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PUSHOUT STABILITY OF EMBEDDINGS, INJECTIVITY AND CATEGORIES OF ALGEBRAS

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ABSTRACT. In several familiar subcategories of the category $\mathcal{T}op$ of topological spaces and continuous maps, embeddings are not pushout-stable. But, an interesting feature, capturable in many categories, namely in categories $\mathcal{B}$ of topological spaces, is the following: For $\mathcal{M}$ the class of all embeddings, the subclass of all pushout-stable $\mathcal{M}$-morphisms (that is, of those $\mathcal{M}$-morphisms whose pushout along an arbitrary morphism always belongs to $\mathcal{M}$) is of the form A^{Inj} for some space A, where A^{Inj} consists of all morphisms $m : X \to Y$ such that the map $\mathrm{Hom}(m, A)$: $\mathrm{Hom}(Y, A) \to \mathrm{Hom}(X, A)$ is surjective. We study this phenomenon. We show that, under mild assumptions, the reflective hull of such a space A is the smallest $\mathcal{M}$-reflective subcategory of $\mathcal{B}$; furthermore, the opposite category of this reflective hull is equivalent to a reflective subcategory of the Eilenberg-Moore category $\mathcal{S}et^{\mathbb{T}}$, where $\mathbb{T}$ is the monad induced by the right adjoint $\mathrm{Hom}(-, A) : \mathcal{T}op^{\mathrm{op}} \to \mathcal{S}et$. We also find conditions on a category $\mathcal{B}$ under which the pushout-stable $\mathcal{M}$-morphisms are of the form $\mathcal{A}^{\mathrm{Inj}}$ for some category $\mathcal{A}$.

0. INTRODUCTION

It is well-known that in the category $\mathcal{T}op$ of topological spaces and continuous maps, the class of all embeddings is pushout-stable, that is, if we

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have a pushout in $\mathcal{T}op$,

$$\begin{tikzcd} \bullet \arrow[r, "m"] \arrow[d, "f"'] & \bullet \arrow[d, "\overline{f}"] \\ \bullet \arrow[r, "\overline{m}"] & \bullet \end{tikzcd}$$

with m an embedding, then $\overline{m}$ is also an embedding. But this is not true for several subcategories of $\mathcal{T}op$, namely, it is not true for various epireflective subcategories of $\mathcal{T}op$.

Let $\mathcal{B}$ be a subcategory* of $\mathcal{T}op$, let $\mathcal{M}$ denote the class of embeddings in $\mathcal{B}$. By $P_{\mathcal{B}}(\mathcal{M})$ we denote the class of all morphisms $m \in \mathcal{M}$ such that every pushout in $\mathcal{B}$ of m along any morphism belongs to $\mathcal{M}$. In this paper, this class of morphisms is shown to have an important rôle concerning some nice behaviours of the reflective hull of a topological space.

In several reflective subcategories of $\mathcal{T}op$, it is possible to select a subcategory $\mathcal{A}$ such that $P_{\mathcal{B}}(\mathcal{M}) = \mathcal{A}^{\mathrm{Inj}_{\mathcal{B}}}$, where $\mathcal{A}^{\mathrm{Inj}_{\mathcal{B}}}$ denotes the class of all morphisms m of $\mathcal{B}$ such that each $A \in \mathcal{A}$ is m-injective, that is, the map $\mathrm{Hom}(m, A) : \mathrm{Hom}(Y, A) \to \mathrm{Hom}(X, A)$ is surjective. Under smooth conditions, this fact implies that the reflective hull of $\mathcal{A}$ is just the smallest $\mathcal{M}$-reflective subcategory of the epireflective hull of $\mathcal{A}$. We recall that an $\mathcal{M}$-reflective subcategory is a reflective subcategory whose corresponding reflections belong to $\mathcal{M}$. The most interesting case, which is rather common, is when $\mathcal{A}$ is chosen as one-object subcategory. In fact, if A is a topological space for which $P_{\mathcal{B}}(\mathcal{M})$ consists of all morphisms m such that A is m-injective, then, it often holds that the dual category of the reflective hull of A in $\mathcal{T}op$ is a reflective subcategory of $\mathcal{S}et^{\mathbb{T}}$ for $\mathbb{T}$ the monad induced by $\mathrm{Hom}(-, A)$. Furthermore, under convenient requisites, it is equivalent to $\mathcal{S}et^{\mathbb{T}}$.

As very helpful tools, two Dikranjan-Giuli closure operators will be used: the regular closure operator (introduced in [8, 2]) and the orthogonal closure operator (introduced in [10]). We shall see that the equality $P_{\mathcal{B}}(\mathcal{M}) = \mathcal{A}^{\mathrm{Inj}_{\mathcal{B}}}$, combined with mild conditions, implies that the two closure operators coincide in $P_{\mathcal{B}}(\mathcal{M})$, and this plays an important rôle in the main result of this paper, Theorem 3.1.

1. Injectivity and pushout-stability

Let A be a topological space and let $f : X \to Y$ be a morphism in $\mathcal{T}op$. We say that A is *f-injective* (respectively, *orthogonal to f*) whenever the map $\mathrm{Hom}(f, A) : \mathrm{Hom}(Y, A) \to \mathrm{Hom}(X, A)$ is surjective (respectively, bijective). For $\mathcal{A}$ a class of spaces, that is, a full subcategory of $\mathcal{T}op$, we denote by $\mathcal{A}^{\mathrm{Inj}}$ the class of those morphisms f such that all objects of $\mathcal{A}$ are f-injective. In case $\mathcal{A}$ has just one object A, we write A^{Inj}. Analogously, $\mathcal{A}^{\perp}$ is the class of

*Along this paper we assume that all subcategories are full.

morphisms f such that any object of $\mathcal{A}$ is orthogonal to f. Given a class $\mathcal{N}$ of morphisms, the subcategory of n-injective objects (respectively, objects orthogonal to n), for all $n \in \mathcal{N}$, is named $\mathcal{N}_{\mathrm{Inj}}$ (respectively, $\mathcal{N}_{\perp}$).

Let $\mathcal{B}$ be a subcategory of $\mathcal{T}op$. We denote by $\mathcal{A}^{\mathrm{Inj}_{\mathcal{B}}}$ the class $\mathcal{A}^{\mathrm{Inj}}$ restricted to $\mathcal{B}$-morphisms. $P_{\mathcal{B}}(\mathcal{M})$ designates the class of all $\mathcal{M}$-morphisms whose pushout in $\mathcal{B}$ along any morphism exists and belongs to $\mathcal{M}$.

Throughout this paper, unless something is said on the contrary, $\mathcal{M}$ denotes the class of embeddings of $\mathcal{T}op$.

An interesting feature, capturable in various reflective subcategories $\mathcal{B}$ of $\mathcal{T}op$, is that the class $P_{\mathcal{B}}(\mathcal{M})$ coincides with $\mathcal{A}^{\mathrm{Inj}_{\mathcal{B}}}$ for some subcategory $\mathcal{A}$, in many cases this subcategory $\mathcal{A}$ having just one object. Next, some examples of this occurrence are given.

Examples 1.1.

(1) Since in $\mathcal{T}op$ embeddings are pushout-stable, we have the equality $P_{\mathcal{T}op}(\mathcal{M}) = \mathcal{M}$. Moreover, $\mathcal{M} = A^{\mathrm{Inj}}$ for A the topological space $\{0, 1, 2\}$ whose only non trivial open is $\{0\}$.
(2) For the subcategory $\mathcal{T}op_0$ of T_0-spaces, again $P_{\mathcal{T}op_0}(\mathcal{M}) = \mathcal{M}$. In this case, it holds $\mathcal{M} = S^{\mathrm{Inj}_{\mathcal{T}op_0}}$, where S is the Sierpiński space.
(3) For $\mathcal{B}$, the subcategory of 0-dimensional spaces, $\mathcal{M} \neq P_{\mathcal{B}}(\mathcal{M})$, but again $P_{\mathcal{B}}(\mathcal{M}) = A^{\mathrm{Inj}_{\mathcal{B}}}$, where A is the space $\{0, 1, 2\}$ whose topology has as only non trivial opens $\{0\}$ and $\{1, 2\}$. The morphisms of $P_{\mathcal{B}}(\mathcal{M})$ are just those embeddings $m : X \to Y$ such that for each clopen set G of X there is some clopen H in Y such that $G = m^{-1}(H)$ (cf. [11]).
(4) If $\mathcal{B}$ is the subcategory of 0-dimensional Hausdorff spaces, $\mathcal{M} \neq P_{\mathcal{B}}(\mathcal{M}) = D^{\mathrm{Inj}_{\mathcal{B}}}$, where D is the space $\{0, 1\}$ with the discrete topology. Analogously to the above example, the morphisms of $P_{\mathcal{B}}(\mathcal{M})$ are those embeddings $m : X \to Y$ in $\mathcal{B}$ such that each clopen in X is the inverse image by m of some clopen in Y (cf. [11]).
(5) For $\mathcal{T}ych$ the subcategory of Tychonoff spaces, $\mathcal{M} \neq P_{\mathcal{T}ych}(\mathcal{M}) = I^{\mathrm{Inj}_{\mathcal{T}ych}}$, where I is the unit interval. The $P_{\mathcal{T}ych}(\mathcal{M})$-morphisms are just the C^*-embeddings (cf. [11]).
(6) In the category $\mathcal{I}nd$ of indiscrete spaces embeddings are pushout-stable, and we have that $P_{\mathcal{I}nd}(\mathcal{M}) = \mathcal{M} = \{0, 1\}^{\mathrm{Inj}_{\mathcal{I}nd}}$.

Examples 1.2. This phenomenon arises in several other categories, not just reflective subcategories of $\mathcal{T}op$. Some examples are given in the following.

(1) For $\mathcal{A}b$ the category of abelian groups and homomorphisms of group, and $\mathcal{M}$ the class of all monomorphisms, $P(\mathcal{M}) = \mathcal{M} = \mathcal{A}^{\mathrm{Inj}}$, where $\mathcal{A}$ is the subcategory of divisible abelian groups. Similarly, in the category of torsion-free abelian groups and homomorphisms of group, and $\mathcal{M}$ the class of all monomorphisms, $P(\mathcal{M}) = \mathcal{M} = \mathcal{A}^{\mathrm{Inj}}$, where $\mathcal{A}$ is the subcategory of divisible groups.

(2) For $\mathcal{V}ec_{\mathbb{K}}$ the category of linear spaces and linear maps over $\mathbb{K}$, and $\mathcal{M}$ the class of all monomorphisms, $P(\mathcal{M}) = \mathcal{M} = \mathbb{K}^{\mathrm{Inj}}$.

(3) Recall that a *separated quasi-metric space* is a pair (X, d), where X is a set and d is a function with domain $X \times X$ which satisfies the same axioms as a metric in X, but d may take the value ∞. A separated quasi-metric space is said to be *complete* if all its Cauchy sequences converge. A function f between two separated quasi-metric spaces (X, d) and (Y, e) is *non-expansive* provided that $e(f(x), f(y)) \leq d(x, y)$ for all $(x, y) \in X \times X$. Let $\mathcal{SM}et$ denote the category of separated quasi-metric spaces and non-expansive maps and let $\mathcal{CSM}et$ denote its subcategory of complete separated quasi-metric spaces. Then, for $\mathcal{M}$ the class of all non-expansive monomorphisms, $\mathcal{M}$ is pushout-stable, so it coincides with $P(\mathcal{M})$, and it holds $\mathcal{M} = \mathcal{CSM}et^{\mathrm{Inj}}$.

(4) For the category $\mathcal{M}et$ of metric spaces and non-expansive maps and $\mathcal{M}$ defined similarly as in 3, we have that

$$P(\mathcal{M}) = \{m \in \mathcal{M} |\text{ the pushout of } m \text{ along any morphism exists}\}$$

and it coincides with $\mathcal{CM}et^{\mathrm{Inj}}$, where $\mathcal{CM}et$ denotes the subcategory of complete metric spaces. An analogous situation holds for the category of normed spaces and non-expansive maps and the subcategory of Banach spaces.

One first question which comes from these examples is the following: When is $P_{\mathcal{B}}(\mathcal{M})$ of the form $\mathcal{A}^{\mathrm{Inj}_{\mathcal{B}}}$ for some subcategory $\mathcal{A}$ of a given category, in particular, for $\mathcal{A}$ a subcategory of $\mathcal{T}op$? The next proposition is a partial answer for this.

Let us recall that, if $\mathcal{N}$ is a class of morphisms in a category $\mathcal{B}$, containing all isomorphisms and being closed under composition with isomorphisms, we say that $\mathcal{B}$ *has enough* $\mathcal{N}$*-injectives* provided that, for each $B \in \mathcal{B}$, there is an $\mathcal{N}$-morphism $n : B \to A$ with the codomain A in $\mathcal{N}_{\mathrm{Inj}}$. A morphism $n : B \to A$ of $\mathcal{N}$ is said to be $\mathcal{N}$*-essential* whenever any composition $k \cdot n$ belongs to $\mathcal{N}$ only if $k \in \mathcal{N}$. The category $\mathcal{B}$ is said to have $\mathcal{N}$*-injective hulls* if, for each $B \in \mathcal{B}$, there is some $\mathcal{N}$-essential morphism $n : B \to A$ with A $\mathcal{N}$-injective.

The following easy lemma to prove will be useful in this paper.

Lemma 1.1. *For any epireflective subcategory $\mathcal{B}$ of $\mathcal{T}op$, the class $P_{\mathcal{B}}(\mathcal{M})$ has the following properties:*

(i) *It is closed under composition.*

(ii) *If $m \cdot n \in P_{\mathcal{B}}(\mathcal{M})$ then $n \in P_{\mathcal{B}}(\mathcal{M})$.*

Proposition 1.1. *Let $\mathcal{B}$ be an epireflective subcategory of $\mathcal{T}op$.*

(1) *If $\mathcal{B}$ has enough $P_{\mathcal{B}}(\mathcal{M})$-injectives, then $P_{\mathcal{B}}(\mathcal{M}) = \mathcal{A}^{Inj_{\mathcal{B}}}$ for some subcategory $\mathcal{A}$ of $\mathcal{B}$.*

(2) *If $P_{\mathcal{B}}(\mathcal{M}) = \mathcal{A}^{Inj_{\mathcal{B}}}$ for some $\mathcal{M}$-reflective subcategory $\mathcal{A}$ of $\mathcal{B}$, then $\mathcal{B}$ has $P_{\mathcal{B}}(\mathcal{M})$-injective hulls, and $[P_{\mathcal{B}}(\mathcal{M})]_{Inj_{\mathcal{B}}} = \mathcal{A}$.*

Proof. The fact that $\mathcal{B}$ is an epireflective subcategory of $\mathcal{T}op$ ensures that $\mathcal{B}$ has pushouts and an (Epi, Embedding)-factorization system for morphisms.

(1). Let $\mathcal{A} = [P_{\mathcal{B}}(\mathcal{M})]_{\mathrm{Inj}_{\mathcal{B}}}$; it is clear that $P_{\mathcal{B}}(\mathcal{M}) \subseteq \mathcal{A}^{\mathrm{Inj}_{\mathcal{B}}}$. In order to show the converse inclusion, let $f : X \to Y$ belong to $\mathcal{A}^{\mathrm{Inj}_{\mathcal{B}}}$ and let $f = me$ be the (Epi, Embedding)-factorization of f. By hypothesis, there is a $P_{\mathcal{B}}(\mathcal{M})$-morphism $n : X \to A$ with $A \in [P_{\mathcal{B}}(\mathcal{M})]_{\mathrm{Inj}_{\mathcal{B}}}$. Let $\overline{n} : Y \to A$ be such that $\overline{n}f = n$; then the equality $n1_X = (\overline{n}m)e$ guarantees the existensce of a unique morphism t such that $te = 1_X$ and, consequently, e is an isomorphism and $f \in \mathcal{M}$. Now, to conclude that $f : X \to Y$ belongs to $P_{\mathcal{B}}(\mathcal{M})$, let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "\overline{g}"] \\ Z \arrow[r, "\overline{f}"] & W \end{tikzcd}$$

be a pushout in $\mathcal{B}$. Since $\mathcal{B}$ has enough $P_{\mathcal{B}}(\mathcal{M})$-injectives, there is some $P_{\mathcal{B}}(\mathcal{M})$-morphism $p : Z \to A$, with $A \in \mathcal{B}$ being $P_{\mathcal{B}}(\mathcal{M})$-injective. Let $h : Y \to A$ be such that $hf = pg$; then, there is t such that $t\overline{f} = p$ and so the fact that $p \in \mathcal{M}$ implies that also $\overline{f} \in \mathcal{M}$.

(2). Let us first show that each reflection $r_B : B \to RB$ of $B \in \mathcal{B}$ in $\mathcal{A}$ belongs to $P_{\mathcal{B}}(\mathcal{M})$. Let $(\overline{f}, \overline{r})$ be the pushout of (r_B, f) for some morphism $f : B \to C$. Since $Rf \cdot r_B = r_C \cdot f$, there is a unique t such that $t \cdot \overline{r} = r_c$ and $t \cdot \overline{f} = Rf$. By hypothesis, $r_C \in \mathcal{M}$, and this implies that $\overline{r} \in \mathcal{M}$. Then $r_B \in P_{\mathcal{B}}(\mathcal{M})$.

To show that r_B is $P_{\mathcal{B}}(\mathcal{M})$-essential, let $k : RB \to K$ be a morphism such that $k \cdot r_B$ belongs to $P_{\mathcal{B}}(\mathcal{M})$. It is easy to see that the fact that $\mathcal{A}$ is $\mathcal{M}$-reflective in $\mathcal{B}$ implies that each reflection in $\mathcal{A}$ is an epimorphism in $\mathcal{B}$. Consequently, for any morphism $f : RB \to D$, $(1_D, f)$ is a pushout of $(r_B, f \cdot r_B)$. Now, let $f : RB \to D$ be a morphism and let $(\overline{k}, \overline{f})$ be the pushout of (k, f). Then, since $(1_D, f)$ is a pushout of $(r_B, f \cdot r_B)$, $\overline{k}$ is the pushout of $k \cdot r_B$ along $f \cdot r_B$. Thus, $\overline{k}$ belongs to $\mathcal{M}$ and, therefore, $k \in P_{\mathcal{B}}(\mathcal{M})$.

In order to conclude that $[P_{\mathcal{B}}(\mathcal{M})]_{\mathrm{Inj}_{\mathcal{B}}} = \mathcal{A}$, since the inclusion $\mathcal{A} \subseteq [P_{\mathcal{B}}(\mathcal{M})]_{\mathrm{Inj}_{\mathcal{B}}}$ is clear, it remains to show the reverse inclusion. Let $B \in [P_{\mathcal{B}}(\mathcal{M})]_{\mathrm{Inj}_{\mathcal{B}}}$; the fact that the reflection $r_B : B \to RB$ belongs to $P_{\mathcal{B}}(\mathcal{M})$ implies that there is some $t : RB \to B$ such that $t \cdot r_B = 1_B$, and this implies that r_B is an isomorphism. $\square$

Given a subcategory $\mathcal{A}$ of $\mathcal{T}op$, let $\mathcal{R}(\mathcal{A})$ denote the reflective hull of $\mathcal{A}$ in $\mathcal{T}op$, provided it exists. It is well known that, when $\mathcal{A}$ is small, $\mathcal{R}(\mathcal{A})$ exists and coincides with the limit closure of $\mathcal{A}$.

The subcategories $\mathcal{A}$ such that $P_{\mathcal{B}}(\mathcal{M}) = \mathcal{A}^{\mathrm{Inj}_{\mathcal{B}}}$, for $\mathcal{B}$ its epireflective hull, have a very special property, as established next.

Proposition 1.2. *Let $\mathcal{B}$ be the epireflective hull of $\mathcal{A}$ in $\mathcal{T}op$. If $P_{\mathcal{B}}(\mathcal{M}) = \mathcal{A}^{Inj_{\mathcal{B}}}$, then $\mathcal{R}(\mathcal{A})$ is the smallest $\mathcal{M}$-reflective subcategory of $\mathcal{B}$ (provided $\mathcal{R}(\mathcal{A})$ exists).*

Proof. Since $\mathcal{B}$ is the epireflective hull of $\mathcal{A}$, it is clear that $\mathcal{R}(\mathcal{A})$ is $\mathcal{M}$-reflective in $\mathcal{B}$ (cf. [1]). Let $\mathcal{C}$ be another $\mathcal{M}$-reflective subcategory of $\mathcal{B}$. Then, as shown in the proof of 2 of Proposition 1.1, each reflection $s_B : B \to SB$ of $B \in \mathcal{B}$ into $\mathcal{C}$ belongs to $P_{\mathcal{B}}(\mathcal{M})$. Therefore, given $B \in \mathcal{A}$, since $\mathcal{A} \subseteq [P_{\mathcal{B}}(\mathcal{M})]_{\mathrm{Inj}_{\mathcal{B}}}$, there is some $t : SB \to B$ such that $t \cdot s_B = 1_B$, and so s_B is an isomorphism. Consequently, $\mathcal{A} \subseteq \mathcal{C}$ and, thus, $\mathcal{R}(\mathcal{A}) \subseteq \mathcal{C}$. □

2. On closure operators

In this section we recall some notions concerning closure operators, in the sense of Dikranjan-Giuli ([2, 3, 4]), and obtain some results, which are going to be useful in the next section.

Let $\mathcal{N}$ be a class of monomorphisms in a category $\mathcal{X}$, which contains all isomorphisms, is closed under composition and is left-cancellable, (i.e., if $m \cdot n$, $m \in \mathcal{N}$ then $n \in \mathcal{N}$). Considering $\mathcal{N}$ as a subcategory of the category $\mathcal{X}^2$ (of all morphisms of $\mathcal{X}$), let $u : \mathcal{N} \to \mathcal{X}$ be the codomain functor, that is, the functor which, to each morphism $(r, s) : (m : X \to Y) \longrightarrow (n : Z \to W)$ of $\mathcal{N}$, assigns the morphism $s : Y \to W$ of $\mathcal{X}$.

A *closure operator* in $\mathcal{X}$ with respect to $\mathcal{N}$ consists of a functor $c : \mathcal{N} \to \mathcal{N}$ such that $u \cdot c = u$ and of a natural transformation $\delta : Id_{\mathcal{N}} \to c$ such that $u \cdot \delta = Id_u$. Therefore, a closure operator determines, for each $m : X \to Y$ in $\mathcal{N}$, morphisms $c(m)$ and $d(m)$ and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "d(m)"] \arrow[d, "m"'] & \overline{X} \arrow[d, "c(m)"] \\ Y \arrow[r, "1_Y"] & Y \end{tikzcd}$$

where $\delta_m = (d(m), 1_Y) : m \to c(m)$.

We recall that, if $\mathcal{N}$ is part of an $(\mathcal{E}, \mathcal{N})$-factorization system for morphisms, then a closure operator $c : \mathcal{N} \to \mathcal{N}$ may be equivalently described by a family of functions

$$(c_X : \mathcal{N}_X \to \mathcal{N}_X)_{X \in \mathcal{X}},$$

where $c_X(m) = c(m)$ for each m, and $\mathcal{N}_X$ denotes the class of all $\mathcal{N}$-morphisms with codomain in X, fulfilling the conditions:

(1) $m \le c_X(m)$, $m \in \mathcal{N}_X$;
(2) if $m \le n$, then $c_X(m) \le c_X(n)$, $m, n \in \mathcal{N}_X$; and
(3) $c_X(f^{-1}(m)) \le f^{-1}(c_Y(m))$, for each morphism $f : X \to Y$ and $m \in \mathcal{N}_Y$.

If c is a closure operator in $\mathcal{X}$ with respect to $\mathcal{N}$, then a morphism $(n : X \to Y) \in \mathcal{N}$ is said to be *c-dense* if $c(n) \cong 1_Y$. It is said to be *c-closed* when $c(n) \cong n$. We say that c is *weakly hereditary*, if $d(n)$ is c-dense for any $n \in \mathcal{N}$, and *idempotent* if $c(n)$ is c-closed for any $n \in \mathcal{N}$. An object $X \in \mathcal{X}$ is said to be *absolutely c-closed* whenever each $\mathcal{N}$-morphism with domain in X is c-closed.

The regular closure operator. Let $\mathcal{A}$ be a reflective subcategory of $\mathcal{T}op$. The *regular closure operator* induced by $\mathcal{A}$, which will be denoted by $c^{\mathrm{reg}}_{\mathcal{A}} : \mathcal{M} \to \mathcal{M}$ (just c^{reg}, without the index $\mathcal{A}$, when there is no confusion about the subcategory $\mathcal{A}$ which is involved), assigns, to each $m : X \to Y$ in $\mathcal{M}$, the equalizer $\mathrm{eq}(u \cdot r_W, v \cdot r_W)$ where $u, v : Y \to W$ form the cokernel pair of m and r_W is a reflection in $\mathcal{A}$. The regular closure operator, which is an idempotent closure operator, in the sense described above, has been widely studied, not only in the topological context, but in a more general setting (see [2]).

It is easy to conclude that if $\mathcal{B}$ is the epireflective hull of $\mathcal{A}$ in $\mathcal{T}op$ (or, equivalently, $\mathcal{A}$ is $\mathcal{M}$-reflective in $\mathcal{B}$), then $c^{\mathrm{reg}}_{\mathcal{A}} = c^{\mathrm{reg}}_{\mathcal{B}}$.

From now on, given a subcategory $\mathcal{A}$ of $\mathcal{T}op$, $\mathcal{E}(\mathcal{A})$ always denotes the epireflective hull of $\mathcal{A}$ in $\mathcal{T}op$.

The orthogonal closure operator. Let $\mathcal{A}$ be a reflective subcategory in its epireflective hull $\mathcal{E}(\mathcal{A})$ in $\mathcal{T}op$. Let $r_X : X \to RX$ be the reflection of X in $\mathcal{A}$. To each $m : X \to Y$ in $P_{\mathcal{E}(\mathcal{A})}(\mathcal{M})$, we assign the $\mathcal{M}$-morphism $c^{\mathrm{ort}}_Y(m) : \overline{X} \to Y$ which is the pullback of the pushout, in $\mathcal{E}(\mathcal{A})$, of m along r_X, according to the next diagram.

$$\begin{tikzcd}
X \arrow[rr, "m"] \arrow[dd, "r_X"'] & & Y \arrow[dd, "\overline{r}"] \\
& \overline{X} \arrow[ur, "c^{\mathrm{ort}}_Y(m)"] \arrow[dl] & \\
RX \arrow[rr, "\overline{m}"] & & \bullet
\end{tikzcd}$$

Using results of [10], we get the next theorem.

Theorem 2.1. *For every reflective subcategory $\mathcal{A}$ of $\mathcal{T}op$, if $c^{ort}_Y(m) : \overline{X} \to Y$ belongs to $P_{\mathcal{E}(\mathcal{A})}(\mathcal{M})$ for all $m \in P_{\mathcal{E}(\mathcal{A})}(\mathcal{M})$, then*

$$c^{ort}_{\mathcal{A}} : P_{\mathcal{E}(\mathcal{A})}(\mathcal{M}) \longrightarrow P_{\mathcal{E}(\mathcal{A})}(\mathcal{M})$$

defines a weakly hereditary idempotent closure operator in $\mathcal{E}(\mathcal{A})$, and $\mathcal{A}$ is just the subcategory of all absolutely $c^{ort}_{\mathcal{A}}$-closed objects.

As expressed for $c_{\mathcal{A}}^{\mathrm{reg}}$ before, we remove $\mathcal{A}$ from $c_{\mathcal{A}}^{\mathrm{ort}}$ if there is no reason for ambiguity.

We are going to make use of the following definition.

Definition 2.1. We say that epimorphisms in a category $\mathcal{B}$ are *left-cancellable with respect to* $\mathcal{M}$ provided that $q \in \mathrm{Epi}(\mathcal{B})$ whenever $p \cdot q \in \mathrm{Epi}(\mathcal{B})$ with $p, q \in \mathcal{M}$.

Corollary 2.1. *If $\mathcal{A}$ is a reflective subcategory of $\mathcal{T}op$ such that, in $\mathcal{E}(\mathcal{A})$, epimorphisms are left-cancellable with respect to $\mathcal{M}$, then*

$$c_{\mathcal{A}}^{ort} : P_{\mathcal{E}(\mathcal{A})}(\mathcal{M}) \longrightarrow P_{\mathcal{E}(\mathcal{A})}(\mathcal{M})$$

defines a weakly hereditary idempotent closure operator and $\mathcal{A}$ is just the subcategory of all absolutely $c_{\mathcal{A}}^{ort}$-closed objects.

Proof. It follows from the above proposition and the fact that the left-cancellability of epimorphisms in $\mathcal{E}(\mathcal{A})$ with respect to $\mathcal{M}$ implies that $c_Y^{\mathrm{ort}}(m) : \overline{X} \to Y$ belongs to $P_{\mathcal{E}(\mathcal{A})}(\mathcal{M})$ for all $m \in P_{\mathcal{E}(A)}(\mathcal{M})$, as it is shown in Remark 4.4 of [10]. $\square$

Lemma 2.1. *In any reflective subcategory of $\mathcal{T}op$ where embeddings are pushout-stable, epimorphisms are left-cancellable with respect to embeddings.*

Proof. First, we point out the following fact: In any category with pushouts, a morphism f is an epimorphism iff (f, f) has a pushout of the form (f', f') for some morphism f'; in this case, f' is an isomorphism.

Let $\mathcal{B}$ be a reflective subcategory of $\mathcal{T}op$ and let $m : X \to Y$ and $n : Y \to Z$ be $\mathcal{M}$-morphisms of $\mathcal{B}$ such that nm is an epimorphism. Consider the following diagram, where the smallest four squares are pushouts in $\mathcal{B}$.

$$\begin{tikzcd}
X \arrow[r, "m"] \arrow[d, "m"'] & Y \arrow[r, "n"] \arrow[d, "u"] & Z \arrow[d, "a"] \\
Y \arrow[r, "v"] \arrow[d, "n"'] & \bullet \arrow[r, "r"] \arrow[d, "s"] & \bullet \arrow[d, "b"] \\
Z \arrow[r, "c"] & \bullet \arrow[r, "d"] & \bullet
\end{tikzcd}$$

Then $b \cdot a = d \cdot c$ and we have that

$$(br)u = ban = dcn = dsv = (br)v.$$

Since br is the composition of two embeddings, it is an embedding, and thus $u = v$; consequently, m is an epimorphism. $\square$

Remark 2.1. The above lemma provides several examples of subcategories of $\mathcal{T}op$ where epimorphisms are left-cancellable with respect to embeddings. This occurrence is far from to confine to subcategories under the condition of Lemma 2.1. In fact, it is a very common property in epireflective subcategories of $\mathcal{T}op$. This is the case, for instance, for T_0-spaces, T_1-spaces, Hausdorff spaces, completely regular Hausdorff spaces, compact Hausdorff spaces, compact 0-dimensional Hausdorff spaces. The left-cancellability of epimorphisms with respect to embeddings, in these categories, is easy to conclude, taking into account that in each one of them epimorphisms are the surjective continuous maps or the dense continuous maps (cf. [4, 7]).

We are going to make use of the following lemma.

Lemma 2.2 (cf. [10]). *Let $\mathcal{A}$ be a subcategory of $\mathcal{T}op$. Then, in $\mathcal{E}(\mathcal{A})$,*

$$\begin{aligned} \mathcal{A}^{\perp} &= \left(\left(\mathcal{A}^{\perp}\right)_{\perp}\right)^{\perp} \\ &= \{c_{\mathcal{A}}^{ort}\text{-dense } P_{\mathcal{E}(\mathcal{A})}\text{-morphisms}\} \\ &\subseteq P_{\mathcal{E}(\mathcal{A})}(\mathcal{M}) \cap Epi(\mathcal{E}(\mathcal{A})). \end{aligned}$$

We point out that, although the existence of the reflective hull of any subcategory of $\mathcal{T}op$ cannot be guaranteed ([12]), we can state the following:

Proposition 2.1. *Let $\mathcal{A}$ be a subcategory of $\mathcal{T}op$. If $\mathcal{R}(\mathcal{A})$ exists, then it coincides with $\left(\mathcal{A}^{\perp}\right)_{\perp}$.*

Proof. From Proposition 3.1.2 and Theorem 4.1.3 of [5], one derives that if $\mathcal{X}$ is a category fulfilling the following conditions:

- it is complete, cocomplete and co-wellpowered;
- it has an $(\mathcal{E}, \mathcal{M})$-factorization system for morphisms with $\mathcal{E} = \mathrm{Epi}(\mathcal{X})$;
- it has a generator;
- for every countable family $(m_i : C_i \to B)_{i\in\omega}$ of $\mathcal{M}$-subobjects of $B \in \mathcal{X}$, and any morphism g with codomain in B, it holds the equality $\bigvee_{i\in\omega} g^{-1}(m_i) = g^{-1}(\bigvee_{i\in\omega} m_i)$;

then, for each morphism f of $\mathcal{X}$, the subcategory $\{f\}_{\perp}$ is reflective. The category $\mathcal{T}op$ is under these conditions for $\mathcal{M}$ the class of all embeddings. Consequently, given a subcategory $\mathcal{A}$ of $\mathcal{T}op$ which has a reflective hull, we have the inclusions:

$$\begin{aligned} \mathcal{R}(\mathcal{A}) &\subseteq \textstyle\bigcap_{f\in\mathcal{A}^{\perp}}\{f\}_{\perp} \\ &= \left(\textstyle\bigcup_{f\in\mathcal{A}^{\perp}}\{f\}\right)_{\perp} \\ &= \left(\mathcal{A}^{\perp}\right)_{\perp}. \end{aligned}$$

Therefore, since the inclusion $\left(\mathcal{A}^{\perp}\right)_{\perp} \subseteq \mathcal{R}(\mathcal{A})$ always occurs, we conclude that $\mathcal{R}(\mathcal{A}) = \left(\mathcal{A}^{\perp}\right)_{\perp}$. $\square$

The next proposition establishes conditions under which the closure operators $c_{\mathcal{A}}^{\mathrm{ort}}$ and $c_{\mathcal{A}}^{\mathrm{reg}}$ coincide in $P_{\mathcal{E}(\mathcal{A})}(\mathcal{M})$. This result is going to be very useful in the proof of the theorem of the next section.

Proposition 2.2. *Let $\mathcal{A}$ be a subcategory of $\mathcal{T}op$ with reflective hull. If epimorphisms in $\mathcal{E}(\mathcal{A})$ are left-cancellable with respect to $\mathcal{M}$, $c_{\mathcal{A}}^{reg}$ is weakly hereditary in $\mathcal{E}(\mathcal{A})$ and $P_{\mathcal{E}(\mathcal{A})}(\mathcal{M}) = \mathcal{A}^{Inj_{\mathcal{E}(\mathcal{A})}}$, then*

$$c_{\mathcal{R}(\mathcal{A})}^{reg} = c_{\mathcal{R}(\mathcal{A})}^{ort} \text{ in } P_{\mathcal{E}(\mathcal{A})}(\mathcal{M}).$$

Proof. In order to simplify the writing, let $\mathcal{A}^{\mathrm{Inj}}$ and $\mathcal{A}^{\perp}$ refer to just morphisms in $\mathcal{E}(\mathcal{A})$. It is easy to conclude that the fact that $\mathcal{E}(\mathcal{A})$ is the epireflective hull of $\mathcal{A}$ implies that

$$\mathcal{A}^{\perp} \subseteq P_{\mathcal{E}(\mathcal{A})}(\mathcal{M}) \bigcap \mathrm{Epi}(\mathcal{E}(\mathcal{A}))$$

(see Lemma 2.2). On the other hand, it is clear that

$$\mathcal{A}^{\mathrm{Inj}} \bigcap \mathrm{Epi}(\mathcal{E}(\mathcal{A})) \subseteq \mathcal{A}^{\perp}.$$

Therefore, we have that

$$\begin{aligned} \mathcal{A}^{\perp} &\subseteq P_{\mathcal{E}(\mathcal{A})}(\mathcal{M}) \bigcap \mathrm{Epi}(\mathcal{E}(\mathcal{A})) \\ &= \mathcal{A}^{\mathrm{Inj}} \bigcap \mathrm{Epi}(\mathcal{E}(\mathcal{A})) \\ &\subseteq \mathcal{A}^{\perp}, \end{aligned}$$

and so,

$$\mathcal{A}^{\perp} = P_{\mathcal{E}(\mathcal{A})}(\mathcal{M}) \bigcap \mathrm{Epi}(\mathcal{E}(\mathcal{A})). \tag{1}$$

Furthermore, from Lemma 2.2 and Proposition 2.1,

$$\begin{aligned} \mathcal{A}^{\perp} &= \left((\mathcal{A}^{\perp})_{\perp}\right)^{\perp} \\ &= (\mathcal{R}(\mathcal{A}))^{\perp} \\ &= \{c_{\mathcal{R}(\mathcal{A})}^{\mathrm{ort}}\text{-dense } P_{\mathcal{E}(\mathcal{A})}\text{-morphisms}\}. \end{aligned} \tag{2}$$

It is well-known that the $c_{\mathcal{R}(\mathcal{A})}^{\mathrm{reg}}$-dense morphisms are just the epimorphisms of $\mathcal{E}(\mathcal{A})$ (see, for instance, [4]). Consequently, using (1) and (2), we get that a morphism of $P_{\mathcal{E}(\mathcal{A})}(\mathcal{M})$ is $c_{\mathcal{R}(\mathcal{A})}^{\mathrm{reg}}$-dense iff it is $c_{\mathcal{R}(\mathcal{A})}^{\mathrm{ort}}$-dense.

Let now m be a morphism in $P_{\mathcal{E}(\mathcal{A})}(\mathcal{M})$. Since the orthogonal closure operator is always smaller than the regular one (cf. [10]), there is some morphism d such that $c_{\mathcal{R}(\mathcal{A})}^{\mathrm{ort}}(m) = c_{\mathcal{R}(\mathcal{A})}^{\mathrm{reg}}(m) \cdot d$. Let d' be the morphism such that $m = c_{\mathcal{R}(\mathcal{A})}^{\mathrm{ort}}(m) \cdot d'$. Since $c_{\mathcal{R}(\mathcal{A})}^{\mathrm{reg}}$ is weakly hereditary, the morphism $d \cdot d'$ is $c_{\mathcal{R}(\mathcal{A})}^{\mathrm{reg}}$-dense, and, thus, also $c_{\mathcal{R}(\mathcal{A})}^{\mathrm{ort}}$-dense. Since $c_{\mathcal{R}(\mathcal{A})}^{\mathrm{ort}}$ is a weakly hereditary idempotent closure operator, it follows that it has a ($c_{\mathcal{R}(\mathcal{A})}^{\mathrm{reg}}$-dense, $c_{\mathcal{R}(\mathcal{A})}^{\mathrm{reg}}$-closed)-factorization system with respect to $P_{\mathcal{E}(\mathcal{A})}(\mathcal{M})$ ([3]) and so, the equality

$$c_{\mathcal{R}(\mathcal{A})}^{\mathrm{reg}}(m) \cdot (d \cdot d') = c_{\mathcal{R}(\mathcal{A})}^{\mathrm{ort}}(m) \cdot d' \cdot 1_X$$

implies the existence of a morphism t such that $t \cdot d \cdot d' = d'$. Consequently, d is an isomorphism and $c_{\mathcal{R}(\mathcal{A})}^{\mathrm{reg}}(m) \cong c_{\mathcal{R}(\mathcal{A})}^{\mathrm{ort}}(m)$. $\square$

3. Injectivity and categories of algebras

Let A, $\mathcal{R}(A)$ and $\mathcal{E}(A)$ be a topological space, the reflective hull of A and the epireflective hull of A in $\mathcal{T}op$, respectively. It is well-known that the functor $U = \mathrm{Hom}(-,A) : \mathcal{T}op^{\mathrm{op}} \to \mathcal{S}et$ is a right adjoint whose left adjoint is given by the power functor A^{-}. Let K' be the corresponding comparison functor and let R be the reflection functor from $\mathcal{T}op$ to $\mathcal{R}(A)$. It is known that the restriction of U to $\mathcal{R}(A)$ is also a right adjoint and induces the same monad as U. Furthermore, if K is the comparison functor concerning to the last adjunction, then, up to isomorphism, $K' = K \cdot R^{\mathrm{op}}$, as illustrated by the following diagram.

$$\begin{tikzcd}
\mathcal{T}op^{\mathrm{op}} \arrow[r, "R^{\mathrm{op}}"'] \arrow[dr, "U = \mathrm{Hom}(-,A)"'] & \mathcal{R}(A)^{\mathrm{op}} \arrow[r, "K"'] \arrow[d] & \mathcal{S}et^{\mathbb{T}} \arrow[dl, "U^{\mathbb{T}}"] \\
& \mathcal{S}et &
\end{tikzcd}$$

We are going to show that, under a convenient injectivity assumption on A, $\mathcal{R}(A)^{\mathrm{op}}$ is a reflective subcategory of $\mathcal{S}et^{\mathbb{T}}$. We also give conditions under which K is an equivalence.

We are going to make use of the following result:

Split Monadicity Theorem under presence of coequalizers (G. Janelidze, private communication). *Let $\mathcal{A}$ and $\mathcal{X}$ be categories with coequalizers. A functor $U : \mathcal{A} \to \mathcal{X}$ is monadic if the following conditions hold:*

(a) *U has a left adjoint F;*
(b) *U reflects isomorphisms;*
(c) *the counit $FU \to 1_{\mathcal{A}}$ is a split epimorphism.*

Theorem 3.1. *Let A be a topological space such that epimorphisms in $\mathcal{E}(A)$ are left-cancellable with respect to $\mathcal{M}$ and $c^{reg}_{\mathcal{R}(A)}$ is weakly hereditary in $\mathcal{E}(A)$. Then, if $P_{\mathcal{E}(A)}(\mathcal{M}) = A^{Inj_{\mathcal{E}(A)}}$, the comparison functor K is full and faithful and $\mathcal{R}(A)^{op}$ is equivalent to a reflective subcategory of the Eilenberg-Moore category $\mathcal{S}et^{\mathbb{T}}$.*

Furthermore, if $P_{\mathcal{E}(A)}(\mathcal{M}) = (\mathcal{R}(A))^{Inj_{\mathcal{E}(A)}}$, then K is an equivalence and so $\mathcal{R}(A)^{op}$ is monadic.

Proof. Under the stated conditions, Proposition 2.2 ensures that the closure operators $c^{\mathrm{reg}}_{\mathcal{R}(\mathcal{A})}$ and $c^{\mathrm{ort}}_{\mathcal{R}(\mathcal{A})}$ coincide in $P_{\mathcal{E}(\mathcal{A})}$.

Since $\mathrm{Hom}(-,A)$ is a right adjoint and $\mathcal{R}(\mathcal{A})^{\mathrm{op}}$ has coequalizers (because it is a full reflective subcategory of $\mathcal{T}op^{\mathrm{op}}$), we know that K is a right adjoint. In order to show that K is full and faithful, it suffices to prove that the co-units of the adjunction $\mathrm{Hom}(-,A)$ are regular monomorphisms in $\mathcal{R}(A)$. For each $B \in \mathcal{R}(\mathcal{A})$, the co-unit is given by $\varepsilon_B : B \to A^{\mathrm{Hom}(B,A)}$, defined as the unique morphism such that $\pi_g \cdot \varepsilon_B = g$ for each $g \in \mathrm{Hom}(B,A)$, where π_g is the corresponding projection. Since ε_B clearly belongs to $\mathcal{A}^{\mathrm{Inj}_{\mathcal{E}(\mathcal{A})}}$, it belongs to $P_{\mathcal{E}(\mathcal{A})}(\mathcal{M})$. But, by Theorem 2.1, B is an absolutely $c^{\mathrm{ort}}_{\mathcal{R}(\mathcal{A})}$-closed object. Consequently, ϵ_B is $c^{\mathrm{ort}}_{\mathcal{R}(\mathcal{A})}$-closed and, since $c^{\mathrm{reg}}_{\mathcal{R}(\mathcal{A})}$ and $c^{\mathrm{ort}}_{\mathcal{R}(\mathcal{A})}$ coincide in $P_{\mathcal{E}(\mathcal{A})}$, ε_B is $c^{\mathrm{reg}}_{\mathcal{R}(\mathcal{A})}$-closed, so it is a regular monomorphism.

In order to prove the second part of the theorem, let

$$P_{\mathcal{E}(A)}(\mathcal{M}) = \mathcal{R}(A)^{\mathrm{Inj}_{\mathcal{E}(A)}}.$$

Then each $P_{\mathcal{E}(A)}(\mathcal{M})$-morphism with domain in $\mathcal{R}(A)$ is a split monomorphism. In particular, for each $B \in \mathcal{R}(A)$, the morphism $\varepsilon_B : B \to A^{\mathrm{Hom}(B,A)}$ is a split monomorphism. Thus, using the Split Monadicity Theorem, to conclude that $\mathrm{Hom}(-, A) : \mathcal{R}(A)^{\mathrm{op}} \to \mathcal{S}et$ is monadic, it remains to show that it reflects isomorphisms. Let $f : B \to C$ be such that $\mathrm{Hom}(f, A) : \mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A)$ is a bijective function. This means that $f \in A^{\perp}$. But

$$A^{\perp} = (\mathcal{R}(A))^{\perp} \subseteq \mathcal{R}(A)^{\mathrm{Inj}_{\mathcal{E}}(A)} \cap \mathrm{Epi}(\mathcal{R}(A)).$$

Thus f is a split monomorphism, because it is a morphism of $\mathcal{R}(A)^{\mathrm{Inj}_{\mathcal{E}}(A)}$ with domain in $\mathcal{R}(A)$, and it is an epimorphism in $\mathcal{R}(A)$; therefore it is an isomorphism. $\square$

Remark 3.1. We point out that the equality $P_{\mathcal{E}(A)}(\mathcal{M}) = \mathcal{R}(A)^{\mathrm{Inj}_{\mathcal{E}(A)}}$ implies that the reflective hull of A is just the subcategory of all retracts of powers of A. Furthermore, the converse implication is true, whenever $P_{\mathcal{E}(A)}(\mathcal{M}) = A^{\mathrm{Inj}_{\mathcal{E}(A)}}$.

As an immediate consequence of Theorem 3.1, we get Corollary below, which was proved by M. Sobral in [9]. We need to make use of the following lemma.

Lemma 3.1. *If $\mathcal{A}$ is a subcategory of $\mathcal{T}op$ such that $\mathcal{A}^{Inj_{\mathcal{E}(A)}}$ coincides with the class of embeddings in $\mathcal{E}(\mathcal{A})$, then, in $\mathcal{E}(\mathcal{A})$, embeddings are pushout-stable and $c^{reg}_{\mathcal{R}(\mathcal{A})}$ is weakly hereditary.*

Proof. It is well-known that each object in $\mathcal{E}(\mathcal{A})$ is the domain of some initial monosource with codomain in $\mathcal{A}$ (see [1]). This fact, combined with the assumption of the lemma on $\mathcal{A}$, is easily seen to imply that embeddings are pushout-stable in $\mathcal{E}(\mathcal{A})$.

In order to show that $c^{\mathrm{reg}}_{\mathcal{R}(\mathcal{A})}$ (which coincides with $c^{\mathrm{reg}}_{\mathcal{E}(\mathcal{A})}$) is weakly hereditary in $\mathcal{E}(\mathcal{A})$, let $m : X \to Y$ be an embedding in $\mathcal{E}(\mathcal{A})$ and let (u, v) be the cokernel pair of m in $\mathcal{E}(\mathcal{A})$. Then $\overline{m} = c^{\mathrm{reg}}_{\mathcal{R}(\mathcal{A})}(m) = \mathrm{eq}(u, v)$. Let d be the morphism such that $m = \overline{m} \cdot d$. We want to show that d is an epimorphism. Let f and g be morphisms such that $fd = gd$. As $\mathcal{E}(\mathcal{A})$ is the epireflective hull of $\mathcal{A}$, we may assume, without loss of generality, that the codomain of f and g belongs to $\mathcal{A}$. Since $\overline{m} \in \mathcal{A}^{\mathrm{Inj}(\mathcal{E}(\mathcal{A}))}$, there are morphisms $\overline{f}$ and $\overline{g}$ such that $\overline{f} \cdot \overline{m} = f$ and $\overline{g} \cdot \overline{m} = g$. Then, we get $\overline{f} \cdot m = \overline{f} \cdot \overline{m} \cdot d = \overline{g} \cdot \overline{m} \cdot d = \overline{g} \cdot m$, what implies the existence of a morphism t such that $\overline{f} = t \cdot u$ and $\overline{g} = t \cdot v$. Consequently, $f = \overline{f} \cdot \overline{m} = t \cdot u \cdot \overline{m} = t \cdot v \cdot \overline{m} = \overline{g} \cdot \overline{m} = g$. $\square$

Corollary 3.1 ([9]). *Let A be an $\mathcal{M}$-injective topological space. Then the comparison functor K is full and faithful and $\mathcal{R}(A)^{op}$ is equivalent to a reflective subcategory of the Eilenberg-Moore category $\mathcal{S}et^{\mathbb{T}}$.*

Proof. The fact that A is $\mathcal{M}$-injective implies that $\mathcal{M} = A^{\mathrm{Inj}_{\mathcal{E}(A)}}$. Thus, by Lemma 3.1, $P_{\mathcal{E}(A)}(\mathcal{M})$ is just the class of all embeddings in $\mathcal{E}(A)$, and, from Lemma 2.1, this implies that epimorphisms in $\mathcal{E}(A)$ are left-cancellable with respect to $\mathcal{M}$. Furthermore, Lemma 3.1 assures that $c^{\mathrm{reg}}_{\mathcal{R}(A)}$ is weakly hereditary. Therefore, we are under the conditions of Theorem 3.1. □

Examples 3.1.

(1) The reflective hull of the topological space

$$A = (\{0,1,2\}, \{\emptyset, \{0\}, \{1,2\}, \{0,1,2\}\})$$

is $\mathcal{T}op$ (cf. [6]). Since in $\mathcal{T}op$ embeddings are pushout-stable and A is $\mathcal{M}$-injective, it follows, from Theorem 3.1, that the dual of $\mathcal{T}op$ is a full reflective subcategory of the Eilenberg-Moore category induced by the right adjoint $\mathrm{Hom}(-,A) : \mathcal{T}op^{\mathrm{op}} \to \mathcal{S}et$.

(2) Analogously for $\mathcal{T}op_0$, we have that embeddings are pushout-stable and the Sierpiński space S is $\mathcal{M}$-injective, so that, since the category $\mathcal{S}ob$ of sober spaces is the reflective hull of S, the dual category of $\mathcal{S}ob$ is a reflective subcategory of the Eilenberg-Moore category induced by the right adjoint $\mathrm{Hom}(-,S)$.

(3) In the category 0-$\mathcal{T}op_2$ of 0-dimensional Hausdorff spaces, epimorphisms are just the dense continuous maps, that is, the continuous maps $f : X \to Y$ such that, for each y and each clopen C in Y, it holds that $y \in C \Rightarrow C \cap f(X) \neq \emptyset$. Thus, it is clear that, in 0-$\mathcal{T}op_2$, epimorphisms are left-cancellable with respect to $\mathcal{M}$. On the other hand, given a subspace X of Y in 0-$\mathcal{T}op_2$, being $m : X \to Y$ the corresponding embedding, it is easily seen that the regular closure of X in Y is just the intersection of all clopens which contain X, and that it is just the Kuratowski closure of X in Y. Therefore the morphism, which composed with $c^{\mathrm{reg}}(m)$ is equal to m, is dense, so an epimorphism; that is, c^{reg} is weakly hereditary. As seen in 1.1.4, $P = D^{\mathrm{Inj}}$. The reflective hull of D is the category of 0-dimensional compact Hausdorff spaces (cf. [6]). Therefore, applying the corollary of Theorem 3.1, we conclude that the opposite category of 0-dimensional compact Hausdorff spaces is a full reflective subcategory of an Eilenberg-Moore category.

(4) Another topological space which fulfils the conditions of Theorem 3.1 is the unit interval I. The reflective hull of I is the category of compact Hausdorff spaces, and its epireflective hull is the category $\mathcal{T}ych$ of Tychonoff spaces. As mentioned in 1.1.5, $P_{\mathcal{T}ych}(\mathcal{M}) = I^{\mathrm{Inj}_{\mathcal{T}ych}}$. The regular closure operator $c^{\mathrm{reg}}_{\mathcal{R}(I)}$ in $\mathcal{T}ych$ coincides with the Kuratowski closure (cf. [4]); consequently it is weakly hereditary,

and epimorphisms are dense morphisms, which are left-cancellabble with respect to $\mathcal{M}$.

(5) The category $\mathcal{I}nd$ of indiscrete spaces is the epireflective hull and the reflective hull of the indiscrete space $\{0,1\}$ (cf. [6]). It holds $P_{\mathcal{I}nd}(\mathcal{M}) = \mathcal{M} = \{0,1\}^{\mathrm{Inj}_{\mathcal{I}nd}} = \mathcal{R}(\{0,)^{\mathrm{Inj}_{\mathcal{I}nd}}$. Then $\mathcal{I}nd^{\mathrm{op}}$ is equivalent to the Eilenberg-Moore category induced by the two-points indiscrete space.

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