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FUZZY FUNCTIONS AND AN EXTENSION OF THE CATEGORY L-TOP OF CHANG-GOGUEN L-TOPOLOGICAL SPACES

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Abstract. We study $\mathcal{F}TOP(L)$, a fuzzy category with fuzzy functions in the role of morphisms. This category has the same objects as the category L-TOP of Chang-Goguen L-topological spaces, but an essentially wider class of morphisms—so called fuzzy functions introduced earlier in our joint work with U. Höhle and H. Porst.

Introduction

In research works where fuzzy sets are involved, in particular, in Fuzzy Topology, mostly certain usual functions are taken as morphisms: they can be certain mappings between corresponding sets, or between the fuzzy powersets of these sets, etc. On the other hand, in our joint works with U. Höhle and H.E. Porst [9], [10] a certain class of L-relations (i.e. mappings $F : X \times Y \to L$) was distinguished which we view as (L-)*fuzzy functions* from a set X to a set Y; these fuzzy functions play the role of morphisms in an L-*fuzzy category* of sets $\mathcal{F}SET(L)$, introduced in [10].

Later on we constructed a fuzzy category $\mathcal{F}TOP(L)$ related to topology with fuzzy functions in the role of morphisms, see [18]. Further, in [19] a certain uniform counterpart of $\mathcal{F}TOP(L)$ was introduced. Our aim here is to continue the study of $\mathcal{F}TOP(L)$. In particular, we show that the top frame $\mathcal{F}TOP(L)^\top$ of the fuzzy category $\mathcal{F}TOP(L)$ is a topological category in H. Herrlich's sense [1]) over the top frame $\mathcal{F}SET(L)^\top$ of the fuzzy category $\mathcal{F}SET(L)$.

In order to make exposition self-contained, we start with Section 1 Prerequisities, where we briefly recall the three basic concepts which are essentially

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used in this work: they are the concepts of a GL-monoid (see e.g. [6], [8], etc.); of an L-valued set (see e.g. [7], etc.), and of an L-fuzzy category (see e.g. [14], [15], [16], etc.). In Section 2 we consider basic facts about fuzzy functions and introduce the L-fuzzy category $\mathcal{FSET}(L)$ [9], [10]. The properties of this fuzzy category and some related categories are the subject of Section 3. $\mathcal{FSET}(L)$ is used as the ground category for the L-fuzzy category $\mathcal{FTOP}(L)$ whose objects are Chang-Goguen L-topological spaces [4], [5], and whose morphisms are certain fuzzy functions, i.e. morphisms from $\mathcal{FSET}(L)$. Fuzzy category $\mathcal{FTOP}(L)$ is considered in Section 4. Its crisp top frame $\mathcal{FTOP}(L)^\top$ is studied in Section 5. In particular, it is shown that $\mathcal{FTOP}(L)^\top$ is a topological category over $\mathcal{FSET}(L)^\top$. Finally, in Section 6 we consider the behaviour of the topological property of compactness with respect to fuzzy functions — in other words in the context of the fuzzy category $\mathcal{FTOP}(L)$ and, specifically, in the context of the category $\mathcal{FTOP}(L)^\top$.

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1. Prerequisities

1.1. GL**-monoids.** Let $(L, \leq)$ be a complete infinitely distributive lattice, i.e. $(L, \leq)$ is a partially ordered set such that for every subset $A \subset L$ the join $\bigvee A$ and the meet $\bigwedge A$ are defined and $(\bigvee A) \wedge \alpha = \bigvee\{a \wedge \alpha) \mid a \in A\}$ and $(\bigwedge A) \vee \alpha = \bigwedge\{a \vee \alpha) \mid a \in A\}$ for every $\alpha \in L$. In particular, $\bigvee L =: \top$ and $\bigwedge L =: \bot$ are respectively the universal upper and the universal lower bounds in L. We assume that $\bot \neq \top$, i.e. L has at least two elements.

A $GL-$monoid (see [6], [7], [8]) is a complete lattice enriched with a further binary operation $*$, i.e. a triple $(L, \leq, *)$ such that:

(1) $*$ is monotone, i.e. $\alpha \leq \beta$ implies $\alpha * \gamma \leq \beta * \gamma$, $\forall \alpha, \beta, \gamma \in L$;
(2) $*$ is commutative, i.e. $\alpha * \beta = \beta * \alpha$, $\forall \alpha, \beta \in L$;
(3) $*$ is associative, i.e. $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$, $\forall \alpha, \beta, \gamma \in L$;
(4) $(L, \leq, *)$ is integral, i.e. $\top$ acts as the unity: $\alpha * \top = \alpha$, $\forall \alpha \in L$;
(5) $\bot$ acts as the zero element in $(L, \leq, *)$, i.e. $\alpha * \bot = \bot$, $\forall \alpha \in L$;
(6) $*$ is distributive over arbitrary joins, i.e. $\alpha * (\bigvee_j \beta_j) = \bigvee_j (\alpha * \beta_j)$, $\forall \alpha \in L, \forall \{\beta_j : j \in J\} \subset L$;
(7) $(L, \leq, *)$ is divisible, i.e. $\alpha \leq \beta$ implies existence of $\gamma \in L$ such that $\alpha = \beta * \gamma$.

It is known that every $GL-$monoid is residuated, i.e. there exists a further binary operation "$\longmapsto$" (implication) on L satisfying the following condition:

$$\alpha * \beta \leq \gamma \Longleftrightarrow \alpha \leq (\beta \longmapsto \gamma) \qquad \forall \alpha, \beta, \gamma \in L.$$

Explicitly implication is given by

$$\alpha \longmapsto \beta = \bigvee\{\lambda \in L \mid \alpha * \lambda \leq \beta\}.$$

Below we list some useful properties of $GL-$monoids (see e.g. [6], [7], [8]):

(i) $\alpha \longmapsto \beta = \top \Longleftrightarrow \alpha \leq \beta$;

(ii) $\alpha \longmapsto (\bigwedge_i \beta_i) = \bigwedge_i(\alpha \longmapsto \beta_i)$;
(iii) $(\bigvee_i \alpha_i) \longmapsto \beta = \bigwedge_i(\alpha_i \longmapsto \beta)$;
(v) $\alpha * (\bigwedge_i \beta_i) = \bigwedge_i(\alpha * \beta_i)$;
(vi) $(\alpha \longmapsto \gamma) * (\gamma \longmapsto \beta) \leq \alpha \longmapsto \beta$;
(vii) $\alpha * \beta \leq (\alpha * \alpha) \vee (\beta * \beta)$.

Important examples of GL-monoids are Heyting algebras and MV-algebras. Namely, a *Heyting algebra* is GL-monoid of the type $(L, \leq, \wedge, \vee, \wedge)$ (i.e. in case of a Heyting algebra $\wedge = *$), cf. e.g. [12]. A GL-monoid is called an *MV-algebra* if $(\alpha \longmapsto \bot) \longmapsto \bot = \alpha \quad \forall \alpha \in L$, [2], [3], see also [8, Lemma 2.14]. Thus in an MV-algebra an order reversing involution $^c : L \to L$ can be naturally defined by setting $\alpha^c := \alpha \longmapsto \bot \quad \forall \alpha \in L$.

If X is a set and L is a GL-monoid, then the fuzzy powerset L^X in an obvious way can be pointwise endowed with a structure of a GL-monoid. In particular the L-sets 1_X and 0_X defined by $1_X(x) := \top$ and $0_X(x) := \bot$ $\forall x \in X$ are respectively the universal upper and lower bounds in L^X.

In the sequel L denotes an arbitrary GL-monoid.

1.2. **L-valued sets.** Following U. Höhle (cf. e.g. [7]) by a (global) *L-valued set* we call a pair (X, E) where X is a set and E is an *L-valued equality*, i.e. a mapping $E : X \times X \to L$ such that

(1eq) $E(x, x) = \top$
(2eq) $E(x, y) = E(y, x) \quad \forall x, y \in X$;
(3eq) $E(x, y) * E(y, z) \leq E(x, z) \quad \forall x, y, z \in X$.

A mapping $f : (X, E_X) \to (Y, E_Y)$ is called *extensional* if

$$E_X(x, x') \leq E_Y(f(x), f(x')) \; \forall x, x' \in X.$$

Let $SET(L)$ denote the category whose objects are L-valued sets and whose morphisms are extensional mappings between the corresponding L-valued sets. Further, recall that an L-set, or more precisely, an L-subset of a set X is just a mapping $A : X \to L$. In case (X, E) is an L-valued set, its L-subset A is called *extensional* if

$$\bigvee_{x \in X} A(x) * E(x, x') \leq A(x') \quad \forall x' \in X.$$

1.3. **L-fuzzy categories.**

Definition 1.3.1 ([14], [15], [16], [17]). *An L-fuzzy category is a quintuple $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \omega, \mathcal{M}(\mathcal{C}), \mu, \circ)$ where $\mathcal{C}_\bot = (\mathcal{O}b(\mathcal{C}), \mathcal{M}(\mathcal{C}), \circ)$ is a usual (classical) category called* the bottom frame *of the fuzzy category $\mathcal{C}$; $\omega : \mathcal{O}b(\mathcal{C}) \longrightarrow L$ is an L-subclass of the class of objects $\mathcal{O}b(\mathcal{C})$ of $\mathcal{C}_\bot$ and $\mu : \mathcal{M}(\mathcal{C}) \longrightarrow L$ is an L-subclass of the class of morphisms $\mathcal{M}(\mathcal{C})$ of $\mathcal{C}_\bot$. Besides ω and μ must satisfy the following conditions:*

(1) *if $f : X \to Y$, then $\mu(f) \leq \omega(X) \wedge \omega(Y)$;*
(2) *$\mu(g \circ f) \geq \mu(g) * \mu(f)$ whenever composition $g \circ f$ is defined;*
(3) *if $e_X : X \to X$ is the identity morphism, then $\mu(e_X) = \omega(X)$.*

Given an L-fuzzy category $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \omega, \mathcal{M}(\mathcal{C}), \mu, \circ)$ and $X \in \mathcal{O}b(\mathcal{C})$, the intuitive meaning of the value $\omega(X)$ is the *degree* to which a potential object X of the L-fuzzy category $\mathcal{C}$ is indeed its object; similarly, for $f \in \mathcal{M}(\mathcal{C})$ the intuitive meaning of $\mu(f)$ is the degree to which a potential morphism f of $\mathcal{C}$ is indeed its morphism.

Definition 1.3.2. *Let $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \omega, \mathcal{M}(\mathcal{C}), \mu, \circ)$ be an L-fuzzy category. By an (L-fuzzy) subcategory of $\mathcal{C}$ we call an L-fuzzy category*

$$\mathcal{C}' = (\mathcal{O}b(\mathcal{C}), \omega', \mathcal{M}(\mathcal{C}), \mu', \circ)$$

where $\omega' \le \omega$ and $\mu' \le \mu$. A subcategory $\mathcal{C}'$ of the category $\mathcal{C}$ is called full if $\mu'(f) = \mu(f) \wedge \omega'(X) \wedge \omega'(Y)$ for every $f \in \mathcal{M}_{\mathcal{C}}(X, Y)$, and all $X, Y \in \mathcal{O}b(\mathcal{C})$.

Thus an L-fuzzy category and its subcategory have the same classes of potential objects and morphisms. The only difference of a subcategory from the whole category is in L-fuzzy classes of objects and morphisms, i.e. in the belongness degrees of potential objects and morphisms.

Let $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \mathcal{M}(\mathcal{C}), \circ)$ be a crisp category and $\mathcal{D} = (\mathcal{O}b(\mathcal{D}), \mathcal{M}(\mathcal{D}), \circ)$ be its subcategory. Then for every GL-monoid L the category $\mathcal{D}$ can be identified with the L-fuzzy subcategory

$$\tilde{\mathcal{D}} = (\mathcal{O}b(\mathcal{C}), \omega', \mathcal{M}(\mathcal{C}), \mu', \circ)$$

of $\mathcal{C}$ such that $\omega'(X) = \top$ if $X \in \mathcal{O}b(\mathcal{D})$ and $\omega'(X) = \perp$ otherwise; $\mu'(f) = \top$ if $f \in \mathcal{M}(\mathcal{D})$ and $\mu'(f) = \perp$ otherwise. In particular, $\tilde{\mathcal{D}}_\top = \mathcal{D}$.

On the other hand sometimes it is convenient to identify a fuzzy subcategory

$$\mathcal{C}' = (\mathcal{O}b(\mathcal{C}), \omega', \mathcal{M}(\mathcal{C}), \mu', \circ)$$

of the fuzzy category

$$\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \omega, \mathcal{M}(\mathcal{C}), \mu, \circ)$$

with the fuzzy category

$$\mathcal{D} = (\mathcal{O}b(\mathcal{D}), \omega_{\mathcal{D}}, \mathcal{M}(\mathcal{D}), \mu_{\mathcal{D}}, \circ)$$

where

$$\mathcal{O}b(\mathcal{D}) := \{X \in \mathcal{O}b(\mathcal{C}) \mid \omega'(X) \neq \perp\}$$

and

$$\mathcal{M}(\mathcal{D}) := \{f \in \mathcal{M}(\mathcal{C}) \mid \mu'(f) \neq \perp\}$$

and $\omega_{\mathcal{D}}$ and $\mu_{\mathcal{D}}$ are restrictions of ω' and μ' to $\mathcal{O}b(\mathcal{D})$ and $\mathcal{M}(\mathcal{D})$ respectively.

2. Fuzzy functions and fuzzy category $\mathcal{FSET}(L)$

As it was already mentioned above, the concept of a fuzzy function and the corresponding fuzzy category $\mathcal{FSET}(L)$ were introduced in [9], [10]. There were studied also basic properties of fuzzy functions. In this section we recall those definitions and results from [10] which will be needed in the sequel[1]. Besides some new needed facts about fuzzy functions will be established here, too.

2.1. **Fuzzy functions and category $\mathcal{F}SET(L)$.**

Definition 2.1.1 (cf. [9, 2.1]). *A fuzzy function*[2] *F from an L-valued set (X, E_X) to (Y, E_Y) (in symbols $F : (X, E_X) \rightarrowtail (Y, E_Y)$)) is a mapping $F : X \times Y \to L$ such that*

(1ff) $F(x,y) * E_Y(y,y') \leq F(x,y') \quad \forall x \in X, \forall y, y' \in Y$;
(2ff) $E_X(x,x') * F(x,y) \leq F(x',y) \quad \forall x, x' \in X, \forall y \in Y$;
(3ff) $F(x,y) * F(x,y') \leq E_Y(y,y') \quad \forall x \in X, \forall y, y' \in Y$.

Notice that conditions (1ff)–(2ff) say that F is a certain $L-$relation, while axiom (3ff) together with evaluation $\mu(F)$ (see Subsection 2.2) specify that the L-relation F is a fuzzy *function*.

Remark 2.1.2. Let $F : (X, E_X) \rightarrowtail (Y, E_Y)$ be a fuzzy function, $X' \subset X$, $Y' \subset Y$, and let the L-valued equalities $E_{X'}$ and $E_{Y'}$ on X' and Y' be defined as the restrictions of the equalities E_X and E_Y respectively. Then defining a mapping $F' : X' \times Y' \to L$ by the equality $F'(x,y) = F(x,y)\ \forall x \in X', \forall y \in Y'$ a fuzzy function $F' : (X', E_{X'}) \rightarrowtail (Y', E_{Y'})$ is obtained. We refer to it as the *restriction of* F to the subspaces $(X', E_{X'})$ $(Y', E_{Y'})$

Given two fuzzy functions $F : (X, E_X) \rightarrowtail (Y, E_Y)$ and $G : (Y, E_Y) \rightarrowtail (Z, E_Z)$ we define their *composition* $G \circ F : (X, E_X) \rightarrowtail (Z, E_Z)$ by the formula

$$(G \circ F)(x,z) = \bigvee_{y \in Y} \big(F(x,y) * G(y,z))\big).$$

In [10] it was shown that the composition $G \circ F$ is indeed a fuzzy function and that the operation of composition is associative. Further, if we define the identity morphism by the corresponding L-valued equality: $E_X : (X, E_X) \rightarrowtail (X, E_X)$, we come to a category $\mathcal{F}SET(L)$ whose objects are L-valued sets and whose morphisms are fuzzy functions $F : (X, E_X) \rightarrowtail (Y, E_Y)$.

2.2. **Fuzzy category $\mathcal{F}SET(L)$.** Given a fuzzy function $F : (X, E_X) \rightarrowtail (Y, E_Y)$ let

$$\mu(F) = \inf_x \sup_y F(x,y).$$

Thus we define an L-subclass μ of the class of all morphisms of $\mathcal{F}SET(L)$. In case $\mu(F) \geq \alpha$ we refer to F as a *fuzzy α-function*.

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[1] Actually, the subject of [9], [10] was a more general fuzzy category L-$\mathcal{F}SET(L)$ containing $\mathcal{F}SET(L)$ as a full subcategory. However, since for the merits of this work the category $\mathcal{F}SET(L)$ is of importance, when discussing results from [9], [10] we reformulate (simplify) them for the case of $\mathcal{F}SET(L)$ without mentioning this every time explicitly

[2] Probably, the name *an L-fuzzy function* would be more adequate here. However, since the GL-monoid L is considered to be fixed, and since the prefix "L" appears in the text very often, we prefer to say just *a fuzzy function*

If $F : (X, E_X) \rightarrowtail (Y, E_Y)$ and $G : (Y, E_Y) \rightarrowtail (Z, E_Z)$ are fuzzy functions, then $\mu(G \circ F) \geq \mu(G) * \mu(F)$ [10]. Further, given an L-valued set (X, E) let $\omega(X, E) := \mu(E) = \inf_x E(x, x) = \top$. Thus a *fuzzy category* $\mathcal{F}SET(L)$= $(FSET(L), \omega, \mu)$ is obtained.

Example 2.2.1. Let $* = \wedge$ and E_Y be a crisp equality on Y, i.e. $E_Y(y, y') = \top$ iff $y = y'$, and $E_Y(y, y') = \bot$ otherwise. Then every fuzzy function $F : (X, E_X) \rightarrowtail (Y, E_Y)$ such that $\mu(F) = \top$ is uniquely determined by a usual function $f : X \to Y$. Indeed, let $f(x) = y$ iff $F(x, y) = \top$. Then condition (3ff) implies that there cannot be $f(x) = y$, $f(x) = y'$ for two different y, $y' \in Y$ and condition $\mu(F) = \top$ guarantees that for every $x \in X$ one can find $y \in Y$ such that $f(x) = y$. If besides E_X is crisp, then, vice versa, every mapping $f : X \to Y$ can be viewed as a fuzzy mapping $F : (X, E_X) \rightarrowtail (Y, E_Y)$ (since the conditions of extensionality (2ff) and (3ff) are automatically fulfilled in this case)

Remark 2.2.2. If $F' : (X', E_{X'}) \rightarrowtail (Y, E_Y)$ is the restriction of $F : (X, E_X) \rightarrowtail (Y, E_Y)$ (see Remark above) and $\mu(F) \geq \alpha$, then $\mu(F') \geq \alpha$. However, generally the restriction $F' : (X', E_{X'}) \rightarrowtail (Y', E_{Y'})$ of $F : (X, E_X) \rightarrowtail (Y, E_Y)$ may fail to satisfy condition $\mu(F') \geq \alpha$.

2.3. **Images and preimages of L-sets under fuzzy functions.** Given a fuzzy function $F : (X, E_X) \rightarrowtail (Y, E_Y)$ and L-subsets $A : X \to L$ and $B : Y \to L$ of X and Y respectively, we define the fuzzy set $F^{\to}(A) : Y \to L$ (the image of A under F) by the equality $F^{\to}(A)(y) = \bigvee_x F(x, y) * A(x)$ and the fuzzy set $F^{\leftarrow}(B) : X \to L$ (the preimage of B under F) by the equality $F^{\leftarrow}(B)(x) = \bigvee_y F(x, y) * B(y)$.

Note that if $A \in L^X$ is extensional, then $F^{\to}(A) \in L^Y$ is extensional (by (2ff)) and if $B \in L^Y$ is extensional, then $F^{\leftarrow}(B) \in L^X$ is extensional (by (3ff)).

Proposition 2.3.1 (Basic properties of images and preimages of L-sets under fuzzy functions).

(1) $F^{\to}(\bigvee_{i \in \mathcal{I}}(A_i) = \bigvee_{i \in \mathcal{I}} F^{\to}(A_i) \qquad \forall \{A_i : i \in \mathcal{I}\} \subset L^X$;

(2) $F^{\to}(A_1 \bigwedge A_2) \leq F^{\to}(A_1) \bigwedge F^{\to}(A_2) \qquad \forall A_1, A_2 \in L^X$;

(3) *If L-sets B_i are* extensional*, then*

$$\bigwedge_{i \in \mathcal{I}} F^{\leftarrow}(B_i) * \mu(F)^2 \leq F^{\leftarrow}(\bigwedge_{i \in \mathcal{I}} B_i) \leq \bigwedge_{i \in \mathcal{I}} F^{\leftarrow}(B_i) \qquad \forall \{B_i : i \in \mathcal{I}\} \subset L^Y.$$

In particular, if $\mu(F) = \top$, then $F^{\leftarrow}(\bigwedge_{i \in \mathcal{I}} B_i) = \bigwedge_{i \in \mathcal{I}} F^{\leftarrow}(B_i)$ for every family of extensinal L-sets $\{B_i : i \in \mathcal{I}\} \subset L^Y$.

(4) $F^{\leftarrow}(\bigvee_{i \in \mathcal{I}} B_i) = \bigvee_{i \in \mathcal{I}} F^{\leftarrow}(B_i) \qquad \forall \{B_i : i \in \mathcal{I}\} \subset L^Y$.

(5) $A * \mu(F)^2 \leq F^{\leftarrow}(F^{\to}(A))$ *for every* $A \in L^X$.

(6) $F^{\to}\big(F^{\leftarrow}(B)\big) \leq B$ *for every* extensional *L-set* $B \in L^Y$.

(7) $F^{\leftarrow}(c_Y) \geq \mu(F) * c$ *where $c_Y : Y \to L$ is the constant function taking value $c \in L$. In particular, $F^{\leftarrow}(c_Y) = c$ if $\mu(F) = \top$.*

Proof. (1).

$$\begin{aligned}\big(\bigvee_i F^{\rightarrow}(A_i)\big)(y) &= \bigvee_i \bigvee_x \big(F(x,y) * A_i(x)\big)\\ &= \bigvee_x \bigvee_i \big(F(x,y) * A_i(x)\big)\\ &= \bigvee_x (F(x,y) * (\bigvee_i A_i)(x))\\ &= F^{\rightarrow}(\bigvee_i A_i)(y).\end{aligned}$$

(2). The validity of (2) follows from the monotonicity of $F^{\rightarrow}$.

(3). To prove property 3 we first establish the following inequality

$$\bigvee_{y \in Y} \big(F(x,y)\big)^2 \geq \big(\bigvee_{y \in Y} F(x,y)\big)^2. \tag{2.3.1.1}$$

Indeed, by a property (vii) of a GL-monoid

$$\begin{aligned}\Big(\bigvee_{y \in Y} F(x,y)\Big)^2 &= \Big(\bigvee_{y \in Y} F(x,y)\Big) * \Big(\bigvee_{y' \in Y} F(x,y')\Big)\\ &= \bigvee_{y,y' \in Y} \Big(F(x,y) * F(x,y')\Big)\\ &\leq \bigvee_{y,y' \in Y} \Big(F(x,y)^2 \vee F(x,y')^2\Big)\\ &= \bigvee_{y \in Y} \Big(F(x,y)\Big)^2.\end{aligned}$$

In particular, it follows from (2.3.1.1) that

$$\forall x \in X \; \bigvee_{y \in Y} \big(F(x,y)\big)^2 \geq \mu(F)^2. \tag{2.3.1.2}$$

Now, applying (2.3.1.2) and taking into account extensionality of L-sets B_i, we proceed as follows:

$$\begin{aligned}&\Big(\bigwedge_i F^{\leftarrow}(B_i)\Big)(x) * \big(\mu(F)\big)^2\\ \leq\;& \Big(\bigwedge_i \big(\bigvee_{y_i \in Y} (F(x,y_i) * B_i(y_i))\big)\Big) * \bigvee_{y \in Y} \big(F(x,y)\big)^2\\ =\;& \bigvee_{y \in Y} \Big(\big(F(x,y)\big)^2 * \bigwedge_i \big(\bigvee_{y_i \in Y} F(x,y_i) * B_i(y_i)\big)\\ =\;& \bigvee_{y \in Y} \Big(F(x,y) * \big(\bigwedge_i \big(F(x,y) * \big(\bigvee_{y_i \in Y} F(x,y_i) * B_i(y_i)\big)\big)\big)\Big)\\ =\;& \bigvee_{y \in Y} \Big(F(x,y) * \big(\bigwedge_i \big(\bigvee_{y_i \in Y} (F(x,y) * F(x,y_i))\big) * B_i(y_i)\big)\Big)\\ \leq\;& \bigvee_{y \in Y} \Big(F(x,y) * \big(\bigwedge_i \big(\bigvee_{y_i \in Y} E(y,y_i) * B_i(y_i)\big)\big)\Big)\\ \leq\;& \bigvee_{y \in Y} \big(F(x,y) * \big(\bigwedge_i B_i(y)\big)\big)\\ =\;& F^{\leftarrow}\big(\bigwedge_i B_i\big)\big)(x),\end{aligned}$$

and hence

$$\Big(\bigwedge_i F^{\leftarrow}(B_i)\Big) * \big(\mu(F)\big)^2 \leq F^{\leftarrow}\big(\bigwedge_i B_i\big)\big).$$

To complete the proof notice that the inequality

$$F^{\leftarrow}(\bigwedge_{i \in \mathcal{I}} B_i) \leq \bigwedge_{i \in \mathcal{I}} F^{\leftarrow}(B_i)$$

is obvious.

(4). The proof of (4) is similar to the proof of (1) and is therefore omitted.

(5). Let $A \in L^X$, then

$$\begin{array}{rcl} F^{\leftarrow}(F^{\rightarrow}(A))(x) & = & \bigvee_y (F(x,y) * F^{\rightarrow}(A)(y)) \\ & = & \bigvee_y \big(F(x,y) * \big(\bigvee_{x'} F(x',y) * A(x')\big)\big) \\ & \geq & \bigvee_y (F(x,y)^2 * A(x)) \\ & \geq & (\mu(F))^2 * A(x) \end{array}$$

for every $x \in X$, and hence 5 holds.

(6). To show property 6 assume that $B \in L^Y$ is extensional. Then

$$\begin{array}{rcl} F^{\rightarrow}(F^{\leftarrow}(B))(y) & = & \bigvee_x \big(F(x,y) * F^{\leftarrow}(B)(x)\big) \\ & = & \bigvee_x F(x,y) * \big(\bigvee_{y'} F(x,y') * B(y')\big) \\ & = & \bigvee_{x\in X, y\in Y} \big(F(x,y) * F(x,y') * B(y')\big) \\ & \leq & E_Y(y,y') * B(y') \\ & \leq & B(y), \end{array}$$

and hence $F^{\rightarrow}\big(F^{\leftarrow}(B)\big) \leq B$.

(7). The proof of property 7 is straightforward and therefore omitted. □

Comments 2.3.2.

(1) Properties 1,2 and 4 were proved in [10, Proposition 3.2]. Here we reproduce these proofs in order to make the article self-contained.

(2) The inequality in item 2 of the previous proposition obviously cannot be improved even in the crisp case.

(3) One can show that the condition of extensionality cannot be omitted in items 3 and 6.

(4) The idea of the proof of Property 3 was communicated to the author by U. Höhle in Prague at TOPOSYM in August 2001.

(5) In [10] there was established the following version of Property 3 without the assumption of extensionality of L-sets B_i in case L is completely distributive:

$$\big(\bigwedge_{i\in\mathcal{I}} F^{\leftarrow}(B_i)\big)^5 \leq F^{\leftarrow}\big(\bigwedge_{i\in\mathcal{I}} B_i\big) \leq \bigwedge_{i\in\mathcal{I}} F^{\leftarrow}(B_i) \qquad \forall\{B_i : i \in \mathcal{I}\} \subset L^Y$$

and

$$\bigwedge_{i\in\mathcal{I}} F^{\leftarrow}(B_i) = F^{\leftarrow}\big(\bigwedge_{i\in\mathcal{I}} B_i\big) \qquad \forall\{B_i : i \in \mathcal{I}\} \subset L^Y, \text{ in case } * = \wedge$$

2.4. **Injectivity, surjectivity and bijectivity of fuzzy functions.**

Definition 2.4.1. *A fuzzy function $F : (X, E_X) \rightarrowtail (Y, E_Y)$ is called injective, if*

(inj) $F(x,y) * F(x',y) \leq E_X(x,x') \quad \forall x, x' \in X, \forall y \in Y.$

Definition 2.4.2. *Given a fuzzy function $F : (X, E_X) \rightarrowtail (Y, E_Y)$, we define its degree of surjectivity by the equality:*

$$\sigma(F) := \inf_y \sup_x F(x,y)$$

In particular, a fuzzy function F is called α-surjective, if $\sigma(F) \geq \alpha$.

In case F is injective and α-surjective, it is called α-bijective.

Remark 2.4.3. Let (X,E_X), (Y,E_Y) be L-valued sets and $(X',E_{X'})$, $(Y',E_{Y'})$ be their subspaces. Obviously, the restriction $F'\colon (X',E_{X'}) \rightarrowtail (Y',E_{Y'})$ of an injection $F\colon (X,E_X) \rightarrowtail (Y,E_Y)$ is an injection. The restriction $F'\colon (X,E_X) \rightarrowtail (Y',E_{Y'})$ of an α-surjection $F\colon (X,E_X) \rightarrowtail (Y,E_Y)$ is an α-surjection. On the other hand, generally the restriction $F'\colon (X',E_{X'}) \rightarrowtail (Y',E_{Y'})$ of an α-surjection $F\colon (X,E_X) \rightarrowtail (Y,E_Y)$ may fail to be an α-surjection.

A fuzzy function $F : (X,E_X) \rightarrowtail (Y,E_Y)$ determines a fuzzy *relation* $F^{-1} : X \times Y \to L$ by setting $F^{-1}(y,x) = F(x,y)$ $\forall x \in X$, $\forall y \in Y$.

Proposition 2.4.4 (Basic properties of injections, α-surjections and α-bijections)**.**

(1) *$F^{-1} : (Y,E_Y) \rightarrowtail (X,E_X)$ is a fuzzy function iff F is injective (actually F^{-1} satisfies (3ff) iff F satisfies (inj))*
(2) *F is α-bijective iff F^{-1} is α-bijective.*
(3) *If F is injective, and L-sets A_i are extensional, then*

$$(\bigwedge_i F^{\to}(A_i)) * (\sigma(F))^2 \le F^{\to}(\bigwedge_i A_i) \le \bigwedge_i F^{\to}(A_i) \qquad \forall \{A_i : i \in \mathcal{I}\} \subset L^X.$$

In particular, if F is $\top$-bijective, then

$$(\bigwedge_i F^{\to}(A_i) = F^{\to}(\bigwedge_i A_i) \qquad \forall \{A_i : i \in \mathcal{I}\} \subset L^X.$$

(4) *$F^{\to}(F^{\leftarrow}(B)) \ge \sigma(F)^2 * B$. In particular, if F is $\top$-surjective and B is extensional, then $F^{\to}(F^{\leftarrow}(B)) = B$.*
(5) *$F^{\to}(c_X) \ge \sigma(F) * c$ where $c_X : X \to L$ is the constant function with value c. In particular, $F^{\to}(c_X) = c$ if $\sigma(F) = \top$.*

Proof. Properties 1 and 2 follow directly from the definitions.

(2). The proof of Property 3 is analogous to the proof of item 3 of Proposition 2.3.1:

First, reasoning as in the proof of (2.3.1.1) we establish the following inequality

$$\text{(2.4.4.1)} \qquad \bigvee_{x\in X} \big(F(x,y)\big)^2 \ge \big(\bigvee_{x\in X} F(x,y)\big)^2.$$

In particular, from here it follows that

$$\text{(2.4.4.2)} \qquad \forall y \in Y \ \bigvee_{x\in X} \big(F(x,y)^2\big) \ge \sigma(F)^2.$$

Now, applying (2.4.4.2) and taking into account extensionality of L-sets A_i, we proceed as follows:

$$\begin{array}{rl}
& \Big(\bigwedge_i F^{\to}(A_i)\Big)(x) * \big(\sigma(F)\big)^2 \\
\leq & \Big(\bigwedge_i\big(\bigvee_{x_i\in X}(F(x_i,y) * A_i(x_i))\big)\Big) * \bigvee_{x\in X}\big(F(x,y)\big)^2 \\
= & \bigvee_{x\in X}\Big(\big(F(x,y)\big)^2 * \bigwedge_i\big(\bigvee_{x_i\in X} F(x_i,y) * A_i(x_i)\big) \\
= & \bigvee_{x\in X}\Big(F(x,y) * \big(\bigwedge_i\big(F(x,y) * \big(\bigvee_{x_i\in X} F(x_i,y) * A_i(x_i)\big)\big)\big)\Big) \\
= & \bigvee_{x\in X}\Big(F(x,y) * \big(\bigwedge_i\big(\bigvee_{x_i\in X}(F(x,y) * F(x_i,y))\big) * A_i(x_i)\big)\Big) \\
\leq & \bigvee_{x\in X}\Big(F(x,y) * \big(\bigwedge_i\big(\bigvee_{x_i\in X} E_X(x,x_i) * A_i(x_i)\big)\big)\Big) \\
\leq & \bigvee_{x\in X}\big(F(x,y) * \big(\bigwedge_i A_i(x)\big)\big) \\
= & F^{\to}\big(\bigwedge_i A_i\big)\big)(y),
\end{array}$$

and hence

$$\Big(\bigwedge_i F^{\to}(A_i)\Big) * \big(\sigma(F)\big)^2 \leq F^{\to}\big(\bigwedge_i A_i)\big).$$

To complete the proof notice that the inequality

$$F^{\to}(\bigwedge_{i\in\mathcal{I}} A_i) \leq \bigwedge_{i\in\mathcal{I}} F^{\to}(A_i)$$

is obvious.

(4). Let $B \in L^Y$, then

$$\begin{array}{rcl}
F^{\to}(F^{\leftarrow}(B))(y) & = & \bigvee_x\big(F(x,y) * F^{\leftarrow}(B)(x)\big) \\
& = & \bigvee_x F(x,y) * \big(\bigvee_{y'} F(x,y') * B(y)\big) \\
& \geq & \bigvee_x F(x,y) * F(x,y) * B(y) \\
& \geq & \sigma(F)^2 * B(y),
\end{array}$$

and hence the first inequality in item 4 is proved. From here and Proposition 2.3.1 (6) the second statement of item 4 follows.

(5) The proof of the last property is straightforward and therefore omitted. □

Question 2.4.5. We do not know whether inequalities in items 3, 4 and 5 can be improved.

Comments 2.4.6.

(1) Properties 1 and 2 were first established in [10].

(2) In [10] the following version of Property 3 was proved:
If L is completely distributive and F is injective, then

$$(\bigwedge_i F^{\to}(A_i))^5 \leq F^{\to}(\bigwedge_i A_i) \leq \bigwedge_i F^{\to}(A_i) \qquad \forall\{A_i : i\in\mathcal{I}\} \subset L^X$$

and

$$F^{\to}(\bigwedge_i A_i) = \bigwedge_i F^{\to}(A_i) \quad \text{ in case } \wedge = *$$

No extensionality is assumed in these cases.

(3) In case of an ordinary function $f : X \to Y$ the equality

$$f^{\to}\big(\bigwedge_{i\in\mathcal{I}}(A_i)\big) = \bigwedge_{i\in\mathcal{I}} f^{\to}(A_i)$$

holds just under assumption that f is injective. On the other hand, in case of a fuzzy function F to get a reasonable counterpart of this property we need to assume that F is bijective. The reason for this, as we see it, is that in case of an ordinary function f, when proving the equality, we actually deal only with points belonging to the image $f(X)$, while the rest of $Y \setminus f(X)$ does not play any role. On the other hand, in case of a fuzzy function $F : X \rightarrowtail Y$ the whole Y is "an image of X to a certain extent", and therefore, when operating with images of L-sets, we need to take into account, to what extent a point y is in the "image" of X.

3. Further properties of fuzzy category $\mathcal{F}SET(L)$

In this section we continue to study properties of the fuzzy category $\mathcal{F}SET(L)$. As different from the previous section, were our principal interest was in the "set-theoretic" aspect of fuzzy functions, here shall be mainly interested in their properties of "categorical nature".

First we shall specify the two (crisp) categories related to $\mathcal{F}SET(L)$: namely, its bottom frame $\mathcal{F}SET(L)^{\perp}$ (=$\mathcal{F}SET(L)$ (this category was introduced already in Section 2) and its top frame $\mathcal{F}SET(L)^{\top}$. The last one will be of special importance for us. By definition its morphisms F satisfy condition $\mu(F) = \top$, and as we have seen in the previous section, fuzzy functions satisfying this condition "behave themselves much more like ordinary functions" than general fuzzy functions. Respectively, the results which we are able to establish about $\mathcal{F}SET(L)^{\top}$ and about topological category $\mathcal{F}TOP(L)^{\top}$based on it, are more complete and nice, then their more general counterparts.

Second, note that the "classical" category $SET(L)$ of L-valued sets can be naturally viewed as a subcategory of $\mathcal{F}SET(L)^{\top}$. In case $L = \{0,1\}$ obviously the two categories collapse into the category SET of sets. On the other hand, starting with the category SET (=SET($\{0,1\}$)) (i.e. $L = \{0,1\}$) of sets and enriching it with respective fuzzy functions, we obtain again the category SET as $\mathcal{F}SET(L)^{\top}$ and obtain the category of sets and partial functions as $\mathcal{F}SET(L)^{\perp}$.

3.1. **Preimages of L-valued equalities under fuzzy functions.** Let an L-valued set (Y, E_Y), a set X and a mapping $F : X \times Y \to L$ be given. We are interested to find the largest L-valued equality E_X on X for which $F : (X, E_X) \to (Y, E_Y)$ is a fuzzy function. This L-valued equality will be called *the preimage of* E_Y *under* F and will be denoted $F^{\leftarrow}(E_Y)$.

Note first that the axioms

(1ff) $F(x,y) * E_Y(y,y') \le F(x,y')$, and

(3ff) $F(x,y) * F(x,y') \leq E(y,y')$

do not depend on the L-valued equality on X and hence we have to demand that the mapping F originally satisfies them. To satisfy the last axiom

(2ff) $E_X(x,x') * F(x,y) \leq F(x',y)$

in an "optimal way" we define

$$E_X(x,x') := \bigwedge_y \Big(\big(F(x,y) \longmapsto F(x',y)\big) \wedge \big(F(x',y) \longmapsto F(x,y)\big) \Big).$$

Then $E_X : X \times X \to L$ is an L-valued equality on X. Indeed, the validity of properties $E_X(x,x) = \top$ and $E_X(x,x') = E_X(x',x)$ is obvious. To establish the last property, i.e. $E_X(x,x') * E_X(x',x'') \leq E_X(x,x'')$, we proceed as follows:

$$\begin{aligned}
&E_X(x,x') * E_X(x',x'') \\
&\quad= \textstyle\bigwedge_y \Big(\big(F(x,y) \longmapsto F(x',y)\big) \wedge \big(F(x',y) \longmapsto F(x,y)\big) \Big) \\
&\qquad * \textstyle\bigwedge_y \Big(\big(F(x',y) \longmapsto F(x'',y)\big) \wedge \big(F(x'',y) \longmapsto F(x',y)\big) \Big) \\
&\quad\leq \textstyle\bigwedge_y \Big(\big(F(x,y) \longmapsto F(x',y)\big) \wedge \big(F(x',y) \longmapsto F(x,y)\big) \\
&\qquad * \big(F(x',y) \longmapsto F(x'',y)\big) \wedge \big(F(x'',y) \longmapsto F(x',y)\big) \Big) \\
&\quad\leq \textstyle\bigwedge_y \Big(\big(F(x,y) \longmapsto F(x',y)\big) * \big(F(x',y) \longmapsto F(x,y)\big) \\
&\qquad \wedge (\big(F(x',y) \longmapsto F(x'',y)\big) * \big(F(x'',y) \longmapsto F(x',y)\big) \Big) \\
&\quad\leq \textstyle\bigwedge_y \Big(\big(F(x,y) \longmapsto F(x'',y)\big) \wedge \big(F(x'',y) \longmapsto F(x,y)\big) \Big) \\
&\quad= E_X(x,x'').
\end{aligned}$$

Further, just from the definition of E_X it is clear that F satisfies the axiom *(2ff)* and hence it is indeed a fuzzy function $F : (X, E_X) \rightarrowtail (Y, E_Y)$. Moreover, from the definition of E_X it is easy to note that it is really the largest L-valued equality on X for which F satisfies axiom *(2ff)*.

Finally, note that the value $\mu(F)$ is an inner property of the mapping $F : X \times Y \to L$ and does not depend on L-valued equalities on these sets.

Question 3.1.1. We do not know whether the preimage $F^{\leftarrow}(E_Y)$ is the initial structure for the source $F : X \rightarrowtail (Y, E_Y)$ in $\mathcal{FSET}(L)$. Namely, given an L-valued set (Z, E_Z) and a "fuzzy quasi-function" $G : (Z, E_Z) \rightarrowtail X$ is it true that composition $F \circ G : (Z, E_Z) \rightarrowtail (Y, E_Y)$ is a fuzzy function if and only if $G : (Z, E_Z) \rightarrowtail (X, E_X)$ is a fuzzy function? By a fuzzy quasi-function we mean that G satisfies properties *(1ff)* and *(3ff)* which do not depend on the equality on X.

3.2. **Images of L-valued equalities under fuzzy functions.** Let an L-valued set (X, E_X), a set Y and a mapping $F : X \times Y \to L$ be given. We are interested to find the smallest L-valued equality E_Y on Y for which $F : (X, E_X) \to (Y, E_Y)$ is a fuzzy function. This L-valued equality will be called *the image of E_X under F* and will be denoted $F^{\rightarrow}(E_X)$.

Note first that the axiom

(2ff) $E_X(x,x') * F(x,y) \leq F(x',y)$

does not depend on the L-valued equality on Y and hence we have to demand that the mapping F originally satisfies it. Therefore we have to bother that F satisfies the remaining two axioms:

(1ff) $F(x,y) * E_Y(y,y') \leq F(x,y')$, and

(3ff) $F(x,y) * F(x,y') \leq E(y,y')$

These conditions can be rewritten in the form of the double inequality:

$$\begin{aligned} F(x,y) * F(x,y') &\leq E_Y(y,y') \\ &\leq \big(F(x,y') \longmapsto F(x,y)\big) \wedge \big(F(x,y) \longmapsto F(x,y')\big). \end{aligned}$$

Defining E_Y by the equality

$$E_Y(y,y') = \bigvee_x \big(F(x,y) * F(x,y')\big),$$

we shall obviously satisfy both of them. Moreover, it is clear that E_Y satisfies property (3ff) and besides E_Y cannot be diminished without loosing this property. Hence we have to show only that E_Y is indeed an L-valued equality. However, to prove this we need the assumption that $\sigma(F) = \top$, that is F is $\top$-surjective. Note that

$$E_Y(y,y) = \bigvee_x \big(F(x,y) * F(x,y)\big) \geq (\sigma(F))^2,$$

and hence the first axiom is justified in case $\sigma(F) = \top$.

The equality $E_Y(y,y') = E_Y(y',y)$ is obvious.

Finally, to establish the last property, we proceed as follows. Let $y, y', y'' \in Y$. Then

$$\begin{aligned} & E_Y(y,y') * E_Y(y',y'') \\ = \;& \textstyle\bigvee_x \big(F(x,y) * F(x,y')\big) \; * \; \bigvee_x \big(F(x,y') * F(x,y'')\big) \\ = \;& \textstyle\bigvee_{x,x'} \big(F(x,y) * F(x,y') * F(x',y') * F(x',y'')\big) \\ \leq \;& \textstyle\bigvee_{x,x'} \big(F(x,y) * E_X(x,x') * F(x',y'')\big) \\ \leq \;& \textstyle\bigvee_x \big(F(x,y) * F(x,y'')\big) \\ = \;& E(y,y''). \end{aligned}$$

Question 3.2.1. We do not know whether the image $F^{\rightarrow}(E_X)$ is the final structure for the sink $F : (X, E_X) \rightarrowtail Y$ in $\mathcal{F}SET(L)$ in case $\sigma(F) = \top$. Namely, given an L-valued set (Z, E_Z) and a "fuzzy almost-function" $G : Y \rightarrowtail (Z, E_Z)$ is it true that composition $F \circ G : (Z, E_Z) \rightarrowtail (Y, E_Y)$ is a fuzzy function if and only if $G : (Y, E_Y) \rightarrowtail (Z, E_Z)$ is a fuzzy function? By a fuzzy almost-function we mean that G satisfies property (2ff) which does not depend on the equality on Y.

3.3. **Products in $\mathcal{F}SET(L)^{\top}$.** Let $\mathcal{Y} = \{(Y_i, E_i) : i \in \mathcal{I}\}$ be a family of L-valued sets and let $Y = \prod_i Y_i$ be the product of the corresponding sets. We introduce the L-valued equality $E : Y \times Y \to L$ on Y by setting $E_Y(y,y') = \bigwedge_{i \in \mathcal{I}} E_i(y_i, {y'}_i)$ where $y = (y_i)_{i \in \mathcal{I}}$, $y' = ({y'}_i)_{i \in \mathcal{I}}$. Further, let $p_i : Y \to Y_i$ be the projection. Then the pair (Y, E) thus defined with the family of projections $p_i : Y \to Y_i$, $i \in \mathcal{I}$, is the product of the family $\mathcal{Y}$ in the category $\mathcal{FSET}(L)^\top$.

To show this notice first that, since the morphisms in this category are fuzzy functions, a projection $p_{i_0} : Y \to Y_{i_0}$ must be realized as the fuzzy function $p_{i_0} : Y \times Y_{i_0} \to L$ such that $p_{i_0}(y, y^0_{i_0}) = \top$ if and only if the i_0-coordinate of y is $y^0_{i_0}$ and $p_{i_0}(y, y^0_{i_0}) = \bot$ otherwise.

Next, let $F_i : (X, E_X) \rightarrowtail (Y_i, E_i)$, $i \in \mathcal{I}$ be a family of fuzzy functions. We define the fuzzy function $F : (X, E_X) \rightarrowtail (Y, E_Y)$ by the equality:

$$F(x, y) = \bigwedge_{i \in \mathcal{I}} F_i(x, y_i).$$

It is obvious that $\mu(F) = \top$ and hence F is in $\mathcal{FSET}(L)^\top$. Finally, notice that the composition

$$(X, E_X) \xrightarrow{F} (Y, E_Y) \xrightarrow{p_{i_0}} (Y_{i_0}, E_{i_0})$$

is the fuzzy function

$$F_{i_0} : (X, E_X) \rightarrowtail (Y_{i_0}, E_{i_0}).$$

Indeed, let $x^0 \in X$ and $y^0_{i_0} \in Y_{i_0}$. Then, taking into account that $\mu(F_i) = \top$ for all $i \in \mathcal{I}$, we get

$$\begin{array}{rcl} (p_{i_0} \circ F)(x^o, y^0_{i_0}) & = & \bigvee_{y \in Y} \big(p_{i_0}(y, y^0_{i_0}) \wedge F(x^0, y)\big) \\ & = & \bigvee_{y \in Y} \big(p_{i_0}(y, y^0_{i_0}) \wedge \bigwedge_{i \in \mathcal{I}} F_i(x^0, y_i)\big) \\ & = & F_{i_0}(x^0, y_{i_0}). \end{array}$$

Question 3.3.1. We do not know whether products in $\mathcal{FSET}(L)$ can be defined in a reasonable way.

3.4. **Coproducts in $\mathcal{FSET}(L)$.** Let $\mathcal{X} = \{(X_i, E_i) : i \in \mathcal{I}\}$ be a family of L-valued sets, let $X = \bigcup X_i$ be the disjoint sum of sets X_i. Further, let $q_i : X_i \to X$ be the inclusion map. We introduce the L-equality on X_0 by setting $E(x, x') = E_i(x, x')$ if $(x, x') \in X_i \times X_i$ for some $i \in \mathcal{I}$ and $E(x, x') = \bot$ otherwise (cf. [7]). An easy verification shows that (X, E) is the coproduct of $\mathcal{X}$ in $\mathcal{FSET}(L)$ and hence, in particular, in $\mathcal{FSET}(L)^\top$.

Indeed, given a family of fuzzy functions $F_i : (X_i, E_i) \to (Y, E_Y)$, let the fuzzy function

$$\oplus_{i \in \mathcal{I}} F_i : (X, E) \to (Y, E_Y)$$

be defined by

$$\oplus_{i \in \mathcal{I}} F_i(x, y) = F_{i_0}(x, y) \text{ whenever } x \in X_{i_0}.$$

Then for $x = x_{i_0} \in X_{i_0}$ we have

$$\begin{array}{rcl} \big(\oplus_{i \in \mathcal{I}} F_i \circ q_{i_0}\big)(x, y) & = & \bigvee_{x' \in X} \Big(q_{i_0}(x, x') \wedge \big(\oplus_{i \in \mathcal{I}} F_i(x', y)\big)\Big) \\ & = & F_{i_0}(x, y). \end{array}$$

3.5. **Subobjects in $\mathcal{FSET}(L)$.** Let (X,E) be an L-valued set, let $Y \subset X$ and let $e : Y \to X$ be the natural embedding. Further, let $E_Y := e^{\leftarrow}(E)$ be the preimage of the L-valued equality E. Explicitly, in this case this means that $E_Y(y,y') = E(y,y')$ for all $y,y' \in Y$. One can easily see that (Y, E_Y) is a subobject of (X,E) in the fuzzy category $\mathcal{FSET}(L)$.

4. Fuzzy category $\mathcal{FTOP}(L)$

4.1. Basic concepts.

Definition 4.1.1 (see [18], cf. also [4], [5], [11]). *A family $\tau_X \subset L^X$ of extensional L-sets*[3] *is called an L-topology on an L-valued set (X, E_X) if it is closed under finite meets, arbitrary joins and contains 0_X and 1_X. Corresponding triple (X, E_X, τ_X) will be called an L-valued L-topological space or just an L-topological space for short. A fuzzy function $F : (X, E_X, \tau_X) \rightarrowtail (Y, E_Y, \tau_Y)$ is called continuous if $F^{\leftarrow}(V) \in \tau_X$ for all $V \in \tau_Y$.*

L-topological spaces and continuous fuzzy mappings between them form the fuzzy category which will be denoted $\mathcal{FTOP}(L)$. Indeed, let

$$F\colon (X, E_X, \tau_X) \rightarrowtail (Y, E_Y, \tau_Y) \text{ and } G\colon (Y, E_Y, \tau_Y) \rightarrowtail (Z, E_Z, \tau_Z)$$

be continuous fuzzy functions and let $W \in \tau_Z$. Then

$$\begin{aligned} (G \circ F)^{\leftarrow}(W)(x) &= \bigvee_z \Big((G \circ F)(x,z) * W(z) \Big) \\ &= \bigvee_{z,y} \Big(F(x,y) * G(y,z) * W(z) \Big). \end{aligned}$$

On the other hand, $G^{\leftarrow}(W)(y) = \bigvee_z G(y,z) * W(z)$ and

$$\begin{aligned} F^{\leftarrow}\big(G^{\leftarrow}(W)\big)(x) &= \bigvee_y F(x,y) * \big(\bigvee_z (G(y,z) * W(z))\big) \\ &= \bigvee_{z,y} \Big(F(x,y) * G(y,z) * W(z) \Big). \end{aligned}$$

Thus $(G \circ F)^{\leftarrow}(W) = G^{\leftarrow}(F^{\leftarrow}(W))$ for every W, and hence composition of continuous fuzzy functions is continuous. Besides, we have seen already before that $\mu(G \circ F) \geq \mu(G) * \mu(F)$. Finally, $E_X^{\leftarrow}(B) = B$ for every *extensional* $B \in L^X$ and hence the identity mapping $E_X : (X, E_X, \tau_X) \rightarrowtail (X, E_X, \tau_X)$ is continuous.

Remark 4.1.2. In case when L-valued equality E_X is crisp, i.e. when X is an ordinary set, the above definition of an L-topology on X reduces to the "classical" definition of an L-topology in the sense of Chang and Goguen, [4], [5].

Remark 4.1.3. Some (ordinary) subcategories of the fuzzy category $\mathcal{FTOP}(L)$ will be of special interest for us. Namely, let $\mathcal{FTOP}(L)^{\perp} =:$ FTOP(L) denote the bottom frame of $\mathcal{FTOP}(L)$, let $\mathcal{FTOP}(L)^{\top}$ be the top frame of $\mathcal{FTOP}(L)$, and finally let L-TOP(L) denote the subcategory of $\mathcal{FTOP}(L)$

[3] Since L-topology is defined on an *L-valued set* X the condition of extensionality of elements of L-topology seems natural. Besides the assumption of extensionality is already implicitly included in the definition of a fuzzy function.

whose morphisms are ordinary functions. Obviously the "classical" category L-TOP of Chang-Goguen L-topological spaces can be obtained as a full subcategory L-TOP(L) whose objects carry crisp equalities. Another way to obtain L-TOP is to consider fuzzy subcategory of $\mathcal{FTOP}(L)$whose objects carry crisp equalities and whose morphisms satisfy condition $\mu(F) > \perp$.

In case when L is an MV-algebra and involution $^c : L \to L$ on L is defined in the standard way, i.e. $\alpha^c := \alpha \longmapsto \perp$ we can reasonably introduce the notion of a closed L-set in an L-topological space:

Definition 4.1.4. *An L-set A in an L-topological space (X, E_X, τ_X) is called closed if $A^c \in \tau_X$ where $A^c \in L^X$ is defined by the equality*

$$A^c(x) := A(x) \longmapsto \perp \quad \forall x \in X.$$

Let $\mathcal{C}_X$ denote the family of all closed L-sets in (X, E_X, τ_X). In case when L is an MV-algebra the families of sets τ_X and $\mathcal{C}_X$ mutually determine each other:

$$A \in \tau_X \Longleftrightarrow A^c \in \mathcal{C}_X.$$

4.2. **Analysis of continuity.** Since the operation of taking preimages $F^{\leftarrow}$ commutes with joins, and in case when $\mu(F) = \top$ also with meets (see Proposition 2.3.1), one can easily verify the following

Theorem 4.2.1. *Let (X, E_X, τ_X) and (Y, E_Y, τ_Y) be L-topological spaces, β_Y be a base of τ_Y, ξ_Y its subbase and let $F : X \rightarrowtail Y$ be a fuzzy function. Then the following are equivalent:*

(1con) *F is continuous;*

(2con) *for every $V \in \beta_Y$ it holds $F^{\leftarrow}(V) \in \tau_X$;*

(3con) *$F^{\leftarrow}(Int_Y(B)) \leq Int_X(F^{\leftarrow}(B))$, for every $B \in L^Y$ where Int_X and Int_Y are the corresponding L-interior operators on X and Y respectively.*

In case when $\mu(F) = \top$ these conditions are equivalent also to the following

(4con) *for every $V \in \xi_Y$ it holds $F^{\leftarrow}(V) \in \tau_X$.*

In case when L is an MV-algebra one can characterize continuity of a fuzzy function by means of closed L-sets and L-closure operators:

Theorem 4.2.2. *Let $(L, \leq, \vee, \wedge, *)$ be an MV-algebra, (X, E_X, τ_X) and (Y, E_Y, τ_Y) be L-topological spaces and $F : X \rightarrowtail Y$ be a fuzzy function. Further, let $\mathcal{C}_X$, $\mathcal{C}_Y$ denote the families of closed L-sets and cl_X, cl_Y denote the closure operators in (X, E_X, τ_X) and (Y, E_Y, τ_Y) respectively. Then the following two conditions are equivalent:*

(1con) *F is continuous;*

(5con) *For every $B \in \mathcal{C}_Y$ it follows $F^{\leftarrow}(B) \in \mathcal{C}_X$.*

In case when $\mu(F) = \top$, the previous conditions are equivalent to the following:

(6con) *For every $A \in L^X$ it holds $F^{\to}(cl_X(A)) \leq cl_Y(F^{\to}(A))$.*

Proof. In case when L is equipped with an order reversing involution, as it is in our situation, families of closed and open L-sets mutually determine each other. Therefore, to verify the equivalence of (1con) and (5con) it is sufficient to notice that for every $B \in L^Y$ and every $x \in X$ it holds

$$\begin{aligned} F^{\leftarrow}(B^c)(x) &= \bigvee_y \big(F(x,y) * (B(y) \longmapsto \bot)\big) \\ &= \bigvee_y \big(F(x,y) * B(y) \longmapsto \bot\big) \\ &= \big(\bigvee_y (F(x,y) * B(y))\big)^c \\ &= (F^{\leftarrow}(B))^c(x), \end{aligned}$$

and hence

$$F^{\leftarrow}(B^c) = (F^{\leftarrow}(B))^c \quad \forall B \in L^Y,$$

i.e. operation of taking preimages preserves involution.

To show implication (5con) $\Longrightarrow$ (6con) under assumption $\mu(F) = \top$ let $A \in L^X$. Then, according to Proposition 2.3.1 (5),

$$A \le F^{\leftarrow}(F^{\rightarrow}(A)) \le F^{\leftarrow}(cl_Y(F^{\rightarrow}(A))),$$

and hence, by (5con), also

$$cl_X(A) \le F^{\leftarrow}(cl_Y(F^{\rightarrow}(A))).$$

Now, by monotonicity of the image operator and by Proposition 2.3.1 (6) (taking into account that $cl_X A$ is extensional as a closed L-set), we get:

$$F^{\rightarrow}(cl_X(A)) \le F^{\rightarrow}\big(F^{\leftarrow}(cl_Y(F^{\rightarrow}(A)))\big) \le cl_Y(F^{\rightarrow}(A)).$$

Conversely, to show implication (6con) $\Longrightarrow$ (5con) let $B \in \mathcal{C}_Y$ and let $F^{\leftarrow}(B) := A$. Then, by (6con),

$$F^{\rightarrow}(cl_X(A)) \le cl_Y(F^{\rightarrow}(A)) \le cl_Y(B) = B.$$

In virtue of Proposition 2.3.1 (5) and taking into account that $\mu(F) = \top$, it follows from here that $cl_X(A) \le F^{\leftarrow}(B) = A$, and hence $cl_X(A) = A$. $\square$

4.3. **Fuzzy α-homeomorphisms and fuzzy α-homeomorphic spaces.** The following definition naturally stems from Definitions 2.4.1, 2.4.2 and 4.1.1 and item 2 of Proposition 2.3.1:

Definition 4.3.1. *Given L-topological spaces (X, E_X, τ_X) and (Y, E_Y, τ_Y), a fuzzy function $F : X \rightarrowtail Y$ is called a fuzzy α-homeomorphism if $\mu(F) \ge \alpha$, $\sigma(F) \ge \alpha$, it is injective, continuous, and the inverse fuzzy function $F^{-1} : Y \rightarrowtail X$ is also continuous. Spaces (X, E_X, τ_X) and (Y, E_Y, τ_Y) are called fuzzy α-homeomorphic if there exists a fuzzy α-homeomorphism $F : (X, E_X, \tau_X) \rightarrowtail (Y, E_Y, \tau_Y)$.*

One can easily verify that composition of two fuzzy α-homeomorphisms is a fuzzy α^2-homeomorphism; in particular, composition of fuzzy $\top$-homeomorphisms is a fuzzy $\top$-homeomorphism, and hence fuzzy $\top$-homeomorphisms determine the equivalence relation $\overset{\top}{\approx}$ on the class of all L-topological spaces. Besides, since every (usual) homeomorphism is obviously a fuzzy $\top$-homeomorphism, homeomorphic spaces are also fuzzy $\top$-homeomorphic:

$$(X, E_X, \tau_X) \approx (Y, E_Y, \tau_Y) \Longrightarrow (X, E_X, \tau_X) \overset{\top}{\approx} (Y, E_Y, \tau_Y).$$

The converse generally does not hold:

Example 4.3.2. Let L be the unit interval $[0,1]$ viewed as an MV-algebra (i.e. $\alpha * \beta = max\{\alpha+\beta-1, 0\}$, let (X, ϱ) be an uncountable separable metric space such that $\varrho(x, x') \leq 1 \ \forall x, x' \in X$, and let Y be its countable dense subset. Further, let the L-valued equality on X be defined by $E_X(x, x') := 1 - \varrho(x, x')$ and let E_Y be its restriction to Y. Let τ_X be any L-topology on an L-valued set (X, E_X) (in particular, one can take $\tau_X := \{c_X \mid c \in [0,1]\}$). Finally, let a fuzzy function $F : (X, E_X) \rightarrowtail (Y, E_Y)$ be defined by $F(x, y) := 1 - \varrho(x, y)$. It is easy to see that F is a $\top$-homeomorphism and hence $(X, E_X, \tau_X) \overset{\top}{\approx} (Y, E_Y, \tau_Y)$. On the other hand $(X, E_X, \tau_X) \not\approx (Y, E_Y, \tau_Y)$ just for set-theoretical reasons.

5. Category $\mathcal{F}TOP(L)^\top$

Let $\mathcal{F}TOP(L)^\top$ be the top-frame of $\mathcal{F}TOP(L)$, i.e. $\mathcal{F}TOP(L)^\top$ is a category whose objects are the same as in $\mathcal{F}TOP(L)$, that is L-topological spaces, and morphisms are continuous fuzzy functions $F : (X, E_X, \tau_X) \rightarrowtail (Y, E_Y, \tau_Y)$ such that $\mu(F) = \top$.

Note, that as different from the fuzzy category $\mathcal{F}TOP(L)$, $\mathcal{F}TOP(L)^\top$ is a usual category. Applying Theorem 4.2.1, we come to the following result:

Theorem 5.0.3. *Let (X, E_X, τ_X) and (Y, E_Y, τ_Y) be L-topological spaces, β_Y be a base of τ_Y, ξ_Y its subbase and let $F : X \rightarrowtail Y$ be a fuzzy function. Then the following conditions are equivalent:*

(1con) *F is continuous;*
(2con) *for every $V \in \beta_Y$ it holds $F^{\leftarrow}(V) \in \tau_X$;*
(3con) *$F^{\leftarrow}(Int_Y(B)) \leq Int_X(F^{\leftarrow}(B))$, where Int_X and Int_Y are the corresponding L-interior operators on X and Y respectively;*
(4con) *for every $V \in \xi_Y$ it holds $F^{\leftarrow}(V) \in \tau_X$.*

In case when L is an MV-algebra, we get from 4.2.2

Theorem 5.0.4. *Let $(L, \leq, \vee, \wedge, *)$ be an MV-algebra, (X, E_X, τ_X) and (Y, E_Y, τ_Y) be L-topological spaces and $F : X \rightarrowtail Y$ be a a morphism in $\mathcal{F}TOP(L)^\top$. Further, let $\mathcal{C}_X$, $\mathcal{C}_Y$ denote the families of closed L-sets and cl_X, cl_Y denote the closure operators on (X, E_X, τ_X) and (Y, E_Y, τ_Y) respectively. Then the following two conditions are equivalent:*

(1con) *F is continuous;*
(5con) *For every $B \in \mathcal{C}_Y$ it follows $F^{\leftarrow}(B) \in \mathcal{C}_X$;*
(6con) *For every $A \in L^X$ it holds $F^{\rightarrow}(cl_X(A)) \leq cl_Y(F^{\rightarrow}(A))$.*

Theorem 5.0.5.

$\mathcal{FTOP}(L)^{\top}$is a topological category over the category $\mathcal{FSET}(L)^{\top}$.

Proof. Since intersection of any family of L-topologies is an L-topology, $\mathcal{FTOP}(L)^{\top}$ is fiber complete. Therefore we have to show only that any structured source in $\mathcal{FSET}(L)^{\top}$ $F : (X, E_X) \rightarrowtail (Y, E_Y, \tau_Y)$ has a unique initial lift. Let

$$\tau_X := F^{\leftarrow}(\tau_Y) := \{F^{\leftarrow}(V) \mid V \in \tau_Y\}.$$

Then from theorem 2.3.1 it follows that τ_X is closed under taking finite meets and arbitrary joins. Furthermore, obviously $F^{\leftarrow}(0_Y) = 0_X$ and taking into account condition $\mu(F) = \top$ one easily establishes that $F^{\leftarrow}(1_Y) = 1_X$. Therefore, taking into account that preimages of extensional L-sets are extensional, (see Subsection 2.3) we conclude that the family τ_X is an L-topology on X.

Further, just from the construction of τ_X it is clear that $F : (X, E_X, \tau_X) \rightarrowtail (Y, E_Y, \tau_Y)$ is continuous and, moreover, τ_X is the weakest L-topology on X with this property.

Let now (Z, E_Z, τ_Z) be an L-topological space and $H : (Z, E_Z) \rightarrowtail (X, E_X)$ a fuzzy function such that the composition $G := H \circ F : (Z, E_Z, \tau_Z) \rightarrowtail (Y, E_Y, \tau_Y)$ is continuous. To complete the proof we have to show that H is continuous.

Indeed, let $U \in \tau_X$. Then there exists $V \in \tau_Y$ such that $U = F^{\leftarrow}(V)$. Therefore

$$H^{\leftarrow}(U) = H^{\leftarrow}\big(F^{\leftarrow}(V)\big) = G^{\leftarrow}(V) \in \tau_Z$$

and hence H is continuous. $\square$

5.1. **Products in $\mathcal{FTOP}(L)^{\top}$.** Our next aim is to give an explicite description of the product in $\mathcal{FTOP}(L)^{\top}$.

Given a family $\mathcal{Y} = \{(Y_i, E_i, \tau_i) : i \in \mathcal{I}\}$ of L-topological spaces, let (Y, E) be the product of the corresponding L-valued sets $\{(Y_i, E_i) : i \in \mathcal{I}\}$ in $\mathcal{FSET}(L)^{\top}$and let $p_i : Y \to Y_i$ be the projections.

Further, for each $U_i \in \tau_i$ let $\hat{U}_i := p_i^{-1}(U_i)$. Then the family $\xi := \{\hat{U}_i : U_i \in \tau_i, i \in \mathcal{I}\}$ is a subbase of an L-topology τ on the product L-valued set (X, E_X) which is known to be the product L-topology for L-topological spaces $\{(Y_i, \tau_i) \mid i \in \mathcal{I}\}$ in the category L-TOP. In its turn the triple (Y, E, τ) is the product of L-topological spaces $\{(Y_i, E_i, \tau_i) : i \in \mathcal{I}\}$ in the category $\mathcal{FTOP}(L)^{\top}$. Indeed, let (Z, E_Z, τ_Z) be an L-topological space and $\{F_i : Z \rightarrowtail Y_i \mid i \in \mathcal{I}\}$ be a family of continuous fuzzy mappings. Then, defining a mapping $F : Z \times Y \to L$ by $F(z, y) = \wedge_{i \in \mathcal{I}} F_i(z, y_i)$ we obtain a fuzzy function $F : Z \rightarrowtail Y$ such that $\mu(F) = \bigwedge_{i \in \mathcal{I}} \mu(F_i) = \top$ and besides it can be easily seen that for every $i_0 \in \mathcal{I}$, every $z \in Z$, and for every $U_{i_0} \in \tau_{i_0}$ it holds:

$$\begin{aligned} F^{\leftarrow}(\hat{U}_{i_0})(z) &= \bigvee_{y\in Y}\Big(\big(\bigwedge_{i\in\mathcal{I}} F_i(z,y_i)\big) * U_i(y_i)\Big) \\ &= \bigvee_{y_i\in Y_i}\Big(\bigwedge_{i\in\mathcal{I}}\big(F_i(z,y_i)\big) * U_{i_0}(y_{i_0})\Big) \\ &= \bigwedge_{i\in\mathcal{I}}\Big(\bigvee_{y_i\in Y_i} F_i(z,y_i)\Big) \wedge \bigvee_{y_{i_0}\in X_{i_0}}\big(F_{i_0}(z,y_{i_0}) * U_{i_0}(y_{i_0})\big) \\ &= \top \wedge F_{i_0}^{\leftarrow}(U_{i_0})(z) \\ &= F_{i_0}^{\leftarrow}(U_{i_0})(z). \end{aligned}$$

Hence continuity of all F_i guarantees the continuity of $F : (Z, E_Z, \tau_Z) \rightarrowtail (Y, E, \tau)$. Thus (Y, E, τ) is indeed the product of $\mathcal{Y} = \{(Y_i, E_i, \tau_i) : i \in \mathcal{I}\}$ in $\mathcal{FSET}(L)^{\top}$.

Question 5.1.1. We do not know whether products in the fuzzy category $\mathcal{FTOP}(L)$ exist.

5.2. **Subspaces in $\mathcal{FTOP}(L)^{\top}$.** Let (X, E, τ) be an L-valued L-topological space, $Y \subset X$ and let (Y, E_Y) be the subobject of the L-valued set (X, E) in the category $\mathcal{FSET}(L)$. Further, let τ_Y be the subspace L-topology, that is (Y, τ_Y) is a subspace of the L-topological space (X, τ) in the category L-TOP. Then it is clear that the triple (Y, E_Y, τ_Y) is the subobject of (X, E, τ) in the category $\mathcal{FTOP}(L)^{\top}$(and in the fuzzy category $\mathcal{FTOP}(L)$ as well).

5.3. **Coproducts in $\mathcal{FTOP}(L)^{\top}$.** Given a family $\mathcal{X} := \{(X_i, E_i, \tau_i)\}$ of L-topological spaces let (X, E) be the direct sum of the corresponding L-valued sets (X_i, E_i) in $\mathcal{FSET}(L)$. Further, let τ be the L-topology on X determined by the subbase $\bigcup_{i\in\mathcal{I}} \tau_i \subset 2^X$. In other words (X, τ) is the coproduct of L-topological spaces (X_i, τ_i) in the category L-TOP. Then the triple (X, E, τ) is the coproduct of the family $\mathcal{X} := \{(X_i, E_i, \tau_i)\}$ in the category $\mathcal{FTOP}(L)^{\top}$(and in the fuzzy category $\mathcal{FTOP}(L)$as well). Indeed, let $q_i : (X_i, E_i, \tau_i) \to (X, E, \tau), i \in \mathcal{I}$ denote the canonical embeddings. Further, consider an L-topological space (Y, E_Y, τ_Y) and a family of continuous fuzzy functions $F_i : (X_i, E_i, \tau_i) \rightarrowtail (Y, E_Y, \tau_Y)$. Then, by setting $F(x, y) := F_i(x_i, y)$ whenever $x = x_i \in X_i$, we obtain a continuous fuzzy function $F : (X, E, \tau) \rightarrowtail (Y, E_Y, \tau_Y)$ (i.e. a mapping $F : X \times Y \to L$) such that $F_i = q_i \circ F$ for every $i \in \mathcal{I}$.

5.4. **Quotients in $\mathcal{FTOP}(L)^{\top}$.** Let (X, E_X, τ_X) be an L-topological space, let $q : X \to Y$ be a surjective mapping. Further, let $q^{\to}(E_X) =: E_Y$ be the image of the L-valued equality E_X and let $\tau_Y = \{V \in L^Y \mid q^{-1}(V) \in \tau_X\}$, that is τ_Y is the quotient L-topology determined by the mapping $q : (X, \tau) \to Y$ in the category L-TOP. Then (Y, E_Y, τ_Y) is the quotient object in the category $\mathcal{FTOP}(L)^{\top}$. Indeed, consider a fuzzy function $F : (X, E_X, \tau_X) \rightarrowtail (Z, E_Z, \tau_Z)$ and let $G : (Y, E_Y) \rightarrowtail (Z, E_Z)$ be a morphism in $\mathcal{FSET}(L)$ such that $q \circ G = F$. Then an easy verification shows that the fuzzy function $G : (Y, E_Y, \tau_Y) \rightarrowtail (Z, E_Z, \tau_Z)$ is continuous (i.e. a morphism in $\mathcal{FTOP}(L)^{\top}$) if and only if $F : (X, E_X, \tau_X) \to (Z, E_Z, \tau_Z)$ is continuous (i.e. a morphism in $\mathcal{FTOP}(L)^{\top}$).

Our next aim is to consider the behaviour of some topological properties of L-valued L-topological spaces in respect of fuzzy function. In this work we restrict our interest to the property of compactness. Some other topological properties, in particular, connectedness and separation properties will be studied in a subsequent work.

6. Compactness

6.1. **Preservation of compactness by fuzzy functions.** One of the basic facts of general topology — both classic and "fuzzy", is preservation of compactness type properties by continuous mappings. Here we present a counterpart of this fact in $\mathcal{F}TOP(L)$. However, since in literature on fuzzy topology different definitions of compactness can be found, first we must specify which one of compactness notions will be used.

Definition 6.1.1. *An L-topological space (X, E, τ) will be called (α, β)-compact where $\alpha, \beta \in L$, if for every family $\mathcal{U} \subset \tau$ such that $\bigvee \mathcal{U} \geq \alpha$ there exists a finite subfamily $\mathcal{U}_0 \subset \mathcal{U}$ such that $\bigvee \mathcal{U}_0 \geq \beta$. An (α, α)-compact space will be called just α-compact.*[4]

Theorem 6.1.2. *Let (X, E_X, τ_X), (Y, E_Y, τ_Y) be L-topological spaces, $F : X \rightarrowtail Y$ be a continuous fuzzy function such that $\mu(F) \geq \beta$, and $\sigma(F) \geq \gamma$. If X is $\alpha * \beta$-compact, then Y is $(\alpha, \alpha * \beta * \gamma)$-compact.*

Proof. Let $\mathcal{V} \subset \tau_Y$ be such that $\bigvee \mathcal{V} \geq \alpha$. Then, applying Proposition 2.3.1 (4), (7) and taking in view monotonicity of $F^{\leftarrow}$, we get

$$\bigvee_{V \in \mathcal{V}} F^{\leftarrow}(V) = F^{\leftarrow}\Big(\bigvee_{V \in \mathcal{V}} V\Big) \geq F^{\leftarrow}(\alpha) \geq \alpha * \beta.$$

Now, since (X, E_X, τ_X) is $\alpha * \beta$-compact, it follows that there exists a finite subfamily $\mathcal{V}_0 \subset \mathcal{V}$ such that

$$\bigvee_{V \in \mathcal{V}_0} F^{\leftarrow}(V) \geq \alpha * \beta.$$

Applying Propositions 2.3.1 (6),(4) and 2.4.4 (5) we obtain:

$$\bigvee_{V \in \mathcal{V}_0} V \geq F^{\rightarrow}\Big(F^{\leftarrow}\big(\bigvee_{V \in \mathcal{V}_0} V\big)\Big) = F^{\rightarrow}\Big(\bigvee_{V \in \mathcal{V}_0} \big(F^{\leftarrow}(V)\big)\Big) \geq F^{\rightarrow}\big(\alpha * \beta\big) \geq \alpha * \beta * \gamma.$$

□

Corollary 6.1.3. *Let (X, E_X, τ_X), (Y, E_Y, τ_Y) be L-topological spaces, $F : X \rightarrowtail Y$ be a fuzzy function such that $\mu(F) = \top$ and $\sigma(F) = \top$. If X is α-compact, then Y is also α-compact.*

---

[4]Note that Chang's definition of compactness [4] for a $[0, 1]$-topological space is equivalent to our 1-compactness. An $[0, 1]$-topological space is compact in Lowen's sense [13] if it is (α, β)-compact for all $\alpha \in [0, 1]$ and all $\beta < \alpha$

6.2. **Compactness in case of an MV-algebra.** In case L is an MV-algebra one can characterize compactness by systems of closed L-sets:

Proposition 6.2.1. *Let (X, E_X, τ_X) be an L-topological space and let $\mathcal{C}_X$ be the family of its closed L-sets. Then the space (X, E_X, τ_X) is (α, β)-compact if and only if for every $\mathcal{A} \subset \mathcal{C}_X$ the following implication follows:*

$$\text{if } \bigwedge_{A \in \mathcal{A}_0} A \not\leq \beta^c \text{ for every finite family } \mathcal{A}_0 \subset \mathcal{A}, \text{ then } \bigwedge_{A \in \mathcal{A}} A \not\leq \alpha^c.$$

Proof. One has just to take involutions "$\longmapsto \perp$" in the definition of (α, β)-compactenss and apply De Morgan law. $\square$

6.3. **Perfect mappings: case of an MV-algebra L.** In order to study preservation of compactness by preimages of fuzzy functions we introduce the property of (α, β)-perfectness of a fuzzy function. Since we shall operate with closed L-sets, from the beginning it will be assumed that L is an MV-algebra.

First we shall extend the notion of compactness for L-subsets of L-topological spaces. We shall say that an L-set $S : X \to L$ is (α, β)-compact if for every family $\mathcal{A}$ of closed L-sets of X the following implication holds:

$$\text{if } S \wedge \big(\bigwedge_{A \in \mathcal{A}_0} A\big) \not\leq \beta^c \text{ for every finite } \mathcal{A}_0 \in \mathcal{A} \text{ then } S \wedge \big(\bigwedge_{A \in \mathcal{A}} A\big) \not\leq \alpha^c.$$

Further, since the preimage $F^{\leftarrow}(y_0) : X \to L$ of a point $y_0 \in Y$ under a fuzzy function $F : X \rightarrowtail Y$ is obviously determined by the equality

$$F^{\leftarrow}(y_0)(x) = \bigvee_{y \in Y} F(x, y) * y_0(y) = F(x, y_0),$$

the general definition of (α, β)-compactness of an L-set in this case means the following:

The preimage $F^{\leftarrow}(y_0)$ of a point y_0 under a fuzzy function F is (α, β)-compact if for every family $\mathcal{A}$ of closed sets of X the following implication holds:

$$\bigvee_x \Big(F(x, y_0) \wedge \big(\bigwedge_{A \in \mathcal{A}_0} A(x)\big)\Big) \not\leq \beta^c \quad \forall \mathcal{A}_0 \subset \mathcal{A}, |\mathcal{A}_0| < \aleph_0$$

implies

$$\bigvee_x \big(F(x, y_0) \wedge \big(\bigwedge_{A \in \mathcal{A}} A(x)\big)\big) \not\leq \alpha^c.$$

Now we can introduce the following

Definition 6.3.1. *A continuous fuzzy mapping $F \colon (X, E_X, \tau_X) \rightarrowtail (Y, E_Y, \tau_Y)$ is called (α, β)-perfect if*

- *F is closed, i.e. $F^{\to}(A) \in \mathcal{C}_Y$ for every $A \in \mathcal{C}_X$;*
- *the preimage $F^{\leftarrow}(y)$ of every point $y \in Y$ is (α, β)-compact.*

Theorem 6.3.2. *Let $F : (X, E_X, \tau_X) \rightarrowtail (Y, E_Y, \tau_Y)$ be an (α,γ)-perfect fuzzy function such that $\mu(F) = \top$ and $\sigma(F) = \top$. If the space (Y, E_Y, τ_Y) is (γ,β)-compact, then the space (X, E_X, τ_X) is (α,β)-compact.*

Proof. Let $\mathcal{A}$ be a family of closed L-sets in X such that $\bigwedge_{A\in\mathcal{A}_0} A \not\leq \beta^c$. Without loss of generality we may assume that $\mathcal{A}$ is closed under taking finite meets. Let $B := B_A := F^{\rightarrow}(A)$ and let $\mathcal{B} := \{B_A : A \in \mathcal{A}\}$. Then, since $\mu(F) = \top$, by 2.3.1 (7) it follows that $B \not\leq \beta^c$ $\forall B \in \mathcal{B}$, and moreover, since $\mathcal{A}$ is assumed to be closed under finite meets,

$$\begin{aligned} B_{A_1} \wedge \ldots \wedge B_{A_n} &= F^{\rightarrow}(A_1) \wedge \ldots \wedge F^{\rightarrow}(A_n) \\ &\geq F^{\rightarrow}(A_1 \wedge \ldots \wedge A_n) \\ &= F^{\rightarrow}(A), \end{aligned}$$

for some $A \in \mathcal{A}$, and hence $\bigwedge_{B\in\mathcal{B}_0}(B) \not\leq \beta^c$ for every finite subfamily $\mathcal{B}_0 \subset \mathcal{B}$. Hence, by (γ,β)-compactness of the space (Y, E_Y, τ_Y) we conclude that $\bigwedge_{B\in\mathcal{B}}(B) \not\leq \gamma^c$, and therefore there exists a point $y_0 \in Y$ such that $F^{\rightarrow}(A)(y_0) = B_A(y_0) \not\leq \gamma^c$ for all $A \in \mathcal{A}$. Now, applying (α,γ)-compactness of the preimage $F^{\leftarrow}(y_0)$ and recalling that $\mathcal{A}$ was assumed to be closed under taking finite meets, we conclude that

$$\bigvee_x \big(F(x, y_0) \wedge (\bigwedge_{A\in\mathcal{A}} A(x))\big) \not\leq \alpha^c.$$

and hence, furthermore,

$$\bigwedge_{A\in\mathcal{A}} A \not\leq \alpha^c.$$

□

References

[1] Jiří Adámek, Horst Herrlich, and George E. Strecker, *Abstract and concrete categories*, John Wiley & Sons Inc., New York, 1990. MR **91h:**18001

[2] C. C. Chang, *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc. **88** (1958), 467–490. MR 20 #821

[3] ———, *A new proof of the completeness of the łukasiewicz axioms*, Trans. Amer. Math. Soc. **93** (1959), 74–80. MR 23 #A58

[4] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190. MR 38 #5153

[5] J. A. Goguen, *The fuzzy Tychonoff theorem*, J. Math. Anal. Appl. **43** (1973), 734–742. MR 49 #6116

[6] Ulrich Höhle, *Monoidal closed categories, weak topoi and generalized logics*, Fuzzy Sets and Systems **42** (1991), no. 1, 15–35. MR **92i:**03071

[7] ———, *M-valued sets and sheaves over integral commutative* cl-*monoids*, Applications of category theory to fuzzy subsets (Linz, 1989), Kluwer Acad. Publ., Dordrecht, 1992, pp. 33–72. MR **93c:**03086

[8] ———, *Commutative, residuated l-monoids*, Non-classical logics and their applications to fuzzy subsets (Linz, 1992), Kluwer Acad. Publ., Dordrecht, 1995, pp. 53–106. MR **96f:**03052

[9] Ulrich Höhle, Hans-E. Porst, and Alexander P. Šostak, *Fuzzy functions: A fuzzy extension of the category set*, In: FSTA 2000 (Abstracts of a conference), Liptovski Mikulaš, Slovakia, January 3–February 4, 2000.

[10] ______, *Fuzzy functions: a fuzzy extension of the category SET and some related categories*, Appl. Gen. Topol. **1** (2000), no. 1, 115–127. MR **2001j:**03097

[11] Ulrich Höhle and Alexander P. Šostak, *Axiomatic foundations of fixed-basis fuzzy topology*, Mathematics of fuzzy sets, Kluwer Acad. Publ., Boston, MA, 1999, pp. 123–272. MR 1 788 903

[12] Peter T. Johnstone, *Stone spaces*, Cambridge University Press, Cambridge, 1982. MR **85f:**54002

[13] R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, J. Math. Anal. Appl. **56** (1976), no. 3, 621–633. MR 55 #13357

[14] Alexander P. Šostak, *Towards the concept of a fuzzy category*, Mathematics (Russian), Latv. Univ., Riga, 1991, pp. 85–94. MR **93h:**18006

[15] ______, *On a concept of a fuzzy category*, 1992, 14th Linz Seminar on Fuzzy Set Theory: Non-Classical Logics and Their Applications, Linz, Austria, pp. 63–66.

[16] ______, *Fuzzy categories versus categories of fuzzily structured sets: Elements of the theory of fuzzy categories*, Categorical Methods in Algebra and Topology (A collection of papers in honor of Horst Herrlich), (Bremen, August 1997), Hans-E. Porst ed., Mathematik-Arbeitspapiere, no. 48, Universität Bremen, 1997, pp. 407–438.

[17] ______, *Fuzzy categories related to algebra and topology*, Tatra Mt. Math. Publ. **16** (1999), no. , part I, 159–185, Fuzzy sets, Part I (Liptovský Ján, 1998). MR 1 725 294

[18] ______, *Fuzzy functions as morphisms in an extension of the category l-top*, CAT-MAT2000: Proc. Conf. Categorical Methods in Algebra and Topology (Bremen 2000), Mathematik-Arbeitspapiere, no. 54, Universität Bremen, 2000, pp. 385–402.

[19] ______, *Fuzzy functions as morphisms in an extension of the category $Unif(L)$ of Hutton L-uniform spaces*, Vestn. Kyrg. Gos. Ped. Univ. Ser. Mat. Fiz. Inform. (2001).

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