AUTOMATICALLY GENERATED ENCYCLOPEDIA OF SPECIAL FUNCTIONS
EXTENDED ABSTRACT

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ABSTRACT. We present our on-going work on creating an automatically generated encyclopedia of special functions. The aim is to exploit recent progress in computer algebra in order to synthesize web pages on special functions and provide some level of interactivity.

1. Introduction and Motivations

Abramowitz and Stegun's *Handbook of Mathematical Functions* [2] is one of the most cited references in mathematics. For about one hundred functions, this book lists properties such as series expansion, asymptotic expansion, graph, zeroes, tables of numerical values, ...

The formulæ in this book and other references on special functions were computed, written and proof-read by hand. The progress made in symbolic computation in the last 20 years makes it possible to automate completely the derivation of many of these formulæ. This opens the way to developing an automatically generated easy-to-use encyclopedia of special functions that gathers information and presents it in a uniform way, with the added possibility of interactivity.

We concentrate on special functions that are solutions of a linear differential or difference equation. These functions are called D-finite or holonomic and many of their properties are algorithmically computable [9, 11, 8, 4, 5]. It turns out that this class covers about 60% of the functions described in [2]. In other words, 60% of this reference can now be produced automatically with very little extra coding on top of standard computer algebra packages.

Section 2 presents important properties of holonomic functions that make them a suitable framework for special functions. Section 3 overviews the content of our encyclopedia, while section 4 discusses its interface.

2. Holonomic functions

A univariate function is said to be holonomic when it satisfies a linear ordinary differential equation with polynomial coefficients. The set of holonomic functions enjoys nice closure properties, such as sum, product, differentiation, integration, composition with algebraic functions, Laplace and Fourier transforms [9]. Numerous usual functions and many so-called special functions fall into the holonomic class. Examples are: exponential, logarithm, trigonometric and hyperbolic trigonometric functions, inverse trigonometric and inverse hyperbolic trigonometric functions, hypergeometric, confluent hypergeometric and generalized hypergeometric functions (which include Bessel J, H and K functions, Airy Ai and Bi functions, dilogarithms and polylogarithms), Struve, Weber and Anger function, the large class of algebraic functions, ...

There is no unified definition for the class of special functions. The lack of any underlying algebraic structure thwarts derivation of generic results or universal insight. Embedding a large subset of special functions into the holomic class overrides this drawback and results in a uniform description. Moreover, the holonomic class has been widely investigated from a computer algebra viewpoint and a collection of techniques that is ready for application has been developed.

3. The Encyclopedia

From minimal input, namely a linear differential equation and a set of initial conditions, the algorithms mentioned above produce numerous formulæ concerning the solution thus defined. This includes Taylor expansion, asymptotic expansion, closed-form for the coefficients in these expansions when available, and integral transforms. Moreover, numerical approximations and graphical representations can also be generated by standard methods of numerical analysis, as well as high-precision evaluations at rational numbers. Code dedicated to numerical computation in arbitrary precision can also be automatically produced in various programming languages, such as Maple, or C linked to an existing multiprecision package, such as MPFR [1].

Thus, the engine of our encyclopedia is an algorithmic machinery that inputs a linear differential equation and initial conditions and outputs a HTML page that describes properties of the corresponding holonomic function. All pages of our encyclopedia are automatically generated in this manner. The content of each page is inspired by the *Handbook on Mathematical Functions* [2] and is partially listed below, together with pointers to the corresponding algorithms. Our encyclopedia can also produce LaTeX documents. The first three pages of such a document generated with the current state of our package is presented at the end of this article.

For clarity, we denote by f the holonomic function for which a HTML page is to be generated.

- *Taylor expansion.* It is classical that the Taylor coefficients satisfy a linear recurrence that can be derived from the differential equation. This provides a fast method for Taylor expansion. This is implemented in the gfun package [8].
- *Closed form for Taylor coefficients.* From the recurrence above, Petkovšek's algorithm [6] decides whether the Taylor coefficients u_n are hypergeometric (i.e., u_{n+1}/u_n is a rational function of n). This is implemented in the LREtools package of Maple. Success then leads to a closed form for the coefficients in terms of the Gamma function. Note that failure of this algorithm is a *proof* that the sequence of coefficients is not a linear combination of hypergeometric sequences.
- *Asymptotic expansion.* A basis of formal asymptotic solutions of the equation can be computed by classical methods that are now algorithmic [10]. This is implemented in Maple's diffop package. The actual coefficients that express the asymptotic behaviour of f in this basis are called *connection coefficients*. They depend on the initial conditions and can be computed automatically only for special classes like the generalized hypergeometric functions.
- *Recurrences for the asymptotic expansion.* The formal asymptotic expansions obtained above can be decomposed as the product of a pre-factor and a formal power

series in logarithms of the indeterminate or in one of its powers. The pre-factor is at worse the exponential of a polynomial times an algebraic power of the variable. Thus its reciprocal is solution of a linear differential equation. Since holonomic functions are closed under product, the formal power series is then again in this class. A linear recurrence for its coefficients follows.

- *Closed form for the asymptotic expansion.* Again, Petkovšek's algorithm applies.
- *Numerical values and graphics.* A low-cost evaluation algorithm with arbitrary precision (hence, with guaranteed results) is available for these functions [3]. From there it is possible to produce graphics. Currently, the encyclopedia does not produce numerical code working in *fixed* precision.
- *Expansion in series of orthogonal polynomial families.* The case of Chebyshev polynomials is dealt with by an algebraic substitution, implemented in GFUN. Other families are dealt with using the multivariate approach of MGFUN [4]. This results in a linear recurrence satisfied by the coefficients of these expansions. Again, Petkovšek's algorithm concludes when a closed form is available.
- *Integral transforms.* In the classical cases such as Laplace, Fourier, Hankel and Stieltjes transform, the multivariate algorithms implemented in MGFUN apply. In many cases, it is however possible to derive the differential equation satisfied by the integral transform by simple univariate manipulations. This is the case for the Laplace transform, for instance.

We emphasize that our encyclopedia is *not* a huge database of formulæ on special functions managed by some search engine; an interesting discussion about this alternative design can be found in *the Hypergeometric Database* [7, § 3].

We feel that our approach shows very attractive features. First, it offers the exciting prospect of presenting ultimately an error-free encyclopedia. Indeed, the whole process that leads to the display of formulæ is fully automated. Possible errors are shifted from individual formulæ to general algorithms and thus are more likely to be spotted and fixed. Second, the content of each page of our encyclopedia is easily extended by adding routines. Finally, the encyclopedia is not limited to the known and named special functions: any holonomic function can be queried to our encyclopedia, which generates the corresponding HTML document *on the fly*.

4. Interface

Our encyclopedia is designed to be a useful tool dedicated to special functions, rather than a mere gallery of nice formulæ. Thus, we aim to present it as a collection of automatically generated HTML pages available on the web, so that it is easily accessible and offers some level of interactivity, which cannot be achieved with hardcopy. Also, this approach allows us to keep our encyclopedia up-to-date and benefit from feedback from users.

The returned HTML pages are customizable. For instance, the number of explicit terms of the series expansions, the number of digits of numerical approximations can be changed. Formulæ are proposed in various formats, such as LaTeX or MAPLE. Automatically generated code can be produced in several programming languages, such as C and MAPLE. Indeed, our goal is to develop an encyclopedia that is as easy-to-use as possible.

As mentioned in section 3, pages can be generated *dynamically* from a differential equation. Then, our encyclopedia proposes the ability to generate pages on custom differential equations.

5. Further prospects

Our current work implements properties of *univariate* holonomic functions. As discussed in section 3, this approach yields numerous applications. The multivariate case [4] offers the wider prospects of deriving results for *families* of functions that are given by multivariate linear differential or difference equations. For instance, we expect to produce automatically identities and formulæ on the Bessel functions $J_n(z)$ and the Hermite polynomials $H_n(z)$, now considered as functions of both variables n and z.

References

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The AiryAi function

1 Introduction

The AiryAi function, denoted by y, is defined by the differential equation

$$\frac{d^2}{dx^2}y(x) - xy(x) = 0,$$

whose initial conditions are

$$y(0) = 1/3\,\frac{\sqrt[3]{3}}{\Gamma(2/3)},$$
$$\mathrm{D}(y)(0) = -1/2\,\frac{\sqrt[6]{3}\Gamma(2/3)}{\pi}.$$

2 Taylor expansion

The Taylor expansion of y at 0 is given by

$$y(x) = \sum_{n=0}^{\infty} u(n)x^n.$$

The coefficients $u(n)$ satisfy the recurrence

$$-u(n) + \left(n^2 + 5\,n + 6\right)u(n+3) = 0,$$

whose initial conditions are

$$u(0) = 1/3\,\frac{\sqrt[3]{3}}{\Gamma(2/3)},$$
$$u(1) = -1/2\,\frac{\sqrt[6]{3}\Gamma(2/3)}{\pi},$$
$$u(2) = 0.$$

This recurrence has hypergeometric solutions

$$u(3\,n) = 1/3\,\frac{\sqrt[3]{3}\,(1/3)^n}{\Gamma(n+1)3^n\Gamma(n+2/3)},$$

$$u(3\,n+1) = -1/9\,\frac{3^{2/3}\,(1/3)^n}{\Gamma(n+4/3)3^n\Gamma(n+1)},$$

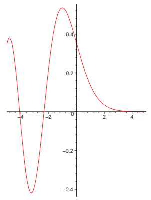
$$u(3\,n+2) = 0,$$

The first few terms of the Taylor expansion are

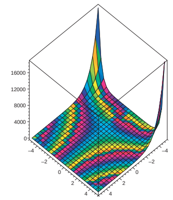
$$y(x) = 1/3\,\frac{\sqrt[3]{3}}{\Gamma(2/3)} - 1/2\,\frac{\sqrt[6]{3}\Gamma(2/3)x}{\pi} + 1/18\,\frac{\sqrt[3]{3}x^3}{\Gamma(2/3)} - 1/24\,\frac{\sqrt[6]{3}\Gamma(2/3)x^4}{\pi} + \frac{1}{540}\,\frac{\sqrt[3]{3}x^6}{\Gamma(2/3)} - \frac{1}{1008}\,\frac{\sqrt[6]{3}\Gamma(2/}{\pi}$$

3 Graphs

3.1 Real axis



3.2 Complex plane



4 Wronskian

Up to a multiplicative constant, the Wronskian of the differential equation that defines y is given by

$$W(x) \propto 1.$$

5 Expansion in Chebyshev Polynomials

The function y can be expanded in terms of Chebyshev polynomials, which are considered as a basis of functions. The coefficients $u(n)$ of the expansion

$$y(x) = \sum_{n=0}^{\infty} u(n)T_n(x)$$

satisfy the following recurrence

$$-u(n) + 2\,u(n+2) + \left(8\,n^2 + 40\,n + 48\right)u(n+3) + \left(-88\,n - 240 - 8\,n^2\right)u(n+5) - 2\,u(n+6) + u(n+8)$$

This recurrence applies for both first and second kind polynomials. The initial conditions are given by their integral forms and require numerical approximation; we are working on producing them as well.