Steps in Differential Geometry, Proceedings of the Colloquium on Differential Geometry, 25–30 July, 2000, Debrecen, Hungary

ONE KIND OF MULTISYMPLECTIC STRUCTURES ON 6-MANIFOLDS

JIŘÍ VANŽURA

1. Introduction

Let us recall first that a *multisymplectic structure* of order $k+1$ on a finite dimensional real vector space V is a $(k+1)$-form $\omega \in \Lambda^{k+1}V^*$ such that the homomorphism

$$V \to \Lambda^k V^*, \quad v \mapsto \iota_v \omega = \omega(v, \cdot, \dots, \cdot)$$

is a monomorphism. The first examples of multisymplectic structures are symplectic structures and non-zero n-forms, where $n = \dim V$. For further information on multisymplectic strucrures we refer the reader to [CIL]. We shall use the following definition.

1. Definition. Let ω be a multisymplectic $(k+1)$-form on $\mathbb{R}^n$ and let ξ be an n-dimensional real vector bundle over a base X. A *multisymplectic structure of type* ω on ξ is a continuous $(k+1)$-form $\Omega \in \Lambda^{k+1}\xi^*$ with the following property: There exists an open cover $\{U_i\}_{i\in I}$ of X such that for every $i \in I$ there is an isomorphism

$$f_i : U_i \times \mathbb{R}^n \to \xi | U_i$$

such that for each $x \in U_i$

$$(f_i \kappa_{i,x})^* \Omega = \omega,$$

where $\kappa_{i,x} : \mathbb{R}^n \to U_i \times \mathbb{R}^n$ is defined by the formula $\kappa_{i,x}(v) = (x, v)$.

2. Remark. If $\xi = TX$ is the tangent bundle of a differentiable manifold X, it is natural to assume that the form Ω is differentiable. The standard definition of multisymplectic structure in this framework requires also $d\Omega = 0$. Such a condition in our setting makes no sense, and from the point of view of differential geometry we should call our structure almost multisymplectic structure. Nevertheless, for the sake of brevity, we speak about multisymplectic structures. The goal of this note is to investigate special multisymplectic structures of order 3 on 6-dimensional real vector bundles. It is known (see [C]) that on $\mathbb{R}^6$ there exist up to isomorphism only three (mutually non-isomorphic) multisymplectic 3-forms. If we denote $e_1, \dots, e_6$ the canonical basis of $\mathbb{R}^6$, and $\alpha_1, \dots, \alpha_6$ the dual basis, the above mentioned three forms are

1991 *Mathematics Subject Classification.* 53C15, 58A10.

Supported by the Grant Agency of the Czech Republic under the grant No. 201/99/0675.

(1) $\omega_1 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6$,
(2) $\omega_2 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6$,
(3) $\omega_3 = \omega = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6$.

If a 3-form α belongs to the isomorphism class of the form ω_i, we shall often say that α *is of type* ω_i. We are interested only in the form ω_3, and therefore we shall denote it simply by ω. This means that we have always

$$\omega = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6.$$

2. Algebraic properties

We shall introduce the subspaces

$$V_0 = [e_1, e_2, e_3] \quad \text{and} \quad V_0^{\perp} = [e_4, e_5, e_6].$$

For any 3-form α on $\mathbb{R}^6$ we define

$$\Delta(\alpha) = \{v \in \mathbb{R}^6 ; \iota_v \alpha \text{ has rank 2.}\}$$

3. Lemma. $\Delta(\omega) = V_0$.

Proof. Let us consider a vector $v = c_1 e_1 + \cdots + c_6 e_6$. We get

$$\iota_v \omega = c_1 \alpha_4 \wedge \alpha_5 + c_2 \alpha_4 \wedge \alpha_6 + c_3 \alpha_5 \wedge \alpha_6 - c_4 \alpha_1 \wedge \alpha_5 - c_4 \alpha_2 \wedge \alpha_6 + \\ + c_5 \alpha_1 \wedge \alpha_4 - c_5 \alpha_3 \wedge \alpha_6 + c_6 \alpha_2 \wedge \alpha_4 + c_6 \alpha_3 \wedge \alpha_5.$$

Further we have

$$(\iota_v \omega) \wedge (\iota_v \omega) = -c_4 c_6 \alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \alpha_5 + c_4 c_5 \alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - c_4^2 \alpha_1 \wedge \alpha_2 \wedge \alpha_5 \wedge \alpha_6 - \\ - c_5 c_6 \alpha_1 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5 + c_5^2 \alpha_1 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_6 - c_4 c_5 \alpha_1 \wedge \alpha_3 \wedge \alpha_5 \wedge \alpha_6 + \\ + (c_2 c_4 + c_3 c_5) \alpha_1 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6 - c_6^2 \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5 + c_5 c_6 \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_6 - \\ - c_4 c_6 \alpha_2 \wedge \alpha_3 \wedge \alpha_5 \wedge \alpha_6 + (-c_1 c_4 + c_3 c_6) \alpha_2 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6 - (c_1 c_5 + c_2 c_6) \alpha_3 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6.$$

Obviously, $(\iota_v \omega) \wedge (\iota_v \omega) = 0$ implies $c_4 = c_5 = c_6 = 0$. On the other hand if $c_4 = c_5 = c_6 = 0$, then $(\iota_v \omega) \wedge (\iota_v \omega) = 0$. This finishes the proof. □

4. Remark. In Lemma 3 we have determined $\Delta(\omega_3)$. Similarly we can find $\Delta(\omega_1)$ and $\Delta(\omega_2)$. We get

$$\Delta(\omega_1) = V_0 \cup V_0^{\perp},$$
$$\Delta(\omega_2) = \{0\},$$
$$\Delta(\omega_3) = V_0.$$

This shows that ω_1, ω_2, and ω_3 represent three different isomorphism classes of 3-forms. It is also easy to see how, using $\Delta(\alpha)$, we can recognize to which isomorphism class a multisymplectic 3-form α belongs. Let us mention here that there is an invariant $\lambda(\alpha)$ defined for any 3-form α, which was introduced by N. Hitchin (see [H]). A multisymplectic 3-form α belongs to the isomorphism class of

ω_1 if and only if $\lambda(\alpha) > 0$
ω_2 if and only if $\lambda(\alpha) < 0$
ω_3 if and only if $\lambda(\alpha) = 0$.

It is also important to notice that $V_0 \subset \mathbb{R}^6$ is a subspace naturally associated with the form ω. This is no more true for the subspace $V_0^{\perp}$.

We shall introduce the group $O(\omega)$ consisting of all automorphisms $\varphi \in GL(6, \mathbb{R})$ preserving the form ω, i. e. such that for every vectors $v, v', v'' \in V$ there is

$$(*) \qquad \omega(\varphi v, \varphi v', \varphi v'') = \omega(v, v', v'').$$

It is clear that $O(\omega)$ is a closed subgroup of $GL(6, \mathbb{R})$. Consequently, $O(\omega)$ is a Lie group.

5. Lemma. *For every element $\varphi \in O(\omega)$ there is $\varphi(V_0) = V_0$.*

Proof. Let $v \in V_0$, and let $\varphi \in O(\omega)$. For arbitrary $v_1, v_2, v_3, v_4 \in \mathbb{R}^6$ we get

$$((\iota_{\varphi v}\omega) \wedge (\iota_{\varphi v}\omega))(v_1, v_2, v_3, v_4) = \frac{1}{6!} \sum_{\pi \in S_4} \operatorname{sign}\pi \cdot \omega(\varphi v, v_{\pi 1}, v_{\pi 2}) \cdot \omega(\varphi v, v_{\pi 3}, v_{\pi 4}) =$$

$$= \frac{1}{6!} \sum_{\pi \in S_4} \operatorname{sign}\pi \cdot \omega(\varphi v, \varphi\varphi^{-1} v_{\pi 1}, \varphi\varphi^{-1} v_{\pi 2}) \cdot \omega(\varphi v, \varphi\varphi^{-1} v_{\pi 3}, \varphi\varphi^{-1} v_{\pi 4}) =$$

$$= \frac{1}{6!} \sum_{\pi \in S_4} \operatorname{sign}\pi \cdot \omega(v, \varphi^{-1} v_{\pi 1}, \varphi^{-1} v_{\pi 2}) \cdot \omega(v, \varphi^{-1} v_{\pi 3}, \varphi^{-1} v_{\pi 4}) =$$

$$= ((\iota_v\omega) \wedge (\iota_v\omega))(\varphi^{-1}v_1, \varphi^{-1}v_2, \varphi^{-1}v_3, \varphi^{-1}v_4) = 0.$$

$\square$

For $\varphi \in O(\omega)$ we denote $\varphi_0 = \varphi | V_0$. Further we denote $O_0(\omega) = \{\varphi \in O(\omega); \varphi_0 = id\}$. We obtain in this way a sequence

$$0 \to O_0(\omega) \to O(\omega) \xrightarrow{\rho} GL(V_0) \to 0,$$

where ρ denotes the restriction homomorphism. It is obvious that this sequence is exact at $O_0(\omega)$ and at $O(\omega)$. At this moment it is not clear whether it is exact at $GL(V_0)$. Our next aim is to investigate this sequence.

We can introduce a subspace $A_{V_0} \subset \Lambda^2\mathbb{R}^{6*}$ as follows.

$$A_{V_0} = \{\alpha \in \Lambda^2\mathbb{R}^{6*}; \iota_v\alpha = 0 \text{ for every } v \in V_0\}$$

We have the following obvious lemma.

6. Lemma. *If $v, v' \in V_0$, then $\iota_v\iota_{v'}\omega = 0$.*

This lemma shows that we can define a homomorphism

$$\beta : V_0 \to A_{V_0}, \quad \beta(v) = \iota_v\omega.$$

7. Lemma. *The homomorphism $\beta : V_0 \to A_{V_0}$ is an isomorphism.*

Proof. The 3-form ω is multisymplectic, and consequently β is a monomorphism. A direct computation shows that A_{V_0} has a basis $\alpha_4 \wedge \alpha_5, \alpha_4 \wedge \alpha_6, \alpha_5 \wedge \alpha_6$, and consequently $\dim A_{V_0} = 3$. This proves the lemma. $\square$

Further, we set $V_1 = \mathbb{R}^6/V_0$, and $p : \mathbb{R}^6 \to V_1$ will denote the projection. It can be easily seen that that $\Lambda^2 p^* : \Lambda^2 V_1^* \to \Lambda^2 \mathbb{R}^{6*}$ has the image A_{V_0}. We can thus define an isomorphism

$$\gamma : V_0 \xrightarrow{\beta} A_{V_0} \xrightarrow{(\Lambda^2 p^*)^{-1}} \Lambda^2 V_1^*.$$

The isomorphism γ enables us to introduce a regular pairing

$$V_0 \times \Lambda^2 V_1 \to \mathbb{R}$$

by the formula

$$< v_0, \tilde{v}_1 >= (\gamma v_0)(\tilde{v}_1), \quad \text{where } v_0 \in V_0, \tilde{v}_1 \in \Lambda^2 V_1.$$

If $\hat{v}_1 \in \Lambda^2 \mathbb{R}^6$, then it is easy to see that

$$< v_0, (\Lambda^2 p)\hat{v}_1 >= \omega(v_0, \hat{v}_1).$$

Let us remark that for any endomorphism $\psi_0 : V_0 \to V_0$ there exists a unique endomorphism $\tilde{\psi}_0^t : \Lambda^2 V_1 \to \Lambda^2 V_1$ such that

$$< \psi_0 v_0, \tilde{v}_1 >=< v_0, \tilde{\psi}_0^t \tilde{v}_1 > \quad \text{for every } v_0 \in V_0, \tilde{v}_1 \in \Lambda^2 V_1.$$

We consider now an element $\varphi \in O(\omega)$, and $v, v' \in \mathbb{R}^6$. It is obvious that φ induces an endomorphism $\varphi_1 : V_1 \to V_1$. We have

$$\omega(\varphi v_0, v, v') = \omega(v_0, \varphi^{-1} v, \varphi^{-1} v')$$

$$< \varphi_0 v_0, pv \wedge pv' >=< v_0, (\varphi_1^{-1} \wedge \varphi_1^{-1})(pv \wedge pv') > .$$

This shows that we have $\tilde{\varphi}_0^t = \Lambda^2(\varphi_1^{-1})$. From this we get

$$\det(\varphi_0) = \det(\tilde{\varphi}_0^t) = \det(\Lambda^2(\varphi_1^{-1})) = (\det \varphi_1^{-1})^2 > 0.$$

We can see that the short sequence under consideration can be written in the form

$$0 \to O_0(\omega) \to O(\omega) \xrightarrow{\rho} GL^+(V_0) \to 0.$$

Let us consider now any complementary subspace W to the subspace V_0, i. e. any subspace W such that $\mathbb{R}^6 = V_0 \oplus W$. Similarly as above we get an isomorphism

$$\gamma_W : V_0 \to \Lambda^2 W^*$$

defined by the formula $\gamma_W v_0 = (\iota_{v_0}\omega)|W$. Using this isomorphism we get the regular pairing

$$V_0 \times \Lambda^2 W \to \mathbb{R}, \quad < v_0, \tilde{w} >= (\gamma_W v_0)(\tilde{w}), \text{ where } v_0 \in V_0, \tilde{w} \in \Lambda^2 W.$$

Consequently, to each endomorphism $\psi_0 : V_0 \to V_0$ there exists a unique endomorphism ψ_0^W such that for every $v_0 \in V_0$ and $\tilde{w} \in \Lambda^2 W$ we have

$$< \psi_0 v_0, \tilde{w} >=< v_0, \psi_0^W \tilde{w} > .$$

Let us consider now an element $\varphi \in O(\omega)$. We define a homomorphism $\varphi_W : W \to W$ as the composition

$$W \xrightarrow{\varphi|W} V \xrightarrow{\pi_W} W,$$

where π_W denotes the projection of $\mathbb{R}^6 = V_0 \oplus W$ onto W. It is easy to see that φ_W is an isomorphism. For any $v_0 \in V_0$ and $w, w' \in W$ we have

$$< \varphi_0 v_0, w \wedge w' >= \omega(\varphi_0 v_0, w, w') = \omega(\varphi v_0, \varphi\varphi^{-1} w, \varphi\varphi^{-1} w') =$$

$$= \omega(v_0, \varphi^{-1} w, \varphi^{-1} w') = \omega(v_0, \pi_W \varphi^{-1} w, \pi_W \varphi^{-1} w') =$$

$$= \omega(v_0, \varphi_W^{-1} w, \varphi_W^{-1} w') =< v_0, \Lambda^2(\varphi_W^{-1}) >,$$

which shows that

$$\varphi_0^W = \Lambda^2(\varphi_W^{-1}).$$

8. Definition. A complementary subspace W is called special if $\omega|W = 0$.

Obviously, the subspace $V_0^\perp$ is a special complementary subspace. It is easy to see that complementary subspaces are in a bijective correspondence with the elements τ of the vector space $\mathrm{Hom}(V_0^\perp, V_0)$. Moreover, τ corresponds to a special complementary subspace if and only if for every $w, w', w'' \in W$ there is

$$\omega(\tau w, w', w'') + \omega(w, \tau w', w'') + \omega(w, w', \tau w'') = 0.$$

We define a subgroup $D(\omega) \subset O(\omega)$ by the formula

$$D(\omega) = \{\varphi \in O(\omega); \varphi(V_0^\perp) = V_0^\perp\}.$$

Obviously, $D(\omega)$ is a closed subgroup of $O(\omega)$, and consequently a Lie group.

Before we continue with our considerations, we shall need the following lemma.

9. Lemma. *Let Z be a 3-dimensional real vector space, and let us consider a homomorphism*

$$\lambda : GL(Z) \to GL(\Lambda^2 Z), \quad \lambda(\varphi) = \Lambda^2\varphi \quad \text{for } \varphi \in GL(Z).$$

Then λ is an epimorphism onto $GL^+(\Lambda^2 Z)$, and its kernel is $\{id, -id\}$.

Proof. Because $\det(\Lambda^2(\varphi)) = (\det(\varphi))^2$, it is obvious that λ is a homomorphism into $GL^+(Z)$. We shall now investigate the kernel of this homomorphism. Let us choose a basis z_1, z_2, z_3 of Z, and for $\varphi \in GL(Z)$ let us write

$$\varphi(z_i) = \sum_{j=1}^{3} \varphi_{ij} z_j, \quad i = 1, 2, 3.$$

Let us assume that $\varphi \in \ker \lambda$. Then we have

$$\varphi z_1 \wedge \varphi z_2 = z_1 \wedge z_2, \quad \varphi z_2 \wedge \varphi z_3 = z_2 \wedge z_3, \quad \varphi z_3 \wedge \varphi z_1 = z_3 \wedge z_1.$$

From these identities we get the following nine equations:

$$
\begin{aligned}
&(1) \quad \varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21} = 1\\
&(2) \quad \varphi_{12}\varphi_{23} - \varphi_{13}\varphi_{22} = 0\\
&(3) \quad \varphi_{13}\varphi_{21} - \varphi_{11}\varphi_{23} = 0\\
&(4) \quad \varphi_{21}\varphi_{32} - \varphi_{22}\varphi_{31} = 0\\
&(5) \quad \varphi_{22}\varphi_{33} - \varphi_{23}\varphi_{32} = 1\\
&(6) \quad \varphi_{23}\varphi_{31} - \varphi_{21}\varphi_{33} = 0\\
&(7) \quad \varphi_{31}\varphi_{12} - \varphi_{32}\varphi_{11} = 0\\
&(8) \quad \varphi_{32}\varphi_{13} - \varphi_{33}\varphi_{12} = 0\\
&(9) \quad \varphi_{33}\varphi_{11} - \varphi_{31}\varphi_{13} = 1
\end{aligned}
$$

The equations (7) and (4) can be written in the form

$$
\begin{aligned}
\varphi_{11}\varphi_{32} - \varphi_{12}\varphi_{31} = 0\\
\varphi_{21}\varphi_{32} - \varphi_{22}\varphi_{31} = 0
\end{aligned}
$$

Considering $\varphi_{32}, \varphi_{31}$ as unknown quantities, and $\varphi_{11}, \varphi_{12}, \varphi_{21}\varphi_{22}$ as coefficients, we have a system of two homogeneous linear equations with two unknown quantities and with a nonvanishing determinant (see (1)). Consequently, we get

$$\varphi_{32} = 0, \quad \varphi_{31} = 0.$$

Proceeding along the same lines, we get

$$\varphi_{ij} = 0 \quad \text{for all } i \neq j.$$

The equation (1), (5), and (9) have now the form

$$\varphi_{11}\varphi_{22} = 1, \quad \varphi_{22}\varphi_{33} = 1, \quad \varphi_{33}\varphi_{11} = 1.$$

It can be easily seen that this system has only two solutions. Either $\varphi_{11} = \varphi_{22} = \varphi_{33} = 1$ or $\varphi_{11} = \varphi_{22} = \varphi_{33} = -1$. This shows that

$$\ker \lambda = \{id, -id\}.$$

Because $\dim GL(Z) = \dim GL(\Lambda^2 Z)$, and $GL^+(\Lambda^2 Z)$ is connected, it follows easily that λ maps already $GL^+(Z)$ onto $GL^+(\Lambda^2 Z)$. $\square$

10. Remark. The above lemma shows that we have a short exact sequence

$$0 \to \mathbb{Z}_2 \to GL(Z) \xrightarrow{\lambda}\!\!\!\to GL^+(\Lambda^2 Z) \to 0,$$

and that the restriction

$$\lambda^+ : GL^+(Z) \to GL^+(\Lambda^2 Z)$$

is an isomorphism. Let us remark that because $\dim Z$ is odd, we have $GL(Z) = GL^+(Z) \times \mathbb{Z}_2$.

If $Z = V_1$ resp. $Z = V_0^\perp$ we denote the isomorphism λ^+ by $\tilde{\lambda}_0$ resp. λ_0. Therefore, we have the isomorphisms

$$\tilde{\lambda}_0 : GL^+(V_1) \to GL^+(\Lambda^2 V_1) \text{ resp. } \lambda_0 : GL^+(V_0^\perp) \to GL^+(\Lambda^2 V_0^\perp).$$

Taking $W = V_0^\perp$, we have the regular pairing

$$V_0 \times \Lambda^2 V_0^\perp \to \mathbb{R}$$

satisfying

$$< v_0, w \wedge w' >= \omega(v_0, w, w'), \quad \text{where } v_0 \in V_0 \text{ ,and } w, w' \in V_0^\perp.$$

Let now $\varphi \in D(\omega)$, and let us denote $\varphi_0^\perp = \varphi | V_0^\perp$. It is obvious that $\varphi_W = \varphi_0^\perp$. We denote for simplicity $\varphi_0^t = \varphi_0^W$. There is $\varphi = \varphi_0 \oplus \varphi_0^\perp$, and we have

$$\varphi_0^t = \Lambda^2((\varphi_0^\perp)^{-1}), \text{ which implies } \lambda_0^{-1}((\varphi_0^t)^{-1}) = \operatorname{sign}\det(\varphi_0^\perp) \cdot \varphi_0^\perp$$

It is now easy to prove the following lemma.

11. Lemma. *The homomorphism*

$$\mu : GL(V_0) \to D(\omega)$$

defined by the formula

$$\mu(\varphi_0) = (|\varphi_0|, \operatorname{sign}\det(\varphi_0) \cdot \lambda_0^{-1}((\varphi_0^t)^{-1})), \quad \textit{where } |\varphi_0| = \operatorname{sign}\det(\varphi_0) \cdot \varphi_0,$$

is an isomorphism.

Proof. Let us define a homomorphism $\bar{\mu} : D(\omega) \to GL(V_0)$ by the formula

$$\bar{\mu}((\varphi_0, \varphi_0^\perp)) = \operatorname{sign}\det(\varphi_0^\perp) \cdot \varphi_0.$$

Because $\operatorname{sign}\det(\operatorname{sign}\det(\varphi_0) \cdot \lambda_0^{-1}((\varphi_0^t)^{-1}))) = \operatorname{sign}\,\det(\varphi_0)$, we have

$$\bar{\mu}\mu(\varphi_0) = \operatorname{sign}\det(\varphi_0) \cdot |\varphi_0| = \varphi_0.$$

Further, because

$$\operatorname{sign}\det(\operatorname{sign}\det(\varphi_0^\perp) \cdot \varphi_0) = \operatorname{sign}\det(\varphi_0^\perp)$$

we have

$$\mu\bar{\mu}((\varphi_0, \varphi_0^\perp)) = (\varphi_0, \operatorname{sign}\det(\varphi_0^\perp) \cdot \operatorname{sign}\det(\varphi_0^\perp) \cdot \varphi_0^\perp) = (\varphi_0, \varphi_0^\perp).$$

□

12. Remark. The above isomorphism can be formulated also in the following way.

$$\mu : GL^+(V_0) \times \mathbb{Z}_2 \to D(\omega),$$

where $\mathbb{Z}_2$ denotes the multiplicative group $\{1, -1\}$, and μ is defined by the formula

$$\mu(\varphi_0, \varepsilon) = (\varphi_0, \varepsilon \cdot \lambda_0^{-1}((\varphi_0^t)^{-1})).$$

We can see that the Lie group $D(\omega)$ has two connected components, and the connected component of the unit element is isomorphic with $GL^+(3, \mathbb{R})$.

We have shown that $\rho D(\omega) = GL^+(V_0)$. This enables us to introduce the short exact sequence

$$0 \to D_0(\omega) \to D(\omega) \overset{\rho_D}{\to}\to GL^+(V_0) \to 0,$$

where $\rho_D = \rho|D(\omega)$. From our previous considerations we can easily get $D_0(\omega) = \{(id, id), (id, -id)\}$. Moreover, we obtain easily the following lemma.

13. Lemma. *The short sequence*

$$0 \to O_0(\omega) \to O(\omega) \overset{\rho}{\to}\to GL^+(V_0) \to 0.$$

is exact.

We now introduce another subgroup $K(\omega) \subset O(\omega)$, namely

$$K(\omega) = \{\varphi \in O(\omega); \varphi_0 = id, \varphi_1 = id\}.$$

We recall that $\varphi_0 = \varphi|V_0$ and φ_1 is the automorphism of V_1 induced by φ. It is again easy to see that $K(\omega)$ is a closed subgroup of $O(\omega)$, and consequently is a Lie group.

14. Lemma. *$K(\omega)$ is a normal subgroup of $O(\omega)$.*

Proof. Let $\varphi \in K(\omega)$ and $\psi \in O(\omega)$. We get

$$(\psi\varphi\psi^{-1})_0 = \psi_0\varphi_0\psi_0^{-1} = \psi_0\psi_0^{-1} = id,$$
$$(\psi\varphi\psi^{-1})_1 = \psi_1\varphi_1\psi_1^{-1} = \psi_1\psi_1^{-1} = id,$$

which finishes the proof. □

We can easily see that there is

$$D(\omega) \cap K(\omega) = \{id\}.$$

Let now $\varphi \in O(\omega)$ be an arbitrary element, and let us define $\varphi_D \in D(\omega)$ by the formula

$$\varphi_D = (\varphi_0, \operatorname{sign}\det\varphi_1 \cdot \lambda_0^{-1}((\varphi_0^t)^{-1}))).$$

Our next aim is to prove that $\varphi\varphi_D^{-1} \in K(\omega)$. Obviously, we have

$$\varphi_D^{-1} = (\varphi_0^{-1}, \operatorname{sign}\det\varphi_1 \cdot \lambda_0^{-1}(\varphi_0^t)).$$

For $v_0 \in V_0$ we have

$$\varphi\varphi_D^{-1}v_0 = \varphi\varphi_0^{-1}v_0 = v_0.$$

Before we continue with our considerations, we shall present the following commutative diagram,

$$\begin{array}{ccc} GL^+(V_0^\perp) & \xrightarrow{\lambda_0} & GL^+(\Lambda^2 V_0^\perp) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \Lambda^2\pi} \\ GL^+(V_1) & \xrightarrow{\tilde\lambda_0} & GL^+(\Lambda^2 V_1) \end{array}$$

where the isomorphisms π and $\Lambda^2\pi$ are defined in the obvious way using the isomorphism $p|V_0^\perp : V_0^\perp \to V_1$. From this commutative diagram we get the relation

$$\lambda_0^{-1} = \pi^{-1}\tilde{\lambda}_0^{-1}(\Lambda^2\pi).$$

Further equality, which we shall need at the end of the following computation, follows from the formula $\tilde{\varphi}_0^t = (\Lambda^2\pi)\varphi_0^t$. Applying $\tilde{\lambda}_0^{-1}$ to this equality, we get

$$\tilde{\lambda}_0^{-1}(\tilde{\varphi}_0^t) = \operatorname{sign}\det(\varphi_1)\cdot\varphi_1^{-1}.$$

Let now $\tilde{v} \in V_1$ be arbitrary, and let $v \in V_0^\perp$ be such that $pv = \tilde{v}$. We have

$$\begin{aligned}\varphi_1(\varphi_D)_1^{-1}\tilde{v} &= \varphi_1 p(\operatorname{sign}\det\varphi_1\cdot(\lambda_0^{-1}(\varphi_0^t))v) = \operatorname{sign}\det\varphi_1\cdot\varphi_1 p(\lambda_0^{-1}(\varphi_0^t))v = \\ &= \operatorname{sign}\det\varphi_1\cdot\varphi_1 p(\pi^{-1}\tilde{\lambda}_0^{-1}(\Lambda^2\pi)(\varphi_0^t))v = \operatorname{sign}\det\varphi_1\cdot\varphi_1 p(\pi^{-1}\tilde{\lambda}_0^{-1}(\tilde{\varphi}_0^t))v = \\ &= \operatorname{sign}\det\varphi_1\cdot\varphi_1(\tilde{\lambda}_0^{-1}(\varphi_0^t))\tilde{v} = \operatorname{sign}\det\varphi_1\cdot\varphi_1(\operatorname{sign}\det\varphi_1\cdot\varphi_1^{-1})\tilde{v} = \tilde{v}.\end{aligned}$$

We have thus shown that $\varphi\varphi_D^{-1} \in K(\omega)$, which proves the following lemma.

15. Lemma. *Every element $\varphi \in O(\omega)$ can be uniquely expressed in the form*

$$\varphi = \delta\zeta \quad \textit{where } \delta \in D(\omega) \textit{ and } \zeta \in K(\omega).$$

Using the above lemma we can define a homomorphism $r : O(\omega) \to D(\omega)$ in the following way. Let $\varphi \in O(\omega)$ be an element, and let $\varphi = \delta\zeta$ be its decomposition with $\delta \in D(\omega)$ and $\zeta \in K(\omega)$. We define

$$r(\varphi) = \delta.$$

Because $\delta_1\zeta_1\delta_2\zeta_2 = \delta_1\delta_2(\delta_2^{-1}\zeta_1\delta_2)\zeta_2$, we can see that r is a homomorphism. Obviously, we have an exact sequence

$$0 \to K(\omega) \to O(\omega) \xrightarrow{r} D(\omega) \to 0.$$

We shall now investigate the subgroup $K(\omega)$. It is obvious that any element of $K(\omega)$ has the form $id + \tau$, where $\tau : \mathbb{R}^6 \to \mathbb{R}^6$ is a homomorphism satisfying

$$\tau(V_0) = 0, \quad \tau(V_0^\perp) \subset V_0.$$

It can be immediately seen that $id + \tau \in O(\omega)$ if and only if for any $w, w', w'' \in V_0^\perp$ there is

$$\omega(\tau w, w', w'') + \omega(w, \tau w', w'') + \omega(w, w', \tau w'') = 0.$$

One can easily verify that for any homomorphism τ with the above properties the trilinear form β on $V_0^\perp$

$$\beta(w, w', w'') = \omega(\tau w, w', w'') + \omega(w, \tau w', w'') + \omega(w, w', \tau w'')$$

is antisymmetric, i. e. it is a 3-form. Consequently, there exist $c(\tau) \in \mathbb{R}$ such that

$$\omega(\tau w, w', w'') + \omega(w, \tau w', w'') + \omega(w, w', \tau w'') = c(\tau)\cdot(\alpha_4 \wedge \alpha_5 \wedge \alpha_6)(w, w', w'')$$

for every $w, w', w'' \in V_0^\perp$. Now, we can see that $c(\tau)$ is a linear form on the vector space $\operatorname{Hom}(V_0^\perp, V_0)$, and $id + \tau \in O(\omega)$ if and only if $c(\tau) = 0$. We get the following lemma.

16. Lemma. *The group $K(\omega)$ is a closed subgroup of $O(\omega)$ which is isomorphic (as a Lie group) with the commutative Lie group $\mathbb{R}^8$.*

17. Remark. Lemma 18 shows that the Lie group $O(\omega)$ considered as a differentiable manifold is diffeomorphic with the differentiable manifold $D(\omega) \times K(\omega)$. Using Lemma 11 and Lemma 16, we can see that $O(\omega)$ is diffeomorphic with $GL(3,\mathbb{R}) \times \mathbb{R}^8$. This shows that $O(\omega)/GL(3,\mathbb{R})$ is diffeomorphic with $\mathbb{R}^8$, and consequently contractible. Moreover, taking into account the equality following Definition 8 we can immediately see that the group $K(\omega)$ operates simply transitively on the set of all special complementary subspaces. Consequently, the group $O(\omega)$ operates transitively on this set.

An easy consequence of the previous considerations is the following proposition.

18. Proposition. *The Lie group $O(\omega)$ is a semi-direct product of $D(\omega)$ and $K(\omega)$. The Lie group $O_0(\omega)$ is a semi-direct product of $D_0(\omega)$ and $K(\omega)$.*

19. Definition. A basis $v_1, \dots, v_6$ is called *adapted basis* with respect to the form ω if and only if the following conditions are satisfied:

$$\begin{gathered}
\omega(v_i, v_j, v_k) = 0 \text{ if at least two indices are } \leq 3, \\
\omega(v_1, v_4, v_5) = 1, \quad \omega(v_2, v_4, v_5) = 0, \quad \omega(v_3, v_4, v_5) = 0, \\
\omega(v_1, v_4, v_6) = 0, \quad \omega(v_2, v_4, v_6) = 1, \quad \omega(v_3, v_4, v_6) = 0, \\
\omega(v_1, v_5, v_6) = 0, \quad \omega(v_2, v_5, v_6) = 0, \quad \omega(v_3, v_5, v_6) = 1, \\
\omega(v_4, v_5, v_6) = 0.
\end{gathered}$$

It is obvious that if $v_1, \dots, v_6$ is an adapted basis, then $\varphi v_1, \dots, \varphi v_6$ is an adapted basis if and only if $\varphi \in O(\omega)$. Because $e_1, \dots, e_6$ is an adapted basis, and $O(\omega)$ preserves V_0, we get easily the following lemma.

20. Lemma. *For every adapted basis $v_1, \dots, v_6$ there is $v_1, v_2, v_3 \in V_0$.*

The following considerations will be used later on. Let us choose any isomorphism

$$h : V_0 \to \Lambda^2 V_0^{\perp}.$$

Such an isomorphism exists because $\dim V_0 = \dim \Lambda^2 V_0^{\perp}$. Now, we can define a bilinear form

$$b : V_0 \times V_0 \to \mathbb{R}, \quad b(v_0, v_0') = \omega(v_0, h v_0').$$

It is easy to see that the bilinear form b is regular. On the other hand, we have obviously

$$\omega(v_0, w, w') = b(v_0, h^{-1}(w \wedge w')) \quad \text{for } v_0 \in V_0, w, w' \in V_0^{\perp}.$$

21. Lemma. *Let $b : V_0 \times V_0 \to \mathbb{R}$ be a regular bilinear form, and let $h : V_0 \to \Lambda^2 V_0^{\perp}$ be an isomorphism. Then there exists a unique 3-form α on $\mathbb{R}^6$ with the following* *properties*

$$\alpha(v_0, v_0', v) = 0 \text{ for } v_0, v_0' \in V_0 \text{ and } v \text{ is arbitrary},$$
$$\alpha(v_0, w, w') = b(v_0, h^{-1}(w \wedge w')) \text{ for } v_0 \in V_0, w, w' \in V_0^\perp,$$
$$\alpha(w, w', w'') = 0 \text{ for } w, w', w'' \in V_0^\perp.$$

Moreover, α is multisymplectic, and belongs to the isomorphism class of ω.

Proof. We prove first that α is multisymplectic. Let $v \in \mathbb{R}^6$ be a non-zero vector. We can write $v = v_0 + w$, where $v_0 \in V_0$ and $w \in V_0^\perp$. Let us assume first that $w = 0$. The regularity of b implies that there exists $v_0' \in V_0$ such that $b(v_0, v_0') \neq 0$. Because every element of $\Lambda^2 V_0^\perp$ is decomposable, there are $w, w' \in V_0^\perp$ such that $h^{-1}(w \wedge w') = v_0'$. We get then $\alpha(v_0, w, w') \neq 0$. Next, let us assume that $w \neq 0$. Obviously, we can find $w' \in V_0^\perp$ such that $h^{-1}(w \wedge w') \neq 0$ and $v_0' \in V_0$ such that $b(v_0', h^{-1}(w \wedge w')) \neq 0$. Then we have

$$\alpha(v_0 + w, v_0', w') = \alpha(w, v_0', w') = -\alpha(v_0', w, w') = -b(v_0', h^{-1}(w \wedge w')) \neq 0.$$

Now, we are going to prove that α belongs to the same isomorphism class as ω. First we define the isomorphism

$$h_0 : V_0 \to \Lambda^2 V_0^\perp$$

by the formulas

$$h_0(e_1) = e_4 \wedge e_5, \quad h_0(e_2) = e_4 \wedge e_6, \quad h_0(e_3) = e_5 \wedge e_6.$$

When we take in the role of b the canonical scalar product on $\mathbb{R}^6$, which we denote b_0, then applying our construction, we obtain the form ω. There is a unique automorphism $\psi_0 : V_0 \to V_0$ such that $b(v_0, v_0') = b_0(\psi v_0, v_0')$ for every $v_0, v_0' \in V_0$. Then we get

$$b(v_0, h^{-1}(w \wedge w')) = b_0(\psi_0 v_0, h_0^{-1}(h_0 h^{-1})(w \wedge w')),$$

where $h_0 h^{-1} : \Lambda^2 V_0^\perp \to \Lambda^2 V_0^\perp$ is an automorphism. If $\det(h_0 h^{-1}) > 0$, then according to Lemma 9 there exists an automorphism $\chi : V_0^\perp \to V_0^\perp$ such that $h_0 h^{-1} = \Lambda^2 \chi$. We have then

$$b(v_0, h^{-1}(w \wedge w')) = b_0(\psi_0 v_0, h_0^{-1}(\chi w \wedge \chi w')).$$

If $\det(h_0 h^{-1}) < 0$, then there exists an automorphism $\chi' : V_0^\perp \to V_0^\perp$ such that $-h_0 h^{-1} = \Lambda^2 \chi$. We have then

$$b(v_0, h^{-1}(w \wedge w')) = b_0(-\psi_0 v_0, h_0^{-1}(\chi' w \wedge \chi' w')).$$

This means that the automorphism $\psi_0 \oplus \chi$ resp. the automorphism $(-\psi_0) \oplus \chi'$ transforms the form ω into the form α. $\square$

3. Topological and geometrical properties

The first results presented in this section are formulated in the category of topological spaces. But it is easy to verify that all of them remain valid in the category of differentiable manifolds of class C^∞. At the end we prove some results which make sense only in the category of differentiable manifolds. In this section we shall call adapted basis adapted frame, which is terminology more common in geometry.

We shall consider now an orientable 6-dimensional real vector bundle ξ over a base X endowed with a continuous form $\Omega \in \Lambda^3\xi^*$ with the following property: for every $x \in X$ there is an isomorphism

$$g_x : \mathbb{R}^6 \to \xi_x \text{ such that } g_x^*\Omega = \omega.$$

In other words, we assume that for each $x \in X$ the restriction $\Omega_x = \Omega|\xi_x$ on the fiber ξ_x is a 3-form of type ω. Similarly as in the algebraic part, we can define a 3-dimensional subbundle

$$\xi_0 = \{v \in \xi; \iota_v\Omega_x \text{ is decomposable }\},$$

where Ω_x denotes the restriction of Ω onto the fiber of ξ passing through v. We denote $\xi_1 = \xi/\xi_0$, and we have again an isomorphism

$$\gamma : \xi_0 \to \Lambda^2\xi_1^*.$$

First we prove the following lemma.

22. Lemma. *The 3-form Ω defines defines on ξ a multisymplectic structure of type ω.*

Proof. Our aim is to prove the local triviality in the sense of Definition 1. Let $x_0 \in X$. First we choose an adapted basis $v_1, \dots, v_6$ in the fiber ξ_{x_0}. On a neighborhood of x_0 we choose local sections $s_1'', \dots, s_6''$ of ξ such that

(i) $s_i''(x_0) = v_i, \quad i = 1, \dots, 6,$
(ii) $s_1'', \dots, s_6''$ are linearly independent,
(iii) s_1'', s_2'', s_3'' are sections of ξ_0.

Now, we are going to prove that it is possible to substitute s_4'', s_5'', s_6'' by local sections s_4', s_5', s_6' (and simultaneously we set $s_1' = s_1''$, $s_2' = s_2''$, and $s_3' = s_3''$) in such a way that $s_1', \dots, s_6'$ satisfy the conditions of type (i)-(iii), and moreover

$$\omega(s_4', s_5', s_6') = 0.$$

We denote η the vector subbundle spanned by the sections s_4'', s_5'', s_6'', and by $\mathrm{Hom}(\eta, \xi_0)$ the vector bundle of homomorphisms. Taking a section τ of this bundle, we set

$$s_4' = s_4'' + \tau s_4'', \quad s_5' = s_5'' + \tau s_5'', \quad s_6' = s_6'' + \tau s_6''.$$

The conditions of type (i)-(iii) are satisfied, and the remaining condition is satisfied if and only if

$$\omega(\tau s_4'', s_5'', s_6'') + \omega(s_4'', \tau s_5'', s_6'') + \omega(s_4'', s_5'', \tau s_6'') = -\omega(s_1'', s_2'', s_3'').$$

Because the homomorphism $\mathrm{Hom}(\eta, \xi_0) \to \varepsilon$ of the vector bundle $\mathrm{Hom}(\eta, \xi_0)$ into the trivial line bundle ε defined for $t \in \mathrm{Hom}(\eta, \xi_0)_x$ by the formula

$$t \mapsto \omega(ts_4''(x), s_5''(x), s_6''(x)) + \omega(s_4''(x), ts_5''(x), s_6''(x)) + \omega(s_4''(x), s_5''(x), ts_6''(x))$$

is obviously surjective for $x = x_0$, it is surjective in a neighborhood of x_0. Consequently, we can see that a local section τ satisfying the above condition exists. Summarizing, we have got sections $s_1', \dots, s_6'$ such that

(a) $s_1', \dots, s_6'$ are linearly independent,
(b) s_1', s_2', s_3' are sections of ξ_0,
(c) $\omega(s_4', s_5', s_6') = 0$,

with all these conditions being satisfied on a neighborhood of x_0. Finally, we set $s_4 = s_4'$, $s_5 = s_5'$, and $s_6 = s_6'$, and we define

$$s_1 = \gamma^{-1}(s_4'^* \wedge s_5'^*), \quad s_2 = \gamma^{-1}(s_4'^* \wedge s_6'^*), \quad s_3 = \gamma^{-1}(s_5'^* \wedge s_6'^*),$$

where $s_1'^*, \dots, s_6'^*$ are sections dual to the sections $s_1', \dots, s_6'$. Now, it is obvious that for every x from a neighborhood of x_0 the basis $s_1(x), \dots, s_6(x)$ is an adapted basis in ξ_x. From this the lemma follows easily. □

Because the vector bundle $\Lambda^2 \xi_1^*$ is orientable (we recall that for an n-dimensional vector bundle ζ there is $w_1(\Lambda^2 \zeta) = (n-1) w_1(\zeta)$), the vector bundle ξ_0 is also orientable. We choose a complementary subbundle $\eta \subset \xi$ to ξ_0, i. e. we have $\xi = \xi_0 \oplus \eta$. It is obvious that η is also orientable. As a consequence of this we get

$$\eta \cong \Lambda^2 \eta^*.$$

Again, similarly as in the algebraic part, we get an isomorphism

$$\gamma_\eta : \xi_0 \to \Lambda^2 \eta^*.$$

Now, we can prove the following proposition.

23. Proposition. *On an orientable 6-dimensional vector bundle ξ over a base X there exists a multisymplectic structure of type ω if and only if there exists an orientable 3-dimensional vector bundle η over X such that*

$$\xi \cong \eta \oplus \eta.$$

Proof. We have already seen that the condition is necessary. It remains to prove that it is also sufficient. Thus, let us assume that $\xi = \eta \oplus \eta$. The following considerations are in fact a bundlification of Lemma 21. We choose arbitrary regular bilinear form B on η, e. g. a riemannian metric. Further, we choose an isomorphism $H : \eta \to \Lambda^2 \eta$. We define a 3-form $\Omega \in \Lambda^2 \xi^*$ in the following way. For any $x \in X$ we set

$$\Omega_x((v_1, 0), (v_2, 0), (v_3, v_4)) = 0$$
$$\Omega_x((v_1, 0), (0, v_2), (0, v_3)) = B(v_1, H^{-1}(v_2 \wedge v_3))$$
$$\Omega_x((0, v_1), (0, v_2), (0, v_3)) = 0$$

for every $v_1, v_2, v_3, v_4 \in \eta_x$. It is easy to see that Ω is continuous and of type ω in every fiber of ξ. Consequently, by virtue of Lemma 22, it defines a multisymplectic structure on ξ. This finishes the proof. $\square$

Let us recall that a tangent structure on a 6-dimensional real vector bundle ξ is a continuous tensor field F of type $(1,1)$ on ξ, i. e. an endomorphism

$$F : \xi \to \xi \text{ such that } F^2 = 0 \text{ and } \ker F = \mathrm{im} F.$$

It is obvious that if there is on ξ a tangent structure F, then we get a 3-dimensional subbundle $\xi_0 = \ker F$, and for any complementary subbbundle η we have the isomorphism $F|\eta : \eta \to \xi_0$. This shows that the bundle ξ is a direct sum of two isomorphic subbundles. If the vector bundle ξ and the subbundle ξ_0 are orientable, then obviously ξ carries a multisymplectic structure of type ω. Conversely, if ξ is endowed with a multisymplectic structure of type ω, then $\xi \cong \eta \oplus \eta$. On $\eta \oplus \eta$ we have a tangent structure defined by the formula

$$F_x(v_1, v_2) = (v_2, 0).$$

Then we get a tangent structure also on the isomorphic bundle ξ. We have thus proved the following corollary.

24. Corollary. *On an orientable 6-dimensional vector bundle ξ over a base X there exists a multisymplectic structure of type ω if and only if there exists a tangent structure F such that the subbundle $\ker F$ is orientable.*

Considering the principal $GL(6,\mathbb{R})$-bundle $Fr(\xi)$ consisting of all frames of the vector bundle ξ, we can state the following lemma.

25. Lemma. *On an orientable 6-dimensional vector bundle ξ over a base X there exists a multisymplectic structure of type ω if and only if the principal $GL(6,\mathbb{R})$-bundle $Fr(\xi)$ can be reduced to the subgroup $O(\omega) \subset GL(6,\mathbb{R})$.*

Proof. If there is on ξ a multisymplectic structure Ω of type ω, the corresponding reduction to the subgroup $O(\omega)$ consists of all adapted frames. Conversely, if there exists a reduction of $Fr(\xi)$ to the subgroup $O(\omega)$, we use any frame from this reduction, and define a multisymplectic form Ω of type ω on ξ using the formulas defining an adapted frame. $\square$

We now pass completely to the category of differentiable manifolds. Let M be a 6-dimensional differentiable manifold, and let Ω be a diffrentiable 3-form on M, or in another words on the vector bundle $\xi = TM$, defining there a multisymplectic structure of type ω. Let us recall that we do not suppose that the 3-form Ω is closed. We have just seen that with this multisymplectic structure there is associated a G-structure, where $G = O(\omega)$. We shall call a multisymplectic structure Ω of type ω *integrable* if and only if the associated $O(\omega)$-structure is integrable. The subbbundle $\xi_0 \subset \xi = TM$ is in this situation a distribution on M, and we shall denote it by D_0.

26. Proposition. *A multisymplectic structure Ω of type ω is integrable if and only if the following two conditions are satisfied:*

(i) *the distribution D_0 is integrable,*

(ii) *the 3-form Ω is closed, i. e. $d\Omega = 0$.*

Proof. Let Ω be integrable. This means that in a neighborhood of every point $x \in M$ we can find coordinates $(x_1, \dots, x_6)$ such that on this neighborhood we have

$$\Omega = dx_1 \wedge dx_4 \wedge dx_5 + dx_2 \wedge dx_4 \wedge dx_6 + dx_3 \wedge dx_5 \wedge dx_6.$$

It is obvious that the distribution D_0 is spanned by the vector fields $\partial/\partial x_1$,$\partial/\partial x_2$, $\partial/\partial x_3$, and consequently it is integrable. Moreover, obviously $d\Omega = 0$. Conversely, let us assume that the conditions (i) and (ii) are satisfied. Because the distribution D_0 is integrable, on a neighborhood of any point $x \in M$, we can find functions x_4, x_5, x_6 such that the distribution D_0 is described by the equations

$$dx_4 = 0, \quad dx_5 = 0, \quad dx_6 = 0.$$

Similarly as in the previous section, we can introduce a subbundle $A_{D_0} \subset \Lambda^2 T^*(M)$. We can immediately see that the three 2-forms

$$dx_4 \wedge dx_5, dx_4 \wedge dx_5, dx_5 \wedge dx_6$$

are local sections of A_{D_0}. Now, we define vector fields X_1, X_2, X_3 on the neighborhood under consideration and belonging to the distribution D_0 by the formulas

$$X_1 = \beta^{-1}(dx_4 \wedge dx_5), \quad X_2 = \beta^{-1}(dx_4 \wedge dx_6), \quad X_3 = \beta^{-1}(dx_5 \wedge dx_6).$$

In other words, we have

$$\iota_{X_1}\omega = dx_4 \wedge dx_5, \quad \iota_{X_2}\omega = dx_4 \wedge dx_5, \quad \iota_{X_3}\omega = dx_5 \wedge dx_6.$$

Because $d\Omega = 0$ we have $d\Omega(X, X', Y, Z) = 0$ for any vector fields X, X', Y, Z defined around the point x. Let us assume that the vector fields X and X' belong to the distribution D_0. Then we obtain

$$\begin{gathered} 0 = 4d\Omega(X, X', Y, Z) = \\ = X\Omega(X', Y, Z) - X'\Omega(X, Y, Z) + Y\Omega(X, X', Z) - Z\Omega(X, X', Y) \\ -\Omega([X, X'], Y, Z) + \Omega([X, Y], X', Z) - \Omega([X, Z], X', Y) \\ -\Omega([X', Y], X, Z) + \Omega([X', Z], X, Y) - \Omega([Y, Z], X, X') = \\ = X\Omega(X', Y, Z) - X'\Omega(X, Y, Z) \\ -\Omega([X, X'], Y, Z) + \Omega([X, Y], X', Z) - \Omega([X, Z], X', Y) \\ -\Omega([X', Y], X, Z) + \Omega([X', Z], X, Y). \end{gathered}$$

Hence we get

$$\begin{gathered} \Omega([X, X'], Y, Z) = X\Omega(X', Y, Z) - X'\Omega(X, Y, Z) \\ +\Omega([X, Y], X', Z) - \Omega([X, Z], X', Y) - \Omega([X', Y], X, Z) + \Omega([X', Z], X, Y) = \\ = X((\iota_{X'}\Omega)(Y, Z)) - X'((\iota_X\Omega)(Y, Z)) \end{gathered}$$

$-(\iota_{X'}\Omega)([X,Y],Z)+(\iota_{X'}\Omega)([X,Z],Y)+(\iota_X\Omega)([X',Y],Z)-(\iota_X\Omega)([X',Z],Y).$
Setting now $X=X_1$ and $X'=X_2$ we get

$$\begin{aligned}\Omega([X_1,X_2],Y,Z)&=X_1((dx_4\wedge dx_6)(Y,Z))-X_2((dx_4\wedge dx_5)(Y,Z))\\&\quad-(dx_4\wedge dx_6)([X_1,Y],Z)+(dx_4\wedge dx_6)([X_1,Z],Y)\\&\quad+(dx_4\wedge dx_5)([X_2,Y],Z)-(dx_4\wedge dx_5)([X_2,Z],Y)=\\&=X_1((dx_4\wedge dx_6)(Y,Z))-Y((dx_4\wedge dx_6)(X_1,Z))+Z((dx_4\wedge dx_6)(X_1,Y))\\&\quad-(dx_4\wedge dx_6)([X_1,Y],Z)+(dx_4\wedge dx_6)([X_1,Z],Y)-(dx_4\wedge dx_6)([Y,Z],X_1)\\&\quad-X_2((dx_4\wedge dx_5)(Y,Z))+Y((dx_4\wedge dx_5)(X_2,Z))-Z((dx_4\wedge dx_5)(X_2,Y))\\&\quad+(dx_4\wedge dx_5)([X_2,Y],Z)-(dx_4\wedge dx_5)([X_2,Z],Y)+(dx_4\wedge dx_5)([Y,Z],X_2)=\\&=(d(dx_4\wedge dx_6))(X_1,Y,Z)-(d(dx_4\wedge dx_5))(X_2,Y,Z)=0.\end{aligned}$$

The 3-form Ω is multisymplectic, and this implies that $[X_1,X_2]=0$. Along the same lines we can prove that $[X_1,X_3]=0$ and $[X_2,X_3]=0$. Now, it is easy to see that we can find functions x_1,x_2,x_3 defined in a neighborhood of the point x such that together with the functions x_4,x_5,x_6 they form coordinates, and

$$X_1=\frac{\partial}{\partial x_1},\quad X_2=\frac{\partial}{\partial x_2},\quad X_3=\frac{\partial}{\partial x_3}.$$

See e. g. [S]. Consequently, with respect to the coordinates $(x_1,\dots,x_6)$, we have

$$\Omega=dx_1\wedge dx_4\wedge dx_5+dx_2\wedge dx_4\wedge dx_6+dx_3\wedge dx_5\wedge dx_6.$$

This finishes the proof. □

The aim of the next example is to show that the assumption concerning the integrability of D_0 is is independent of the assumption $d\Omega=0$. We present an example of a multisymplectic 3-form of type ω which is closed but the distribution D_0 of which is not integrable.

27. Example. Let us consider the open subset $U=\{x\in\mathbb{R}^6; x_3\neq x_4\}$. We define on U the following six 1-forms

$$\begin{aligned}A_1&=e^{x_1x_3+x_4}\,dx_1+x_3\,dx_3+dx_6\\A_2&=e^{x_1x_3+x_4}\,dx_1+x_4\,dx_3+dx_5\\A_3&=dx_2\\A_4&=x_1\,dx_3+dx_4\\A_5&=dx_5\\A_6&=dx_6\end{aligned}$$

It is easy to see that at each point $x\in U$ the 1-forms $A_{1x},\dots,A_{6x}$ are linearly independent. Let us set

$$\Omega=A_1\wedge A_4\wedge A_5+A_2\wedge A_4\wedge A_6+A_3\wedge A_5\wedge A_6.$$

We have

$$dA_4=dx_1\wedge dx_3,\quad dA_5=0,\quad dA_6=0,$$

which shows that the distribution D_0 determined by the equations $dA_4 = 0$, $dA_5 = 0$, and $dA_6 = 0$ is not integrable. Moreover, we have

$$\begin{gathered} d\Omega = (-x_1 e^{x_1x_3+x_4} dx_1 \wedge dx_3 - e^{x_1x_3+x_4} dx_1 \wedge dx_4) \wedge (x_1 dx_3 + dx_4) \wedge dx_5 \\ -(e^{x_1x_3+x_4} dx_1 + x_3 dx_3 + dx_6) \wedge dx_1 \wedge dx_3 \wedge dx_5 \\ +(-x_1 e^{x_1x_3+x_4} dx_1 \wedge dx_3 - e^{x_1x_3+x_4} dx_1 \wedge dx_4 - dx_3 \wedge dx_4) \wedge (x_1 dx_3 + dx_4) \wedge dx_6 \\ -(e^{x_1x_3+x_4} dx_1 + x_4 dx_3 + dx_5) \wedge dx_1 \wedge dx_3 \wedge dx_6 = 0. \end{gathered}$$

28. Remark. Topological conditions for the existence of a multisymplectic 3-form of type ω (or equivalently of a tangent structure) on a 6-dimensional vector bundle will be the subject of a forthcoming paper.

References

[1] Capdevielle, B. *Classification des formes trilinéaires alternées en dimension 6*, L'enseignement Math. **18-19** (1972-73), 225-243.

[2] Cantrijn, F.; Ibort, A.; de León, M. *On the geometry of multisymplectic manifolds*, preprint, 1998

[3] Hitchin, N., *The geometry of three-forms in six and seven dimensions*, preprint arXiv.math. DG/0010054 5 Oct 2000

[4] Spivak, M., *A Comprehensive Introduction to Differential Geometry*, vol. 1, Publish or Perish, Berkeley, 1979, Chap. 5, Th. 14, p. 219.

Mathematical Institute, Academy of sciences of the Czech Republic, Žižkova 22, 616 62 Brno, Czech Republic

E-mail address: vanzura@drs.ipm.cz