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QUASIORDERS ON TOPOLOGICAL CATEGORIES

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ABSTRACT. We prove that, for every cardinal number $\alpha \geq \mathfrak{c}$, there exists a metrizable space X with $|X| = \alpha$ such that for every pair of quasiorders $\leq_1$, $\leq_2$ on a set Q with $|Q| \leq \alpha$ satisfying the implication

$$q \leq_1 q' \implies q \leq_2 q'$$

there exists a system $\{X(q) : q \in Q\}$ of non-homeomorphic clopen subsets of X with the following properties:

- $q \leq_1 q'$ if and only if $X(q)$ is homeomorphic to a clopen subset of $X(q')$,
- $q \leq_2 q'$ implies that $X(q)$ is homeomorphic to a closed subset of $X(q')$ and
- $\neg(q \leq_2 q')$ implies that there is no one-to-one continuous map of $X(q)$ into $X(q')$.

1. Introduction and the Main Result

Let $\mathcal{M}$ be a class of morphisms of a category $\mathcal{K}$ containing all isomorphisms and closed with respect to the composition. Then $\mathcal{M}$ determines a quasiorder $\preccurlyeq$ on the class of objects of $\mathcal{K}$ by the rule

$$\begin{gathered} X \preccurlyeq Y \text{ if and only if there exists } m \in \mathcal{M} \\ \text{with the domain } X \text{ and the codomain } Y. \end{gathered}$$

An *$\mathcal{M}$-representation of a quasiordered set $(Q, \leq)$ in $\mathcal{K}$* is any collection $X = \{X(q) : q \in Q\}$ of non-isomorphic objects of $\mathcal{K}$ such that, for every $q, q' \in Q$,

$$q \leq q' \text{ if and only if } X(q) \preccurlyeq X(q').$$

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Which quasiordered sets have $\mathcal{M}$-representations in which categories $\mathcal{K}$ for which classes $\mathcal{M}$? In topology, this investigation with $\mathcal{M}$ being the class of all homeomorphic embeddings has rather long tradition. In 1926, C. Kuratowski and W. Sierpiński proved (see [2, 3]) that the antichain of the cardinality $2^{\mathfrak{c}}$ and the ordinal $\mathfrak{c}^+$ have such representations within the category of all subspaces of the real line $\mathbb{R}$. In 1993, P.T. Matthews and T.B.M. McMaster refreshed this field of problems and proved (see [5]) that every partially ordered set of the cardinality $\mathfrak{c}$ has such representation. In 1999, A.E. McCluskey, T.B.M. McMaster and W.S. Watson proved (see [8]) that the set $(\exp \mathbb{R}, \subseteq)$ of all subsets of the real line $\mathbb{R}$ ordered by the inclusion also has such a representation, i.e. a representation by subspaces of $\mathbb{R}$ with respect to the embeddability. Since $(\exp \mathbb{R}, \subseteq)$ contains an antichain of the cardinality $2^{\mathfrak{c}}$, their result implies the first result of C. Kuratowski and W. Sierpiński. Analogously it implies the previous result of P.T. Matthews and T.B.M. McMaster because $(\exp \mathbb{R}, \subseteq)$ contains an isomorphic copy of any partially ordered set of cardinality at most $\mathfrak{c}$. The authors of [8] also announced that they have a counterexample consistent with ZFC to the statement that every partially ordered set of the cardinality $2^{\mathfrak{c}}$ can be represented by subspaces of $\mathbb{R}$ with respect to the embeddability. In [6], A.E. McCluskey and T.B.M. McMaster generalized the construction of [8] and they proved that, for any infinite cardinal numbers $\mathfrak{a}, \mathfrak{b}$ such that $\mathfrak{b}^{\mathfrak{a}} = \mathfrak{b}$, every partially ordered set $(P, \leq)$ with $|P| \leq \mathfrak{b}$ has a representation (with respect to the embeddability) by subsets of *every* T_3-space X containing a dense subset D with $|D| \leq \mathfrak{a}$ whenever every non-empty open set of X has the cardinality $\mathfrak{b}$. Moreover, if X admits a homeomorphism into itself such that every $x \in X$ has an infinite orbit, then every quasiordered set obtained from a partially ordered set $(P, \leq)$ with $|P| \leq \mathfrak{b}$ by splitting every $p \in P$ into $2^{\mathfrak{b}}$ mutually comparable points has also such a representation.

Given a cardinal number α, A.E. McCluskey and T.B.M. McMaster also present (in [7]) a construction of a T_0-space X with $|X| = \delta(\alpha)$ such that every quasiordered set $(Q, \leq)$ with $|Q| \leq \alpha$ has a representation by subsets of X (with respect to the embeddability), where $\delta(\alpha)$ is the smallest cardinal number δ such that there exist α distinct cardinal numbers γ (not necessarily infinite) smaller than δ. If $\alpha = \aleph_0$, then $\delta(\alpha) = \aleph_0$; hence there exists a countable T_0-space X such that every countable quasiordered set has a representation (with respect to the embeddability) by subsets of X. For α uncountable, the size of X is rather high. The statement below offers a stronger result for any $\alpha \geq \mathfrak{c}$.

For every cardinal number $\alpha \geq \mathfrak{c}$ there exists a metrizable space X with $|X| = \alpha$ such that every quasiordered set $(Q, \leq)$ with $|Q| \leq \alpha$ has an $\mathcal{M}$-representation by retracts of X whenever

(1) $\mathcal{M}$ consists of one-to-one continuous maps and it contains all coproduct injections (= homeomorphic embeddings onto clopen subspaces) or

(2) $\mathcal{M}$ consists of continuous surjections and it contains all product projections.

The claim (1) implies e.g. the representability of every quasiordered set $(Q, \leq)$ with $|Q| \leq \alpha$ by retracts of X with respect to the embeddability or the embeddability onto closed subspaces or onto retracts or onto clopen subspaces. The claim (2) implies the representability e.g. by being continuous image or a quotient or a continuous open image or a factor in a product. Here we prove a result much stronger than (1) above, namely the following.

Theorem. *For every cardinal number $\alpha \geq \mathfrak{c}$, there exists a metrizable space X with $|X| = \alpha$ such that, for every pair $\leq_1$, $\leq_2$ of quasiorders on a set Q with $|Q| \leq \alpha$ satisfying the implication*

$$q \leq_1 q' \implies q \leq_2 q',$$

there exists a system $\{X(q) : q \in Q\}$ of non-homeomorphic clopen subsets of X with the following properties:

(1) *$q \leq_1 q'$ if and only if $X(q)$ is homeomorphic to a clopen subset of $X(q')$,*
(2) *$q \leq_2 q'$ implies that $X(q)$ is homeomorphic to a closed subset of $X(q')$ and*
(3) *$\neg(q \leq_2 q')$ implies that there is no one-to-one continuous map of $X(q)$ into $X(q')$.*

The proof of this theorem is presented in the part II of the present paper. Inspecting the proof, one can see that none of the spaces $X(q)$ has an isolated point. Hence if one adds α isolated points to every $X(q)$ (and also to the space X), she (he) gets the following result:

For every cardinal number $\alpha \geq \mathfrak{c}$ there exists a metrizable space X with $|X| = \alpha$ such that every quasiordered set $(Q, \leq)$ with $|Q| \leq \alpha$ has an $\mathcal{M}$-representation by clopen subspaces of X whenever $\mathcal{M}$ is the class of all continuous bijections.

We do not present here the proof of the above claim (2) and of its stronger variant concerning simultaneous representation of a pair $\leq_1$, $\leq_2$ of quasiorders by product projections and continuous surjections. The proof will appear elsewhere.

Finally, let us refresh here some results which have applications in the present field of problems. In [1], J. Adámek and V. Koubek introduced a sum-productive representation of a partially ordered commutative semigroups as follows: If $(S, \circ, \leq)$ is a partially ordered commutative semigroup (i.e. if $(S, \circ)$ is a commutative semigroup and $\leq$ is a partial order on S such that

$$(a \leq b) \text{ and } (a' \leq b') \implies a \circ a' \leq b \circ b'),$$

then its sum-productive representation in a category $\mathcal{K}$ is any collection $\{X(s) : s \in S\}$ of objects of $\mathcal{K}$ such that

($\times$) $X(s \circ s')$ is always isomorphic to the product $X(s) \times X(s')$ and

($\leq$) $s \leq s'$ if and only if $X(s)$ is isomorphic to a summand of $X(s')$ (i.e. $X(s')$ is a coproduct of $X(s)$ and an object of $\mathcal{K}$).

In [11], J. Vinárek proved that every partially ordered commutative semigroup has a sum-productive representation in the category of all metric zero-dimensional spaces. Every partially ordered set $(P, \leq)$ can be enlarged to a partially ordered set $(S, \leq)$ in which every pair of elements has an infimum. Putting $p \circ p' = \inf\{p, p'\}$, one gets the partially ordered commutative semigroup $(S, \circ, \leq)$ and the Vinárek's result can be applied. Hence

every partially ordered set has a representation by zero-dimensional metrizable spaces with respect to the embeddability onto clopen subspaces.

In [10], sum-productive representations in the category of all $F_{\sigma\delta}$-and-$G_{\sigma\delta}$ subspaces of the Cantor discontinuum are examined. Omitting the product forming again, we get the result that

if $\mathbb{C}$ is a countable set, then $(\exp \mathbb{C}, \subseteq)$ has a representation by $F_{\sigma\delta}$ and $G_{\sigma\delta}$ subspaces of the Cantor discontinuum with respect to the embeddability onto clopen subspaces.

These scattered results are surrounded by many unsolved questions, such as: which *quasiordered* sets can be represented by metrizable zero-dimensional spaces or by closed or Borelian or all subsets of the real line with respect to the embeddability onto clopen subspaces or onto Borelian subspaces or onto closed subspaces or onto retracts; and many others.

2. Proof of the Theorem

Let $\alpha \geq \mathfrak{c}$ be given. Let Q be a set with $|Q| = \alpha$. In the part A of the proof, we suppose that we have sets $S^{(1)}, \ldots, S^{(4)}$ of metrizable spaces of cardinality α with the five properties below and we prove Theorem using such sets. In the part B, we prove that such sets $S^{(1)}, \ldots, S^{(4)}$ really do exist.

Part A.

a. Thus, let us suppose that $S^{(1)}, \ldots, S^{(4)}$ are sets of metrizable spaces with $|S^{(i)}| = \alpha$ and $|Y| = \alpha$ for every $Y \in S^{(i)}$, $i = 1, \ldots, 4$, such that the statements (1)–(5) below are satisfied:

(1) $S^{(1)} = \{A_q, B_q : q \in Q\}$ are spaces such that, for every $q \in Q$, A_q is homeomorphic to a clopen subspace of B_q and B_q is homeomorphic to a closed subspace but to no clopen subspace of A_q; moreover, if $q \neq q'$, there exists no continuous one-to-one map of any A_q, B_q into any $A_{q'}, B_{q'}$;

(2) $S^{(2)} = \{D_q, E_q : q \in Q\}$ are such that D_q is homeomorphic to a closed subspace of E_q and E_q is homeomorphic to a closed subspace of D_q but there exists no homeomorphism of D_q onto a clopen subspace of E_q and no homeomorphism of E_q onto a clopen subspace of D_q; moreover, if $q \neq q'$, then there exists no continuous one-to-one map of any D_q, E_q into any $D_{q'}, E_{q'}$;

(3) $S^{(3)} = \{M_q, N_q : q \in Q\}$ are spaces such that M_q is homeomorphic to a clopen subspace of N_q and N_q is homeomorphic to a closed subspace but to no clopen subspace of M_q (i.e. they are mutually situated as A_q and B_q in (1)); moreover, if $q \neq q'$, there exists no continuous one-to-one map of any M_q, N_q into any $M_{q'}, N_{q'}$;

(4) $S^{(4)} = \{G_q, H_q : q \in Q\}$ are spaces such that G_q is homeomorphic to a clopen subspace of H_q and H_q is homeomorphic to a clopen subspace of G_q but G_q is not homeomorphic to H_q; moreover, if $q \neq q'$, there exists no continuous one-to-one map of any G_q, H_q into any $G_{q'}, H_{q'}$;

(5) if $i, j = 1, \dots, 4$, $i \neq j$, $Z \in S^{(i)}$, $Y \in S^{(j)}$, then there exists no continuous one-to-one map of Z into Y.

We put

$$X = \coprod_{q \in Q} (A_q \amalg B_q \amalg (D_q \times \omega) \amalg (E_q \times \omega) \amalg M_q \amalg N_q \amalg G_q)$$

where $\amalg$ and $\coprod$ denote the coproduct (= disjoint union as clopen subspaces) and ω is a countable discrete space (i.e. $D_q \times \omega$ is a coproduct of countably many copies of D_q and analogously for E_q).

Hence X is a metrizable space and $|X| = \alpha$. We show that X has all the required properties. In the reasoning below, we shall frequently use the statement (5) without mentioning it.

b. Let $(Q, \leq_1, \leq_2)$ be a set with two quasiorders $\leq_1, \leq_2$ such that

$$q \leq_1 q' \implies q \leq_2 q'.$$

To begin with, let us suppose, moreover, that $\leq_1$ is a partial order, i.e.

$$(q \leq_1 q') \text{ and } (q' \leq_1 q) \implies q = q' \tag{2.1}$$

(this requirement will be removed at the end of part A of the proof).

By means of $S^{(1)}$–$S^{(3)}$, we construct a system $\{X(q) : q \in Q\}$ of clopen subspaces of X such that

- $q \leq_1 q'$ if and only if $X(q)$ is homeomorphic to a clopen subset of $X(q')$,
- $q \leq_2 q'$ if and only if $X(q)$ is homeomorphic to a closed subset of $X(q')$ and
- if $\neg(q \leq_2 q')$, then there is no continuous one-to-one map of $X(q)$ into $X(q')$.

First, let us define $<$ by the rule

$$q < q' \text{ if and only if } (q \leq_1 q') \text{ and } (\neg(q' \leq_1 q)) \text{ and } (q' \leq_2 q),$$

and $q \leq q'$ denotes $q < q'$ or $q = q'$. (By (2.1), $\leq$ is a partial order.) Let $\mathcal{C}$ be the system of all components of $\leq$ (i.e. $q, q' \in C \in \mathcal{C}$ if and only if there exist $q_0, \dots, q_n$ in Q such that $q = q_0 \leq q_1 \geq q_2 \leq \dots q_n = q'$).

For every $q \in C \in \mathcal{C}$ put

$$S_q^{(1)} = \{B_{q'} : q' \leq q\} \cup \{A_{q'} : q' \in C \text{ and } \neg(q' \leq q)\},$$
$$X_q^{(1)} = \coprod S_q^{(1)}.$$

Moreover, let us denote

$$B_C = \{B_q : q \in C\},$$
$$D_C = \{D_q \times \omega : q \in C\},$$
$$E_C = \{E_q \times \omega : q \in C\}.$$

Observation. If $q \in C$ and $q' \in C'$ with $C, C' \in \mathcal{C}$, $C \neq C'$, then there exists no continuous one-to-one map of $X_q^{(1)}$ into $X_{q'}^{(1)}$. If $q, q' \in C$, then $q < q'$ if and only if $X_q^{(1)}$ is homeomorphic to a clopen subspace of $X_{q'}^{(1)}$ and $X_{q'}^{(1)}$ is homeomorphic to a closed but not to any clopen subspace of $X_q^{(1)}$.

c. On Q, let us define $q\,R\,q'$ if and only if either

$q < q'$ or
$q \leq_2 q'$ and $q' \leq_2 q$ but neither $q \leq_1 q'$ nor $q' \leq_1 q$.

Let $\prec$ be the transitive envelope of R, and let $\preceq$ mean $\prec$ or $=$. Then $\preceq$ is a quasiorder on Q; let $\mathcal{L}$ be the system of all its components. Clearly, each $C \in \mathcal{C}$ is a subset of precisely one $L \in \mathcal{L}$. For every $q \in C \in \mathcal{C}$, $C \subseteq L$, put

$$S_q^{(2)} = S_q^{(1)} \cup \bigcup_{C' \in \mathcal{C}, C \neq C' \subseteq L} (B_{C'} \cup E_{C'} \cup D_{C'}) \cup E_C;$$
$$X_q^{(2)} = \coprod S_q^{(2)}$$

Observation. If $q, q' \in C \in \mathcal{C}$, $q \neq q'$, then $S_q^{(2)}$ differs from $S_q^{(1)}$ by the same summand

$$\bigcup_{C' \in \mathcal{C}, C \neq C' \subseteq L} (B_{C'} \cup E_{C'} \cup D_{C'}) \cup E_C$$

as $S_{q'}^{(2)}$ differs from $S_{q'}^{(1)}$. Hence $X_q^{(2)}$ and $X_{q'}^{(2)}$ are in the same relations as $X_q^{(1)}$ and $X_{q'}^{(1)}$. If $C, C' \in \mathcal{C}$, $C \neq C'$ and both C, C' are subsets of $L \in \mathcal{L}$, $q \in C$, $q' \in C'$, then X_q is homeomorphic to a closed subspace but not to any clopen subspace of $X_{q'}$ and vice versa. This is because $\coprod(E_C \cup D_C \cup E_{C'})$ is homeomorphic to a closed (but not to any clopen!) subspace of $\coprod(E_{C'} \cup D_{C'} \cup E_C)$, by (2) (and $\coprod(S_q^{(1)} \cup B_{C'})$ is homeomorphic to a closed subspace of $\coprod(S_{q'}^{(1)} \cup B_C)$, by (1)).

Hence $\{X_q^{(2)} : q \in L\}$ is such that

$q \leq_1 q'$ if and only if $X_q^{(2)}$ is homeomorphic to a clopen subset of $X_{q'}^{(2)}$

and

$q \leq_2 q'$ if and only if $X_q^{(2)}$ is homeomorphic to a closed subset of $X_{q'}^{(2)}$.

(For every $q, q' \in L$, we have $q \leq_2 q'$ and $q' \leq_2 q$, of course.) Moreover, if $L, L' \in \mathcal{L}$, $L \neq L'$, $q \in L$, $q' \in L'$, then there exists no one-to-one continuous map of $X(q)$ into $X(q')$ and vice versa. Let us denote

$$S_L^{(2)} = \{B_C \cup E_C \cup D_C : C \in \mathcal{C}, C \subseteq L\}.$$

d. On Q, let us define $q\dot{R}q'$ if and only if either

$$\begin{aligned} &q \prec q' \text{ or} \\ &q \neq q', q \leq_1 q' \text{ or} \\ &q \neq q', q \leq_2 q'. \end{aligned}$$

Let $\dot{\prec}$ be the transitive envelope of $\dot{R}$, and let $\dot{\preceq}$ mean $\dot{\prec}$ or $=$. Let $\mathcal{T}$ be the system of all components of the quasiorder $\dot{\preceq}$. Every $L \in \mathcal{L}$ is a subset of precisely one $T \in \mathcal{T}$. If $L, L' \in \mathcal{L}$, $L \neq L'$, $q \in L$, $q' \in L'$ and $q \leq_2 q'$, then for no $\bar{q} \in L$, $\bar{q}' \in L'$ is $\bar{q}' \leq_2 \bar{q}$ because $L \neq L'$. Hence in fact, $\dot{\preceq}$ determines a partial order on the components in $\mathcal{L}$, let us denote it also by $\dot{\preceq}$. For every $L \in \mathcal{L}$ define

$$\widetilde{S}_q^{(2)} = \bigcup_{L' \dot{\prec} L} \widetilde{S}_{L'}^{(2)}$$

and for $q \in L$,

$$\begin{aligned} \widetilde{S}_q^{(2)} &= S_q^{(2)} \cup \widetilde{S}_L^{(2)}, \\ \widetilde{X}_q^{(2)} &= \coprod \widetilde{S}_q^{(2)}. \end{aligned}$$

If $q \in L, q' \in L'$ and L, L' are components of $\mathcal{L}$ incomparable in the partial order $\dot{\preceq}$, then there is no continuous one-to-one map of $\widetilde{X}_q^{(2)}$ into $\widetilde{X}_{q'}^{(2)}$ and vice versa. If $q, q' \in L$, then $\widetilde{X}_q^{(2)}$ is in the same position with respect to $\widetilde{X}_{q'}^{(2)}$ as $X_q^{(2)}$ to $X_{q'}^{(2)}$. If $q' \in L' \dot{\preceq} L \ni q$, then $\widetilde{X}_{q'}^{(2)}$ is homeomorphic to a clopen subspace of $\widetilde{X}_q^{(2)}$ and there exists no continuous one-to-one map of $\widetilde{X}_q^{(2)}$ into $\widetilde{X}_{q'}^{(2)}$. This is what we need whenever $q' \leq_1 q$. However, if $q' \leq_2 q$ and $\neg(q' \leq_1 q)$, we still have to modify the systems $\widetilde{S}_q^{(2)}$ and the spaces $\widetilde{X}_q^{(2)}$ by means of the spaces in $S^{(3)}$ in (3). For $q \in L$, we put

$$\begin{aligned} &S_q^{(3)} = \widetilde{S}_q^{(2)} \cup \{N_{q'} : q' \in T \text{ and } q' \leq_1 q\} \cup \{M_{q'} : q' \in T \text{ and } \neg(q' \leq_1 q)\}; \\ &X(q) = \coprod S_q^{(3)}. \end{aligned}$$

First, we outline why these "new summands" do not destroy the mutual position of $\widetilde{X}_{q_1}^{(2)}$ and $\widetilde{X}_{q_2}^{(2)}$ whenever $q_1, q_2 \in L \subseteq T$:

- $q_1 \leq_1 q_2$: if $N_{q'} \in S_{q_1}^{(3)}$, i.e. $q' \leq_1 q_1$, we get $q' \leq_1 q_2$ so that $N_{q'} \in S_{q_2}^{(3)}$; since $S_{q_2}^{(3)}$ always contains either $M_{q'}$ or $N_{q'}$, the case $M_{q'} \in S_{q_1}^{(3)}$ is clear.
- $q_1 \leq_2 q_2$: if $N_{q'} \in S_{q_1}^{(3)}$, i.e. $q' \leq_1 q_1$, then either $q' \leq_1 q_2$ hence $N_{q'} \in S_{q_2}^{(3)}$, or $\neg(q' \leq_1 q_2)$ hence $M_{q'} \in S_{q_2}^{(3)}$; but $N_{q'}$ is homeomorphic to a closed subspace of $M_{q'}$.

No other case for $q_1, q_2 \in L$ is possible (if $\neg(q_1 \leq q_2)$ for $q_1, q_2 \in L$, then already $\widetilde{S}^{(2)}_{q_1}$ is not homeomorphic to a clopen subspace of $\widetilde{S}^{(2)}_{q_2}$).

Next we show that the system $\{X(q) : q \in Q\}$ has all the required properties.

Let $q_1, q_2 \in Q$. If $q_1, q_2 \in L$, we have just proved it. If $q_1 \in T, q_2 \in T'$ where T, T' are distinct elements of $\mathcal{T}$ (i.e. distinct components of the quasiorder $\preceq$), then there is no continuous one-to-one map of $X(q_1)$ into $X(q_2)$ or vice versa. But this precisely corresponds to the fact that $\neg(q_1 \leq_2 q_2)$ and $\neg(q_2 \leq_2 q_1)$. The remaining case is that q_1, q_2 are in distinct components L_1, L_2 of $\mathcal{L}$ but both L_1 and L_2 are subsets of a component $T \in \mathcal{T}$. We discuss the following cases:

α) L_1 and L_2 are incomparable in the partial order $\dot{\preceq}$: then there exists no one-to-one continuous map of $\widetilde{X}^{(2)}_{q_1}$ into $\widetilde{X}^{(2)}_{q_2}$ and adding any summands $M_{q'}, N_{q'}$ cannot change this situation; hence there exists no continuous one-to-one map of $X(q_1)$ into $X(q_2)$ or vice versa.

β) let $L_1 \dot{\prec} L_2$: hence $\widetilde{X}^{(2)}_{q_1}$ is homeomorphic to a clopen subspace of $\widetilde{X}^{(2)}_{q_2}$ and there exists no one-to-one continuous map of $\widetilde{X}^{(2)}_{q_2}$ into $\widetilde{X}^{(2)}_{q_1}$, adding of any summands $M_{q'}, N_{q'}$ cannot change the latter fact; since $\neg(q_2 \leq_2 q_1)$, this is precisely what we need. Thus we have to discuss the cases $q_1 \leq_1 q_2$ and $q_1 \leq_2 q_2$. However the reasoning is precisely the same as in the above discussed cases when $q_1, q_2 \in L$ and $q_1 \leq_1 q_2$ or $q_1 \leq_2 q_2$.

e. Finally, we have to remove the condition (2.1): If $\leq_1$ is a quasiorder, we use the standard trick that we define an equivalence on Q by the rule

$$q \sim q' \text{ if and only if } q \leq_1 q' \text{ and } q' \leq_1 q.$$

Then $\leq_1$ determines a partial order and $\leq_2$ a quasiorder on the set $Q/{\sim}$; let us denote them by $\leq_1$ and $\leq_2$ as well. For every $[q] \in Q/{\sim}$, we construct $S^{(3)}_{[q]}$ and $X^{(3)}_{[q]} = \coprod S^{(3)}_{[q]}$ as described and, for $q \in [q]$, we put

$$S^{(3)}_q = S^{(3)}_{[q]} \cup \{G_{q'} : q' \in Q \setminus \{q\}\} \cup \{H_q\}$$

where $G_{q'}, H_q$ are from $S^{(4)}$, i.e. they satisfy the condition (4). Then, for $X_q = \coprod S^{(3)}_q$, the system $\{X_q : q \in Q\}$ has all the required properties.

Part B.

To finish the proof, it remains to show that the systems $S^{(1)}, \dots, S^{(4)}$ with the properties (1)–(5) do exist. For this, we use the fact (see [9]) that for every $\alpha \geq \mathfrak{c}$ there exists a system $\mathbb{S}$ of metrizable spaces of the cardinality α such that for every $Y, Z \in \mathbb{S}$ and every continuous map $f : Y \to Z$, either f is constant or $Y = Z$ and f is the identity, and the system $\mathbb{S}$ itself is large enough: for our purpose it suffices $|\mathbb{S}| = \alpha$ (although, as shown in [9], such $\mathbb{S}$ with $|\mathbb{S}| = 2^\alpha$ does exist). Let

$$\{A_q^*, D_q^*, E_q^*, M_q^*, G_q^* : q \in Q\}$$

be a subsytem of $\mathbb{S}$.

(1) Choose two distinct points, say a_q, b_q, in the space A_q^* and, in the space $A_q^* \times \omega$, identify (b_q, n) with $(a_q, n+1)$. The obtained space is A_q, then $B_q = A_q^* \coprod A_q$. Clearly, $S^{(1)} = \{A_q, B_q : q \in Q\}$ satisfies (1).

(2) Choose two distinct points, say $d_q^{(1)}, d_q^{(2)}$ in D_q^* and $e_q^{(1)}, e_q^{(2)}$ in E_q^*; in $D_q^* \times \omega$, identify $(d_q^{(2)}, n)$ with $(d_q^{(1)}, n+1)$ and denote $\widetilde{D}_q$ the obtained space; analogously define $\widetilde{E}_q$; then put

$$D_q = D_q^* \coprod \widetilde{D}_q \coprod \widetilde{E}_q$$
$$E_q = E_q^* \coprod \widetilde{D}_q \coprod \widetilde{E}_q$$

Then $S^{(2)} = \{D_q, E_q : q \in Q\}$ satisfies (2).

(3) The spaces M_q and N_q are constructed from M_q^* as A_q and B_q from A_q^* in (1). Then $S^{(3)} = \{M_q, N_q : q \in Q\}$ satisfies (3).

(4) Let K be a closed subset of the Cantor discontinuum such that K is homeomorphic to $K \coprod K \coprod K$ but not to $K \coprod K$. Such a space was constructed in [4]. Put $G_q = K \times G_q^*$ and $H_q = (K \coprod K) \times G_q^*$. Since K is zero-dimensional while G_q^* is connected and every continuous map of G_q^* into itself is either the identity or a constant, every continuous one-to-one map $f : G_q \to H_q$ sends every $K \times \{a\}$ into $(K \coprod K) \times \{a\}$ for an arbitrary $a \in G_q^*$. Since K and $K \coprod K$ are non-homeomorphic, the space G_q is not homeomorphic to H_q. However G_q is homeomorphic to a clopen subspace of H_q and vice versa.

(5) The system $S^{(1)} \cup S^{(2)} \cup S^{(3)} \cup S^{(4)}$ satisfies (5) because

$$\{A_q^*, D_q^*, E_q^*, M_q^*, N_q^*, G_q^* : q \in Q\} \subseteq \mathbb{S},$$

evidently.

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