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COMPACTIFICATION OF A MAP WHICH IS MAPPED TO ITSELF

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Abstract. We prove that if $T : X \to X$ is a selfmap of a set X such that $\bigcap\{T^n X : n \in \mathbb{N}\}$ is a one-point set, then the set X can be endowed with a compact Hausdorff topology so that T is continuous.

1. Introduction

If (X, d) is a compact metric space and $T : X \to X$ is a Banach contraction (there is $c \in [0, 1)$ such that $d(Tx, Ty) \leq cd(x, y)$ for all $x, y \in X$) then the iterates T^n shrink X to a point x^*, the unique fixed point of T. Thus X and T satisfy the set theoretical condition

$$\bigcap\{T^n X : n \in \mathbb{N}\} = \{x^*\} \tag{1.1}$$

In the late 1960's J. deGroot asked about a converse to this, namely if X is an abstract set with cardinality at most that of the continuum and T a selfmap satisfying (1.1) must there be a compact metric d on X so that T is a Banach contraction? This and related questions have been examined see e.g. [1, 3, 4, 5, 2]. In [4] the second author constructed a totally bounded metric whenever (1.1) is satisfied, but the question of compactness of X remained open. In [5] A. Kuběna showed that a compact metric cannot exist by constructing $2^{\mathfrak{c}}$ models of mutually nonisomorphic systems (X_i, T_i) satisfying (1.1) and showing that the cardinality of mutually nonisomorphic compactified systems (X_i, τ_i, T_i) cannot exceed $\mathfrak{c}$.

As a result of this example, A. Iwanik, who only recently passed away, asked if the conjecture of deGroot would be true with the metrizability condition and the cardinality restriction removed. Indeed he, together with the second and third authors, proved the following:

Main Theorem. *If X is a set and $T : X \to X$ satisfies the condition (1.1), then X can be given a compact Hausdorff topology so that T is continuous.*

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In that which follows we provide a simplified proof of this theorem.

2. Auxiliary Lemmas

The main and dominating idea of compactification of sets respecting maps between them comes to light by the following.

Lemma 2.1. *If X and Y are disjoint sets and $T : X \to Y$ a surjective map, then there are compact Hausdorff topologies on X and Y so that T is continuous.*

Proof. Using the axiom of choice we well order the set Y so that it has the last element and endow Y with the corresponding order topology. Thus Y becomes a compact Hausdorff space. Doing the same with each set $T^{-1}y$ for $y \in Y$ we "lift" the topology from Y to X ordering X lexicographically according to the order of Y. Thus X becomes also a compact Hausdorff space and since T is evidently order-preserving it is continuous. □

In the sequel we shall refer to a compact Hausdorff topology obtained by this method as a w.o. topology (well order topology).

This result and the technique in the proof will be applied systematically many times in the sequel. However, there is an obstacle to overcome. The maps we shall deal with are not surjective in general so that Lemma 2.1 is not readily applicable. We must first partition the domains and targets of those maps into a finite number of parts, called "atoms" so that the atoms will be mapped onto atoms. We introduce some definitions concerning partitions of sets and their behavior under mappings. By a partition π of a set X we mean a pairwise disjoint family of sets $\{C^i : i \in I\}$ such that $\bigcup \{C^i : i \in I\} = X$. If π_1, and π_2 are partitions then $\pi_1 \leq \pi_2$ means that π_1 refines π_2 and $\pi_1 \wedge \pi_2$ will denote the common refinement of π_1 and π_2. If $T : X \to Y$ is a map between two disjoint sets X and Y and if λ is a partition of Y defined by $\lambda = \{D^j : j \in J\}$ then $T^{-1}\lambda$ shall denote the partition of X defined by $\{T^{-1}D^j : j \in J\}$.

Definition 2.2. Let $T : X \to Y$ be a map between the disjoint sets X and Y and let π be a partition of X given by $\pi = \{C^i : i \in I\}$. We denote by $T\pi$ the partition of Y given by $T\pi = \{D^j : j \in J\}$ where D^j are classes of the equivalence relation $\sim$ defined on Y by setting $y_1 \sim y_2$ if the sets $\{i : T^{-1}y_1 \bigcap C^i \neq \emptyset\}$ and $\{i : T^{-1}y_2 \bigcap C^i \neq \emptyset\}$ coincide. It is evident that if π is finite so is $T\pi$.

Definition 2.3. If X, Y and $T : X \to Y$ are as above and if π and λ are partitions of X and Y, respectively, we say that π and λ are *T-related* if every class of π is mapped under T onto some class of λ. If π and λ are finite this is the desired situation. We say in this case that π and λ atomise X and Y, respectively.

Lemma 2.4. *Let $T : X \to Y$ be as above and suppose that π is a finite partition of X and λ is a finite refinement of $T\pi$. Then the partitions $T^{-1}\lambda \wedge \pi$ and λ are T-related.*

Proof. Let $\pi = \{C^i : i = 1, \cdots n\}$, $T\pi = \{D^j : j = 1, \cdots m\}$ and $\lambda = \{A^k : k = 1, \cdots r\}$.

Every class of $T^{-1}\lambda \wedge \pi$ is a nonempty set of the form $T^{-1}A^k \bigcap C^i$. This implies that there is some $a_1 \in A^k$ with

$$T^{-1}a_1 \bigcap C^i \neq \emptyset. \tag{2.1}$$

Since λ is a refinement of $T\pi$ there is some j for which $A^k \subseteq D^j$. From this it follows that all elements of A^k are equivalent under the relation $\sim$ on Y induced by π (Definition 2.2). From this and 2.1 it follows that $T^{-1}a \bigcap C^i \neq \emptyset$ for every $a \in A^k$ implying that $T(T^{-1}A^k \bigcap C^i) = A^k$ which concludes our proof. □

Lemma 2.5. *Let $T : X \to Y$ be as above and suppose π and λ are finite T-related partitions of X and Y, respectively. Then one can put on X and Y compact Hausdorff topology so that*

(i) *Each class of π and λ is compact*
(ii) *T is continuous.*

Proof. Let $\pi = \{A^i : i = 1, \cdots n\}, \lambda = \{B^j : j = 1, \cdots m\}$. We compactify Y by putting on each set B^j a w.o. topology and for each $i = 1, \cdots n$ we apply the techniques in the proof of Lemma 2.1 to lift this topology to a topology on A^i. Continuity of T follows from continuity of each of its restriction to the $A^i, i = 1, \cdots n$. □

Definition 2.6. Let $X_n, n = 1, \cdots, N, N \in \mathbb{N}$ be disjoint sets and $T_n : X_n \to X_{n-1}, n = 2, \cdots, N$ be maps. We shall call such finite family of sets and maps a chain of sets and denote it by $\{X_n, T_n\}_1^N$.

Lemma 2.7. *For every chain $\{X_n, T_n\}_1^N$ there exist finite partitions λ_1, $\ldots$, λ_N of X_1, $\ldots$, X_N, respectively so that λ_n and λ_{n-1} are T_n-related for every $n = 2, \cdots, N$.*

Proof. We define inductively finite partitions π_n of $X_n, n = 1, \cdots N$ as follows. Starting with π_N we set $\pi_N = \{X_N\}$ and if π_n is already defined we set $\pi_{n-1} = T_n\pi_n$. Thus $\pi_1, \cdots \pi_N$ are finite partitions of $X_1, \cdots, X_N$, respectively and we define the partition λ_1 of X_1 as π_1. We define λ_2 on X_2 as $T_2^{-1}\lambda_1 \wedge \pi_2$ and if λ_n is already defined we define λ_{n+1} as $T_{n+1}^{-1}\lambda_n \wedge \pi_{n+1}$. From the fact that $\lambda_n \leq \pi_n$ for $n = 1, \cdots N$ and Lemma 2.4 we conclude that the partitions $\lambda_n, n = 1, \cdots, N$ have the desired property, i.e., they "atomize" the chain $\{X_n, T_n\}_1^N$. □

From this result and Lemma 2.5 we obtain the following result:

Theorem 2.8. *Any finite chain $\{X_n, T_n\}_1^N$ of sets and maps can be compactified in the sense that one can put on each X_n a compact Hausdorff topology so that the maps $T_n : X_n \to X_{n-1}, n = 2, \cdots N$ are continuous.*

Proof. By partitioning each X_n by λ_n as described by Lemma 2.4. we apply the argument used in the proof ofLemma 2.7 sequentially to the maps $T_2, \cdots T_n$. □

3. Proof of the Main Theorem

Let $T : X \to X$ satisfy the condition (1.1). This implies that the T-orbits $O(x) = \{T^n x : n \in \mathbb{N}\}$ are either infinite or finite and in the latter case contain the fixed point x^*. This allows us to visualize the system (X, T) as a tree or more precisely as a forest of trees. The individual trees will be defined as classes corresponding to the equivalence relation $\sim$ on the set $X \backslash \{x^*\}$ defined by setting $x \sim y$ if there are $n \geq 0, m \geq 0$ such that $T^n x = T^m y \neq x^*$. If A is a class we say that A is a class of the first kind if it contains an element z of the set $T^{-1}x^* \backslash \{x^*\}$. In this case the class A can be evidently represented as the disjoint union of sets $\{T^{-n}z : n \geq 0\}$; i.e.,

$$A = \bigcup \{T^{-n}z : n \geq 0\} \tag{3.1}$$

We note that this family of sets $\{T^{-n}z : n \geq 0\}$ is finite since if for every n there existed a solution x to the equation $T^n x = z$, (1.1) would imply that $z = x^*$.

We now compactify the class A by applying Theorem refcompactchain to the chain $\{T^{-n}z, T_n\}$ where T_n is defined as the restriction of T to $T^{-n}z$, $n \geq 1$.

Any class which is not of the first kind will be called one of the second kind. If A is of the second kind, let $a \in A$. Now $O(a) = \{b_n = T^n a : n \geq 0\}$ is infinite. For each $b_n \in O(a)$ let

$$B_n^k = \{x \in X : T^k x = b_n \text{ and } T^{k-1}x \notin O(a)\}. \tag{3.2}$$

For each fixed n, $\{B_n^k : k \geq 0\}$ forms a finite disjoint family, since if not, (1.1) implies $b_n = x^*$. We now apply Theorem 2.8 to the chain $\{B_n^k, T_k : k \geq 1\}$, where $T_k : B_n^k \to B_n^{k-1}$ is T restricted to B_n^k. Thus, we compactify the set $B_n = \bigcup \{B_n^k : k \geq 0\}$ which is the whole branch of the tree A growing out of $T^n a$. The whole tree A is now evidently the disjoint union of the sets B_n which implies that A receives a locally compact topology. Thus each class of the first kind is compact and each class of the second kind is locally compact. This implies that the set $X \backslash \{x^*\}$ has a locally compact topology as the disjoint union of locally compact spaces. We now compactify it by adding the point x^*. The map T is continuous on $X \backslash \{x^*\}$ since its restriction to each compact subset is. We now must show its continuity at x^*. This reduces to showing that for each open set U containing x^* the pre-image $T^{-1}U$ is again open. The set U is of the form $X \backslash C$ where C is a compact subset of $X \backslash x^*$. We observe by inspection easily that $T^{-1}C$ is again compact, so that $T^{-1}(X \backslash C) = T^{-1}X \backslash T^{-1}C = X \backslash T^{-1}C$ which is again an open neighborhood of x^* and which concludes the proof of continuity of T, and with it the proof of our theorem.

Corollary 3.1. *The deGroot conjecture is true if the set X is countable.*

Proof. This follows from the fact that a countable compact Hausdorff space is metrizable. □

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