_j-1}$ $(j = 1, 2, \cdots, n)$ in (3.22) to summations of the form $\sum\limits_{i=1}^{m}$ to obtain new expressions, say $T_{n,q}^{(5)}$, as we did in passing from $T_n^{(3)}$ to $T_n^{(4)}$. Just as in the earlier case, the difference between (3.22) and $T_{n,q}^{(5)}$ involves terms in which the number of summations is $\leq n - 1$ and these terms cancel some of the terms in $J_n - T_n$. We note that the terms in the summation over q in (3.20) contain expressions that combine with the first term in (3.20), and the combination of these quantities can be estimated by using an argument similar to that used in (3.13). Finally, by combining the evaluations of (3.20) with the results (3.16), we obtain the formula (3.5), thereby completing the proof of Theorem 3.2.

REMARK. We note that, for dimensions $n \geq 3$, the contributions to (3.21) for $q \geq 2$ contain terms that cancel some of the contributions to (3.21) with smaller values of q. This has the effect of changing the sign of the contributions of the latter terms to (3.20).

4. Characterization of multipliers

We now combine Theorem 3.2 with a modified version of Theorem 2.1 to obtain the following characterization of multipliers for the class of HK_n functions.

THEOREM 4.1. *Let f and g denote real-valued functions defined on I_n. Then, for a given positive integer $n \geq 1$, $fg \in HK_n(I_n)$ for all $f \in HK_n(I_n)$ if and only if g is of strongly bounded essential variation on I_n.*

Before discussing the proof of this theorem, we need to modify the definition of the n-dimensional variations (2.2) to obtain quantities more appropriate for functions of strongly bounded variation. A definition appropriate for this case has been given by Kurzweil in [Ku2]. For partitions of the form (2.1), in addition to considering essential suprema of n^{th} differences as in (2.2), one must also consider essential suprema of all lower order differences involving the "component intervals" of I_n. For example, in the case $n = 2$ we consider:

$$V_{SBV}(\phi; I_2) \equiv ess\sup_{P} \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \Bigg[\left| \Delta_2(\phi; x_{i_1-1}^{(1)}, x_{i_1}^{(1)}; x_{i_2-1}^{(2)}, x_{i_2}^{(2)}) \right| + \sum_{c_1=1}^{2} \left| \Delta_1(\phi(x_{c_1 i_1}^{(1)}, \cdot); x_{i_2-1}^{(2)}, x_{i_2}^{(2)}) \right| + \sum_{c_2=1}^{2} \left| \Delta_1(\phi(\cdot, x_{c_2 i_2}^{(2)}); x_{i_1-1}^{(1)}, x_{i_1}^{(1)}) \right| \Bigg], \tag{4.1}$$

where, as in (2.6), the values $c_j = 1, 2$ for $j = 1, 2$ correspond to the evaluation of the function ϕ at the left or right endpoint, respectively, of the intervals $[x^{(j)}_{i_j-1}, x^{(j)}_{i_j}]$. Then, by analogy with the derivation of the inequalities (2.6), we obtain upper bounds for the n-dimensional $(n = 2, 3, \cdots)$ versions of (4.1) in terms of the one-dimensional essential oscillations ω defined by (2.4):

$$
\begin{aligned}
V_{SBV}(\phi; I_n) \leq n \, ess \sup_P \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} \sum_{\nu=1}^{n} \\
\Big[\sum_{c_{j_1}=1}^{2} \cdots \sum_{c_{j_{n-1}}=1}^{2} \omega(\phi(x^{(j_1)}_{c_{j_1} i_{j_1}}, \cdots, x^{(j_{n-1})}_{c_{j_{n-1}} i_{j_{n-1}}}, \cdot); x^{(\nu)}_{i_\nu - 1}, x^{(\nu)}_{i_\nu}) \Big],
\end{aligned}
\tag{4.2}
$$

where $j_p < j_{p+1} (p = 1, 2, \cdots, n-2)$ and ν has the value in the set $\{1, 2, \cdots, n\}$ that is distinct from the values $\{j_1, j_2, \cdots, j_{n-1}\}$. We see that, whereas the variations (2.2) can be bounded by one-dimensional essential oscillations over any of the one-dimensional "component intervals" of I_n that we wish, as in (2.6), we note from (4.2) that the upper bounds of the variations $V_{SBV}(\phi; I_n)$ involve one-dimensional essential oscillations over *all* of the one-dimensional component intervals. Thus, it will be convenient in the proof of Theorem 4.1, to be discussed below, to decompose the variations $V_{SBV}(\phi; I_n)$ in the following manner:
$V_{SBV}(\phi; I_n) = \sum_{\nu=1}^{n} V^{(\nu)}_{SBV}(\phi; I_n)$ where, for each $\nu \in \{1, 2, \cdots, n\}$ as defined following (4.2), with $j_p < j_{p+1}$ for $p = 1, 2, \cdots, n-2$:

$$
\begin{aligned}
V^{(\nu)}_{SBV}(\phi; I_n) \leq n \, ess \sup_P \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} \sum_{c_{j_1}=1}^{2} \cdots \\
\sum_{c_{j_{n-1}}=1}^{2} \omega(\phi(x^{(j_1)}_{c_{j_1} i_{j_1}}, \cdots, x^{(j_{n-1})}_{c_{j_{n-1}} i_{j_{n-1}}}, \cdot); x^{(\nu)}_{i_\nu - 1}, x^{(\nu)}_{i_\nu}).
\end{aligned}
\tag{4.3}
$$

PROOF OF THEOREM 4.1. It follows from Theorem 3.2 that the condition that g is of strongly bounded essential variation is sufficient to have $fg \in HK_n(I_n)$ for all $f \in HK_n(I_n)$ with $n = 1, 2, \cdots$, so our task is to show that the proof of Theorem 2.1 can be modified to provide a proof of the necessity of this condition. For $n = 1$ the definition of functions of strongly bounded variation reduces to the usual concept of functions of bounded variation, so the result is true in this case.

For $n \geq 2$, following the argument in the proof of Theorem 2.1, we may again reduce to the case in which g is essentially bounded. Now, suppose that g is not equal a.e. to a function of strongly bounded variation. Then, modifying the argument in the proof of Theorem 2.1, for any function $g_2 = g$ a.e., g_2 is not of strongly bounded variation and there exists a point $\kappa \in I_n$ such that g_2 is not of strongly bounded variation on any open n-cell which contains κ. In particular, we may suppose that κ is a singular point of g where, however, we now define the concept of *singular point* in a different manner than in the proof of Theorem 2.1 by using the definition of variation given following the statement of the present theorem (i.e., the n-dimensional analogues of (4.1)). We may assume, without loss in generality, that the point $(a_1, \cdots, a_n) \in I_n$ is a singular point in this modified sense.

The proof now proceeds in a similar manner to that of Theorem 2.1. We obtain by induction a sequence of disjoint n-cells $J_k^{(n)} \equiv [\alpha_1^{(k)}, \beta_1^{(k)}] \times \cdots \times [\alpha_n^{(k)}, \beta_n^{(k)}] \subset I_n$ with $\beta_i^{(k)} < \alpha_i^{(k-1)}$ for $i = 1, 2, \cdots, n$ and $k = 2, 3, \cdots$ such that $\lim\limits_{k \to +\infty} \alpha_i^{(k)} = \lim\limits_{k \to +\infty} \beta_i^{(k)} = a_i$ for each $i \in \{1, 2, \cdots, n\}$. (For convenience, we use the same notation for these n-cells as for the n-cells discussed in the proof of Theorem 2.1.) On each of these n-cells $J_k^{(n)}$, there exists a partition of the form (2.10) such that inequalities analogous to (2.11) hold for each value of ν defined following (4.2):

$$\begin{aligned}&\sum_{i_1=q_{1,k-1}+1}^{q_{1,k}} \cdots \sum_{i_n=q_{n,k-1}+1}^{q_{n,k}} \\ &\sum_{c_{j_1}=1}^{2} \cdots \sum_{c_{j_{n-1}}=1}^{2} \omega(g(x^{(j_1)}_{c_{j_1} i_{j_1}}, \cdots, x^{(j_{n-1})}_{c_{j_{n-1}} i_{j_{n-1}}}, \cdot); x^{(\nu)}_{i_\nu - 1}, x^{(\nu)}_{i_\nu}) \\ &\geq \tfrac{1}{n} V^{(\nu)}_{SBV} \geq \tfrac{k}{n}.\end{aligned} \tag{4.4}$$

Then, on each set $K_k^{(n,\nu)} \equiv \{x^{(j_1)}_{c_{j_1} i_{j_1}}, \cdots, x^{(j_{n-1})}_{c_{j_{n-1}} i_{j_{n-1}}}\} \times [x^{(\nu)}_{i_\nu - 1}, x^{(\nu)}_{i_\nu}] \subset J_k^{(n)}$, there exist disjoint measurable sets $L^{(k,\nu)}_{i_{j_1}, \cdots, i_{j_{n-1}}, i_\nu}, H^{(k,\nu)}_{i_{j_1}, \cdots, i_{j_{n-1}}, i_\nu}$ such that $\eta_{k-1,\nu} > \mu(L^{(k,\nu)}_{i_{j_1}, \cdots, i_{j_{n-1}}, i_\nu}) = \mu(H^{(k,\nu)}_{i_{j_1}, \cdots, i_{j_{n-1}}, i_\nu}) \equiv \eta_{k,\nu} > 0$ and such that analogues of (2.7) hold:

$$\mathbf{z} \in L^{(k,\nu)}_{i_{j_1}, \cdots, i_{j_{n-1}}, i_\nu} \Rightarrow g(\mathbf{z}) \geq \frac{3}{2} M^{(\nu)}_{i_{j_1}, \cdots, i_{j_{n-1}}, i_\nu} + \frac{1}{2} m^{(\nu)}_{i_{j_1}, \cdots, i_{j_{n-1}}, i_\nu}, \tag{4.5a}$$

while

$$\mathbf{z} \in H^{(\nu)}_{i_{j_1}, \cdots, i_{j_{n-1}}, i_\nu} \Rightarrow g(\mathbf{z}) \leq \frac{1}{2} M^{(\nu)}_{i_{j_1}, \cdots, i_{j_{n-1}}, i_\nu} + \frac{3}{2} m^{(\nu)}_{i_{j_1}, \cdots, i_{j_{n-1}}, i_\nu}, \tag{4.5b}$$

where

$$M^{(\nu)}_{i_{j_1}, \cdots, i_{j_{n-1}} i_\nu} \equiv \sum_{c_{j_1}=1}^{2} \cdots \sum_{c_{j_{n-1}}=1}^{2} \operatorname*{ess\,sup}_{y \in [x^{(\nu)}_{i_\nu - 1}, x^{(\nu)}_{i_\nu}]} g(x^{(j_1)}_{c_{j_1} i_{j_1}}, \cdots, x^{(j_{n-1})}_{c_{j_{n-1}} i_{j_{n-1}}}; y), \tag{4.6}$$

$$m^{(\nu)}_{i_{j_1},\cdots,i_{j_{n-1}},i_\nu} \equiv \sum_{c_{j_1}=1}^{2} \cdots \sum_{c_{j_{n-1}}=1}^{2} \operatorname*{ess\,inf}_{y\in[x^{(\nu)}_{i_\nu-1},x^{(\nu)}_{i_\nu}]} g(x^{(j_1)}_{c_{j_1}i_{j_1}},\cdots,x^{(j_{n-1})}_{c_{j_{n-1}}i_{j_{n-1}}};y), \tag{4.7}$$

and, in defining (4.6) and (4.7), we have used results analogous to the one stated following (2.9). In addition, $\lim\limits_{k\to+\infty} \eta_{k,\nu} = 0$ for each value of ν defined following (4.2).

For a given z in I_n, we now define a real-valued function $f(z)$ by

$$f(z) = \begin{cases} p_{k,\nu} & \text{if } z \in L^{(k,\nu)}_{i_{j_1},\cdots,i_{j_{n-1}},i_\nu}, \\ -p_{k,\nu} & \text{if } z \in H^{(k,\nu)}_{i_{j_1},\cdots,i_{j_{n-1}},i_\nu}, \nu = 1,2,\cdots,n, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$\frac{1}{p_{k,\nu}} \equiv \eta_{k,\nu}\sqrt{V^{(\nu)}_{SBV}(g;J^{(n)}_k)}, \tag{4.8}$$

and $q_{j,k-1}+1 \leq i_j \leq q_{j,k}$ for $j = 1,2,\cdots,n$. Then

$$\int_{I_n} f dz = \int_{J^{(n)}_k} f dz = \sum_{i_1=q_{1,k-1}+1}^{q_{1,k}} \cdots \sum_{i_n=q_{n,k-1}}^{q_{n,k}} \sum_{c_{j_1}=1}^{2} \cdots \sum_{c_{j_{n-1}}=1}^{2}$$
$$\sum_{\nu=1}^{n} [\int_{L^{(k,\nu)}_{i_{j_1},\cdots,i_{j_{n-1}},i_\nu}} f dz + \int_{H^{(k,\nu)}_{i_{j_1},\cdots,i_{j_{n-1}},i_\nu}} f dz] = 0,$$

so that $f \in HK_n(I_n)$. Similarly, by (4.5) and using (4.6), (4.7), and (2.4), we have

$$\int_{I_n} fg dz \geq \sum_{i_1=q_{1,k-1}+1}^{q_{1,k}} \cdots \sum_{i_n=q_{n,k-1}+1}^{q_{n,k}} \sum_{c_{j_1}=1}^{2} \cdots \sum_{c_{j_{n-1}}=1}^{2}$$
$$\sum_{\nu=1}^{n} p_{k,\nu}\eta_{k,\nu}[M^{(\nu)}_{i_{j_1},\cdots,i_{j_{n-1}},i_\nu} - m^{(\nu)}_{i_{j_1},\cdots,i_{j_{n-1}},i_\nu}]$$
$$= \sum_{i_1=q_{1,k-1}+1}^{q_{1,k}} \cdots \sum_{i_n=q_{n,k-1}+1}^{q_{n,k}} \sum_{c_{j_1}=1}^{2} \cdots \sum_{c_{j_{n-1}}=1}^{2}$$
$$\sum_{\nu=1}^{n} p_{k,\nu}\eta_{k,\nu}\omega(g(x^{(j_1)}_{c_{j_1}i_{j_1}},\cdots,x^{(j_{n-1})}_{c_{j_{n-1}}i_{j_{n-1}}},\cdot);x^{(\nu)}_{i_\nu-1},x^{(\nu)}_{i_\nu}).$$

Then, using the inequalities (4.3) and the definition (4.8) of $p_{k,\nu}$, we have

$$\int_{I_n} fg dz \geq \sum_{\nu=1}^{n} \sqrt{V^{(\nu)}_{SBV}(g;J^{(n)}_k)}.$$

Since $V^{(\nu)}_{SBV}(g; J^{(n)}_k) \to +\infty$ as $k \to +\infty$ for $\nu \in \{1, 2, \cdots, n\}$ by the inequality on the right-hand side of (4.4), it follows that $fg \notin HK_n(I_n)$. This contradiction completes the proof of Theorem 4.1.

Acknowlegments

One of us (WWZ) wishes to thank Lee Tuo Yeong for reprints of the papers [LCL] and [L2], and to also thank him, as well as Erik Talvila and Luisa Di Piazza, for discussions concerning the work presented herein.

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