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SPECIAL METRICS

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Abstract. This is a survey on special metrics. We shall present some results and open questions on special metrics mainly appeared in the last 10 years.

1. Introduction

This is a survey on special metrics. We shall present some results and open questions on special metrics mainly appeared in the last 10 years. We shall consider *metrizable spaces* and all metrics induce the original topology of a given metrizable space. For a metrizable space X, there are many metrics which induce the original topology of X. Some of them may determine topological properties of X. For example, it is well known that a metrizable space X is *separable* if and only if X admits a *totally bounded* metric, and X is *compact* if and only if X admits a *complete totally bounded* metric.

2. Properties related to midsets

Let (X, ρ) be a metric space and $y \neq z$ be distinct points of X. A set of the form

$$M(y,z) = \{x \in X : \rho(x,y) = \rho(x,z)\}$$

is called a **midset** or a **bisector**.

Midsets are a geometrically intuitive concept, and several topological properties can be approached through midsets.

For example, the covering dimension of a separable metrizable space X can be characterized by midsets:

Theorem 1 (Janos-Martin [11]). *A separable metrizable space X has $\dim X \leq n$ if and only if X admits a totally bounded metric ρ such that $\dim M \leq n-1$ for every midset M in X.*

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Furthermore, metrics which have special midsets may determine the topological structures of spaces.

Definition 1. For a natural number n, a metric space (X, ρ) is said to have the **n-points midset property**, abbreviated as the **n-MP**, if every midset in X has exactly n points.

The 1-MP, 2-MP and 3-MP are sometimes called the **unique midset property**, **double midset property** and **triple midset property**, abbreviated as the **UMP**, **DMP** and **TMP**, respectively.

The real line with the usual metric is an example of the space having the UMP, and the circle in the two-dimensional Euclidean plane is an example of the space having the DMP.

Berard [2] proved that a connected metric space with the UMP is homeomorphic to an interval. Further, Nadler [23] proved that every non-degenerate component of a metric space with the UMP is homeomorphic to an interval and that a separable, locally compact metric space with the UMP is homeomorphic to a subspace of the real line. However, the following question remains open.

Question 1. Is a separable metric space with the UMP homeomorphic to a subspace of the real line?

Concerning the question above, we have the following example.

Example 1 (Hattori-Ohta [9])**.** There is a separable metric space (X, ρ) such that the cardinality of any midset is at most 1, but X is not suborderable, and hence X is not homeomorphic to a subspace of the real line.

Metrizable spaces that are homeomorphic to subspaces of the real line are also characterized as follows.

Theorem 2 (Hattori-Ohta [9])**.** *A separable metrizable space X is homeomorphic to a subspace of the real line if and only if X admits a metric satisfying both of the following conditions:*

(i) *The cardinality of any midset is at most 1.*
(ii) *The cardinality of any subset consisting of points which are equidistant from a point is at most 2.*

Furthermore, if X is locally compact, then X is homeomorphic to a subspace of the real line if and only if X admits a metric satisfying the condition (i) only.

Now, we have an open question.

Question 2 ([9])**.** Is every rim-compact (= each point has a neighborhood base consisting of the open sets with compact boundaries), separable metric space (X, ρ) satisfying the condition (i) homeomorphic to a subspace of the real line?

A metrizable space that admits a metric with n-points midset property is also said to have the n-points midset property. Several spaces are known to have the UMP, or do not have the UMP.

Theorem 3 (Ohta-Ono [28], Ito-Ohta-Ono [10]).

(a) *Let I and J be separated intervals in the real line $\mathbb{R}$. Then $I \cup J$ admits a metric which has the UMP if and only if at least one of I and J is not compact.*

(b) *The union of odd numbers of disjoint closed intervals in $\mathbb{R}$ admits a metric which has the UMP.*

(c) *The subsets $[0,1] \cup \mathbb{Z}$ and $[0,1] \cup \mathbb{Q}$ of $\mathbb{R}$ do not admit metrics which have the UMP, where $\mathbb{Z}$ and $\mathbb{Q}$ denote the sets of the integers and rational numbers, respectively.*

(d) *Let X be the union of at most countably many subsets X_n of $\mathbb{R}$. If each X_n is either an interval or totally disconnected and if at least one of X_n is a noncompact interval, then X admits a metric which has the UMP.*

(e) *A discrete space D admits a metric which has the UMP if and only if $|D| \neq 2, 4$ and $|D| \leq \mathfrak{c}$, where $|D|$ denotes the cardinality of D and $\mathfrak{c}$ is the cardinality of the continuum.*

(f) *Let D be the discrete space with $|D| \leq \mathfrak{c}$. Then the product of countably many copies of D admits a metric which has the UMP. In particular, the Cantor set and the space of irrationals admit metrics which have the UMP.*

Question 3 ([28]). Does every subspace of the real line $\mathbb{R}$ containing a noncompact intervals as a clopen set have the UMP?

Question 4 ([10]). Is there a subspace X of $\mathbb{R}^n$ or the Hilbert space such that X is homeomorphic to the Cantor set or the space of irrationals and the metric inherited from the usual metric has the UMP?

The UMP of a finite discrete space can be considered in terms of graph theory. Let G be a **simple graph** (i.e., a graph which does not contain neither multiple edges nor loops). By a **colouring** of G we mean a map defined on the set of edges $E(G)$ of G. A coloring φ of G is said to have the **unique midset property** if for every pair of distinct vertices x and y there is a unique vertex p such that xp and yp are edges of G and $\varphi(xp) = \varphi(yp)$.

Theorem 4 (Ito-Ohta-Ono [10]). *Let K_n be the complete graph (i.e., each vertex of K_n is adjacent to every other vertices) with n vertices. Then, a finite discrete space with n points has the UMP if and only if there is a colouring φ of the complete graph K_n with the UMP.*

Let G be a finite graph having a colouring with the UMP. Then we denote by $\mathrm{ump}(G)$ the smallest number of colours required for a colouring of G with the UMP; i.e.,

$$\mathrm{ump}(G) = \min\{|\varphi(E(G))| : \ \varphi \text{ is a colouring of } G \text{ with the UMP}\}.$$

Theorem 5 (Ito-Ohta-Ono [10]). *For a complete graph K_n, we have the following.*

(a) *For each $n \geq 0$, $ump(K_{2n+1}) = n$.*
(b) *For each $n \geq 3$, $ump(K_{2n}) \leq 2n - 1$.*

In particular, we have

- $\mathrm{ump}(K_6) = 4$,
- $\mathrm{ump}(K_8) = 5$,
- $\mathrm{ump}(K_{10}) = 5$,
- $\mathrm{ump}(K_{12}) \leq 8$,
- $\mathrm{ump}(K_{14}) \leq 10$.

Question 5 ([10]). Determine the values of $\mathrm{ump}(K_{2n})$ for each $n \geq 6$.

The following *double midset conjecture* seems to be very interesting.

Double Midset Conjecture. A continuum (= non-degenerate connected compact metric space) having the DMP must be a simple closed curve.

The conjecture still remains open. However, several partial results about the conjecture are known.

Theorem 6. *A continuum X with the DMP satisfying the either of the following conditions is a simple closed curve:*

(a) (L. D. Loveland and S. G. Wayment [19]) *X contains a continuum with no cut points.*
(b) (L. D. Loveland [15]) *The midset function $M : \{(x, y) \in X \times X : x \neq y\} \to 2^X$ is continuous.*
(c) (L. D. Loveland [17]) *$X \subset \mathbb{R}^2$ with the Euclidean metric.*

Furthermore, L. D. Loveland and S. M. Loveland proved:

Theorem 7 (L. D. Loveland and S. M. Loveland [18]). *Every continuum in the Euclidean plane with the n-MP for $n \geq 1$ must either be a simple closed curve or an arc.*

We shall consider a more general setting of the double midset conjecture.

$(n-1)$-sphere Midset Conjecture. A non-degenerate compact metric space such that all of its midsets are homeomorphic to an $(n-1)$-sphere S^{n-1} is homeomorphic to an n-sphere S^n.

The double midset conjecture can be considered as the 0-sphere midset conjecture. A space is said to have the **k-sphere midset property** if each midset of X is homeomorphic to a k-sphere S^k. We have a few results in this direction:

Theorem 8.

(a) (L. D. Loveland [16]) *If X is a metric space with 1-sphere midset property, and if X contains a subset homeomorphic to 2-sphere, then X is a 2-sphere.*

(b) (L. D. Loveland [16]) *If X is a nondegenerate compact metric space with 1-sphere midset property and every simple closed curve separates X, then X is a 2-sphere.*

(c) (W. Dębski, K. Kawamura and K. Yamada [4]) *Let $n \geq 3$ and X be a nondegenerate compact subset of the n-dimensional Euclidean space such that each of its midsets is a convex $(n-2)$-sphere (= the boundary of a convex $(n-1)$-cell). Then X is a convex $(n-1)$-sphere.*

3. Metrics having special properties

There are several topology preserving metrics in a metrizable space X. Some of them may have special properties. From this point of view, we have the following results: (But, the results below are also motivated by some results concerning special metrics characterizing topological dimension.)

Theorem 9. *Every metrizable space X admits the following metrics ρ_1, ρ_2, ρ_3 and ρ_4.*

(a) (Nagata [25] or see [8]) *For every $x \in X$ and each sequence $\{y_1, y_2, \dots\}$ in X there is a triplet of indices i, j and k such that $i \neq j$ and $\rho_1(y_i, y_j) \leq \rho_1(x, y_k)$.*

(b) (Nagata, see [26]) *$\{S_\varepsilon(x) : x \in X\}$ is closure-preserving for each $\varepsilon > 0$, where $S_\varepsilon(x) = \{y \in X : \rho_2(x,y) < \varepsilon\}$ is an ε-ball with respect to ρ_2.*

(c) (Hattori [7]) *For every $x \in X$ and every sequence $\{y_1, y_2, \dots\}$ in X with $\rho_3(x, y_i) \geq \delta$ for infinitely many y_i for some $\delta > 0$, there is a pair of distinct indices i and j such that $\rho_3(y_i, y_j) \leq \rho_3(x, y_i)$.*

(d) (Hattori [7]) *X has a σ-locally finite open base consisting of open balls with respect to ρ_4.*

Concerning (c) and (d) in the theorem above, the following are still remain open.

Question 6 ([25] or see [8]). Can we drop the condition "$\rho(x, y_i) \geq \delta$ for infinitely many y_i for some $\delta > 0$" in (c)? I.e., does every metrizable space admit a metric ρ such that for every $x \in X$ and every sequence $\{y_1, y_2, \dots\}$ in X, there is a pair of distinct indices i and j such that $\rho_(y_i, y_j) \leq \rho_(x, y_i)$?

Question 7 ([25]). Does every metrizable space X admit a metric ρ such that $\mathcal{B}_n = \{S_{1/n}(x) : x \in X_n\}$ is discrete in X for some $X_n \subset X$, $n = 1, 2, \dots$ and $\mathcal{B} = \bigcup_{n=1}^\infty \mathcal{B}_n$ is a base for X?

Nagata [27] also asked whether if every metrizable space admits a metric such that $\{S_\varepsilon(x) : x \in X\}$ is hereditarily closure-preserving for each $\varepsilon > 0$. This is answered negatively by Ziqiu-Junnila and Balogh-Gruenhage independently as follows.

Theorem 10 (Ziqiu-Junnila [31]). *The hedgehog space of the weight $\geq 2^{\mathfrak{c}^+}$ does not admit a metric such that $\{S_\varepsilon(x) : x \in X\}$ is hereditarily closure-preserving for each $\varepsilon > 0$.*

Theorem 11 (Balogh-Gruenhage [1]). *A metrizable space X admits a metric such that $\{S_\varepsilon(x) : x \in X\}$ is hereditarily closure-preserving for each $\varepsilon > 0$ if and only if X is strongly metrizable, where a metrizable space X is called strongly metrizable if there is a base for X which is the union of countably many star-finite open coverings. In particular, the hedgehog space of weight ω_1 does not have such metric.*

We notice that there is another chracterization of strongly metrizable spaces by a special metric (Hattori [7]) : A metrizable space X is strongly metrizable if and only if X admits a metric ρ such that for every $\varepsilon > 0$, every $x \in X$ and every sequence $\{y_1, y_2, \dots\}$ in X with $\rho(S_\varepsilon(x), y_i) < \varepsilon$ for all i, there is a pair of distinct indicies i and j such that $\rho(y_i, y_j) < \varepsilon$.

The following questions asked by Nagata in [25] seems to be open.

Question 8. If X is a rim-separable (= every point has a neighborhood base consisting of open sets with separable boundaries) metrizable space, then does X admit a metric ρ such that $\mathrm{Bd}\, S_\varepsilon(x)$ is separable for each $\varepsilon > 0$ and $x \in X$?

Question 9. If X is a rim-compact (= every point has a neighborhood base consisting of open sets with compact boundaries) metrizable space, then does X admit a metric ρ such that $\mathrm{Bd}\, S_\varepsilon(x)$ is compact for each $\varepsilon > 0$ and $x \in X$?

4. Ultrametric spaces

A metric ρ is called an **ultrametric** (or a **non-Archimedean metric**) if ρ satisfies the strong triangle inequality:

$$\max\{\rho(x, y), \rho(y, z)\} \geq \rho(x, z) \text{ for all } x, y, z \in X.$$

Ultrametric spaces are topologically characterized independently by F. Hausdorff, J. de Groot ([3] and also see [5, Problem 4.1.H]) and K. Morita [22]: A metrizable space X admits an ultrametric if and only if $\dim X = 0$, where $\dim X$ denotes the covering dimension of X. The characterization was extended to higher dimension by J. Nagata [24] and P. Ostrand [29]. Ultrametric spaces are also studied in dimension theory from more geometrical points of view.

Let $\mathbb{R}_+$ denote the set of non-negative real numbers and C, $s \in \mathbb{R}_+$.

Definition 2. A metric space (X, ρ) is (C, s)**- homogeneous** if the inequality $|X_0| \leq C(b/a)^s$ holds for $a > 0$, $b > 0$ and $X_0 \subset X$ provided that $b \geq a$ and that $a \leq \rho(x, y) \leq b$ holds for every pair of distinct points x and y of X_0.

The space (X, ρ) is said to be s**-homogeneous** if it is (C, s)-homogeneous for some $C \in \mathbb{R}_+$.

Definition 3. We define the **Assouad dimension** $\dim_A(X, \rho)$ of a metric space (X, ρ) as follows:

$$\dim_A(X, \rho) = \inf\{s \in \mathbb{R}_+ : (X, \rho) \text{ is } s\text{-homogeneous}\},$$

if the infimum exists. Otherwise, we define $\dim_A(X,\rho) = \infty$.

Then we have the following theorem.

Theorem 12. *An ultrametric space (X,ρ) can be bi-Lipschitz embedded in the n-dimensional Euclidean space $\mathbb{R}^n$ if and only if $\dim_A(X,\rho) < n$, where a mapping $f : X \to \mathbb{R}^n$ is said to be a* **bi-Lipschitz embedding** *if there exists a real number $\alpha \geq 1$ such that for every x, $y \in X$, $\frac{1}{\alpha}\rho(x,y) \leq \|f(x) - f(y)\| \leq \alpha\rho(x,y)$ holds.*

The "if" part of the theorem is proved by Luukkainen and Movahedi-Lankarani [21] and the "only if" part is proved by Luosto [20].

Several metrically universal properties of ultrametric spaces are known. A. J. Lemin ([13]) proved that every ultrametric space of weight τ is isometrically embedded in the generalized Hilbert space H^τ. Furthermore, A. J. Lemin and V. A. Lemin proved the following:

Theorem 13 (A. J. Lemin and V. A. Lemin [14])**.** *For every cardinal τ there is an ultrametric space (LW_τ, ρ) such that every ultrametric space of weight $\leq \tau$ is isometrically embedded in (LW_τ, ρ).*

We notice that the weight of the space (LW_τ, ρ) is $\tau^{\aleph_0}$ and this is the best possible if we consider the countable case. In fact, if an ultrametric space (X,ρ) isometrically contains all two-point ultrametric spaces, then the weight of $X \geq \mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of the continuum. On the other hand, J. Vaughan [30] proved the following.

Theorem 14 (J. Vaughan [30])**.** *Under the assumption of the singular cardinal hypothesis, for every cardinal τ satisfying $\mathfrak{c} < \tau < \tau^\omega$ there exists an ultrametric space (LW'_τ, ρ') such that the weight of (LW'_τ, ρ') is τ and every ultrametric space of weight $\leq \tau$ is isometrically embedded in (LW'_τ, ρ').*

In the case of spaces consisting of finite points, we have the following.

Theorem 15 (A. J. Lemin and V. A. Lemin [14])**.** *Every ultrametric space consisting of $n+1$ points is isometrically embedded in the n-dimensional Euclidean space $\mathbb{R}^n$ and there is no ultrametric space X consisting of $n+1$ points such that X is isometrically embedded in the k-dimensional Euclidean space $\mathbb{R}^k$ for $k < n$.*

Recently, the theory of ultrametric spaces has been developped in several branches of mathematics as well as physics, biology and information sciences. In particular, the theory of ultrametric spaces is applied to domain theory and logic programming.

We shall introduce some notion and terminology from the domain theory. Let $(P, \leq)$ be a partially ordered set. A partial order $\leq$ on $(P, \leq)$ is said to be **directed complete** if every directed subset D of P has a least upper bound $\bigvee D$ and a partially ordered set $(P, \leq)$ is said to be a **directed complete partially ordered set** if $\leq$ is directed complete. (A directed complete partially ordered set is often abbreviated as a **dcpo**.) Let P be a dcpo and $x, y \in P$. We say x is **way below** y, written by $x \ll y$, if for every directed subset D of P with $y \leq \bigvee D$ there is $d \in D$ such that $x \leq d$. An element x of a dcpo P is **compact** if $x \ll x$ and we denote the set of compact elements of P by P_c. A dcpo P is called **algebraic** (**ω-algebraic**) if (P_c is countably infinite and) every element of P is a directed sup of compact elements, i.e., for each element $x \in P$, $\{y \in P_c : y \leq x\}$ is directed and $x = \bigvee\{y \in P_c : y \leq x\}$. An algebraic dcpo is called a **domain**. We call that a dcpo P is **ω-continuous** if there is a countably infinite subset B of P such that for every $x \in P$, $\{y : y \in B,\ y \ll x\}$ is directed and $\bigvee\{y : y \in B,\ y \ll x\} = x$.

Definition 4. The **Scott topology** of a partially ordered set P is defined as follows: A subset $U \subset P$ is open if

(i) for every $x \in U$, $\{y \in P : y \geq x\} \subset U$, and
(ii) for every directed subset D in P with $\bigvee D \in U$, $D \cap U \neq \emptyset$.

If P is a domain, then the Scott topology is generated by the subbase consisting of sets of the form $\{y \in P : y \geq x\}$ for $x \in P_c$.

Definition 5. The **Lawson topology** of a partially ordered set P is the supremun of the Scott topology and the weak^d topology, where the **weak^d topology** is the topology determined by the closed base consisting of sets of the form $\{y \in P : y \geq x\}$ for $x \in P$.

Definition 6. Let $\mathrm{Max}\,(P)$ be the set of maximal elements of P. A **computational model** for a topological space X is an ω-continuous diretected complete partially ordered set P with an embedding $i : X \to \mathrm{Max}\,(P)$ which satisfy the following conditions:

(1) The restrictions of the Scott topology and the Lawson topology to $\mathrm{Max}\,(P)$ coincide.
(2) The embedding $i : X \to \mathrm{Max}\,(P)$ is a homeomorphism.

Then, B. Flagg and R. Kopperman ([6]) proved the following.

Theorem 16. *A topological space X has an ω-algebraic computational model if and only if X is a complete separable ultrametric space.*

We refer the reader to [12] for terminology and recent developments of domain theory related to topology. We also refer the reader to [14] for a brief historical introduction to ultrametric spaces.

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