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PROPERTIES OF THE WEYL CONFORMAL CURVATURE OF KÄHLER-NORDEN MANIFOLDS

KARINA SŁUKA

ABSTRACT. Let (M, J, g) be an $n = 2m$-dimensional Kählerian manifold endowed with a Norden metric. It is proved that (M, J, g) is conformally flat if and only if it is holomorphically projectively flat and its scalar curvature vanishes; the $*$-scalar curvature of such a manifold is constant and it is locally symmetric. If (M, J, g) is of recurrent conformal curvature, then it is locally symmetric in case of dimension $n \geqslant 6$, and it is locally symmetric or holomorphically projectively flat in case of dimension $n = 4$. Next, we show that the pseudosymmetry as well as the Weyl-pseudosymmetry and the holomorphically projective-pseudosymmetry of (M, J, g) reduces to the semisymmetry. Moreover, the Ricci-pseudosymmetry of (M, J, g) reduces to the Ricci-semisymmetry. An example of a Ricci-semisymmetric and non-semisymmetric Kähler-Norden structure is stated. Examples of semisymmetric, especially locally symmetric, Kähler-Norden structures are given in [15].

1. PRELIMINARIES

By a Kählerian manifold with Norden metric (Kähler-Norden in short) [9] we mean a triple (M, J, g), where M is a connected differentiable manifold of dimension $n = 2m$, J is a $(1,1)$-tensor field and g is a pseudo-Riemannian metric on M satisfying the conditions

$$J^2 = -I, \qquad g(JX, JY) = -g(X, Y), \qquad \nabla J = 0,$$

for any $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of vector fields on M and ∇ is the Levi-Civita connection of g.

Let (M, J, g) be a Kähler-Norden manifold. Since in dimension 2 such a manifold is flat, we assume in the sequel that $\dim M \geq 4$. Let $R(X, Y)$ be the curvature operator $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and let R be the Riemann-Christoffel curvature tensor, $R(X, Y, Z, W) = g(R(X, Y)Z, W)$. The Ricci tensor S is defined as $S(X, Y) = \operatorname{trace}\{Z \mapsto R(Z, X)Y\}$. These tensors have the following properties [1]

$$\begin{aligned} R(JX, JY) &= -R(X, Y), & R(JX, Y) &= R(X, JY), \\ S(JY, Z) &= \operatorname{trace}\{X \mapsto R(JX, Y)Z\}, & S(JX, Y) &= S(JY, X). \end{aligned} \tag{1}$$

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Let $\widetilde{S}$ be the Ricci operator. Then we have $g(\widetilde{S}X,Y) = S(X,Y)$ and

$$\widetilde{S}Y = -\sum_i \varepsilon_i R(e_i, Y)e_i.$$

In the above and in the sequel, $(e_1, e_2, ..., e_n)$ is an orthonormal frame and ε_i are the indicators of e_i, $\varepsilon_i = g(e_i, e_i) = \pm 1$. The Weyl conformal curvature tensor C is defined in the usual way,

$$C(X,Y) = R(X,Y) + \frac{1}{n-2}\left(\frac{r}{n-1} X \wedge Y - \widetilde{S}X \wedge Y - X \wedge \widetilde{S}Y\right), \tag{2}$$

where r $(= \operatorname{trace}\widetilde{S})$ is the scalar curvature and for any X,Y, the operator $X \wedge Y$ is defined by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y, \quad Z \in \mathfrak{X}(M).$$

Using (2) and (1), we find the following useful equality

$$\sum_i \varepsilon_i C(Je_i, JY)e_i = \frac{n}{n-2}\widetilde{S}Y + \frac{r^*}{n-2}JY - \frac{r}{(n-1)(n-2)}Y, \tag{3}$$

where r^* is the $*$-scalar curvature, which is defined as the trace of $J\widetilde{S}$. In the above, we have applied the identity $\sum_i \varepsilon_i g(Je_i, e_i) = 0$, which is a consequence of the traceless of J.

The holomorphically projective curvature tensor P is defined in the following way ([19], [15])

$$P(X,Y) = R(X,Y) - \frac{1}{n-2}(X \wedge_S Y - JX \wedge_S JY), \tag{4}$$

the operator $X \wedge_S Y$ is defined by

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y, \quad Z \in \mathfrak{X}(M).$$

We notice, for later use, that this tensor has the following properties

$$\begin{aligned} &P(X,Y,Z,W) = -P(Y,X,Z,W), \quad P(JX,JY,Z,W) = -P(X,Y,Z,W), \\ &\sum_i \varepsilon_i P(e_i, Y, Z, Je_i) = 0, \quad \sum_i \varepsilon_i P(X,Y,e_i,e_i) = 0, \end{aligned} \tag{5}$$

A Kähler-Norden manifold (M,J,g) is holomorphically projectively flat if and only if its holomorphically projective curvature tensor P vanishes identically (ibidem).

2. The Weyl conformal curvature

Theorem 1. *A Kähler-Norden manifold (M,J,g) is conformally flat if and only if it is holomorphically projectively flat and its scalar curvature vanishes.*

Proof. Let us assume that $C = 0$. Then, using (3), we get

$$\widetilde{S}Y = -\frac{r^*}{n}JY + \frac{r}{n(n-1)}Y.$$

Hence, taking the trace, we find $r = 0$, and consequently

$$\widetilde{S}Y = -\frac{r^*}{n}\,JY. \tag{6}$$

Then, from (2) by $r = 0$ and (6), we obtain

$$R(X,Y) = -\frac{r^*}{n(n-2)}(X \wedge JY + JX \wedge Y). \tag{7}$$

Applying (6) and (7) into the right hand side of (4) yields $P = 0$.

Conversely, suppose $P = 0$ and $r = 0$. In this case, the Ricci operator and the curvature tensor are as in the formulas (6) and (7) (see [15]). By applying these formulas into (2), we find $C = 0$, which completes the proof. □

Corollary 1. *The curvature tensor and the Ricci tensor of a conformally flat Kähler-Norden manifold (M, J, g) have the shapes* (6) *and* (7)*, respectively. Moreover, the $*$-scalar curvature is constant and the manifold is locally symmetric.*

Proof. Firstly, using a well known identity and $r = 0$, we obtain

$$\sum_i \varepsilon_i(\nabla_{e_i}S)(Y, e_i) = \frac{1}{2}dr(Y) = 0. \tag{8}$$

On the other hand, with the help of (6), we find

$$\sum_i \varepsilon_i(\nabla_{e_i}S)(Y, e_i) = -\frac{1}{n}dr^*(JY) = 0.$$

Hence, r^* is constant. Therefore, the local symmetry follows from (7). □

Theorem 2. *Any Kähler-Norden manifold (M, J, g) with parallel Weyl conformal curvature tensor is locally symmetric.*

Proof. Assume that the Weyl conformal curvature tensor of a Kähler-Norden manifold (M, J, g) is parallel. Covariant differentiation of (3) and our assumption yield

$$0 = n(\nabla_W S)(Y, Z) + dr^*(W)g(JY, Z) - \frac{1}{n-1}dr(W)g(Y, Z). \tag{9}$$

Contracting formula (9) with respect to the pair of arguments Y, Z (that is, taking $Y = Z = e_i$ into (9), multiplying by ε_i and summing up over i), one finds

$$dr(W) = 0. \tag{10}$$

Using the famous identity and (10), we obtain

$$\sum_i \varepsilon_i(\nabla_{e_i}S)(Y, e_i) = \frac{1}{2}dr(Y) = 0. \tag{11}$$

Equality (10) applied into (9) gives

$$(\nabla_W S)(Y, Z) = -\frac{1}{n}dr^*(W)g(JY, Z).$$

Taking JY instead of Y into the last relation, and transvecting the resulting equality with respect to Y, Z, one gets $dr^* = 0$, which implies also $\nabla S = 0$.

In view of (2), the covariant derivative ∇C can be expressed in the following form

$$(\nabla_W C)(X,Y) = (\nabla_W R)(X,Y) + \frac{1}{n-2}\left(\frac{1}{n-1}dr(W)X\wedge Y - X\wedge((\nabla_W\widetilde{S})Y) - ((\nabla_W\widetilde{S})X)\wedge Y\right).$$

To have the proof complete it is now sufficient to apply $\nabla\widetilde{S} = 0$ together with $\nabla C = 0$ and $dr = 0$ into the last expression. □

In the context of the above theorem, it should be said that the symmetric Kähler-Norden manifolds are not yet classified, even locally. Examples of locally symmetric Kähler-Norden structures are given in [15].

In the rest of this section, we study Kähler-Norden manifolds satisfying the condition

$$\nabla C = \varphi \otimes C, \tag{12}$$

where φ is a 1-form.

A pseudo-Riemannian manifold is said to be of recurrent conformal curvature (cf. e.g. [13], [14]) if its conformally curvature Weyl tensor C is non identically zero and satisfies the condition (12), φ is then called the recurrence form of C.

Proposition 1. *If a Kähler-Norden manifold (M, J, g) satisfies the condition (12), then its holomorphically projective curvature tensor P realizes*

$$\nabla P = \varphi \otimes P. \tag{13}$$

Proof. From (2) and (12), we obtain

$$(\nabla_W R)(X,Y) = \varphi(W)R(X,Y) - \frac{1}{n-2}\left(\frac{1}{n-1}(dr(W) - \varphi(W)r)X\wedge Y - X\wedge(\nabla_W\widetilde{S})Y + \varphi(W)X\wedge\widetilde{S}Y - (\nabla_W\widetilde{S})X\wedge Y + \varphi(W)\widetilde{S}X\wedge Y\right). \tag{14}$$

Formula (3) together with (12) let us find the condition

$$\begin{aligned} &n((\nabla_W S)(Y,Z) - \varphi(W)S(Y,Z)) \\ = \quad &-(dr^*(W) - \varphi(W)r^*)g(JY,Z) + \frac{1}{n-1}(dr(W) - \varphi(W)r)g(Y,Z). \end{aligned} \tag{15}$$

Contracting (15) with respect to Y, Z, one finds

$$dr(W) - \varphi(W)r = 0.$$

Therefore, we can rewrite (15) in the following way

$$(\nabla_W S)(Y,Z) = \varphi(W)S(Y,Z) - \frac{1}{n}(dr^*(W) - \varphi(W)r^*)g(JY,Z). \tag{16}$$

When we substitute (16) into (14), we obtain

$$(\nabla_W R)(X,Y) = \varphi(W)R(X,Y) + \eta(X \wedge (JY) + (JX) \wedge Y), \tag{17}$$

with

$$\eta = \frac{1}{n(n-2)}(dr^*(W) - \varphi(W)r^*).$$

Differentiating covariantly (4) and using (16), (17) enable us to check that the holomorphically projective curvature tensor P satisfies the condition (13). □

Theorem 3. *Let (M, J, g) be a Kähler-Norden manifold of recurrent conformal curvature.*

(1) If $\dim M \geqslant 6$, then the manifold (M, J, g) is locally symmetric.

(2) If $\dim M = 4$, then the manifold (M, J, g) is locally symmetric or holomorphically projectively flat.

Proof. In virtue of Proposition 1, the tensor P satisfies the condition (13). As it is known, under such a condition, P vanishes everywhere or nowhere on M [17, Theorem 3.8], [18].

If P does not vanish at every point of the manifold, then (M, J, g) is of recurrent holomorphically projective curvature, and by Theorem 4 of [15] it is locally symmetric. Let us assume that P is identically zero, that is, the manifold (M, J, g) is holomorphically projectively flat. If $\dim M \geqslant 6$, then by Proposition 3 of [15], the manifold is locally symmetric. □

Theorem 4. *Let (M, J, g) be a holomorphically projectively flat Kähler-Norden manifold of dimension 4. If its scalar curvature does not vanish everywhere on M, then (M, J, g) is of recurrent conformal curvature.*

Proof. Let us assume that $P = 0$. By Proposition 3 of [15], the curvature tensor and the Ricci tensor have the shapes

$$\begin{aligned} R(X,Y) &= \frac{r}{n(n-2)}(X \wedge Y - JX \wedge JY) - \frac{r^*}{n(n-2)}(X \wedge JY + JX \wedge Y), \\ S(X,Y) &= \frac{r}{n}g(X,Y) - \frac{r^*}{n}g(JX,Y). \end{aligned}$$

Substituting these relations into (2), we find

$$C(X,Y) = \frac{r}{n(n-1)(n-2)}(X \wedge Y - (n-1)JX \wedge JY).$$

Therefore, the covariant derivative ∇C can be expressed in the following form

$$(\nabla_W C)(X,Y) = \frac{1}{n(n-1)(n-2)}dr(X \wedge Y - (n-1)JX \wedge JY).$$

Now it is easy to see that the covariant derivative of the non-zero tensor C satisfies the condition

$$\nabla C = \varphi \otimes C$$

with $\varphi = d \ln |r|$. This means that the manifold is of recurrent conformal curvature with φ as the recurrence form. $\square$

Remark. In the paper [15], we have constructed examples of 4-dimensional Kähler-Norden manifolds which are holomorphically projectively flat with non-constant scalar curvature as well as locally symmetric.

3. Pseudosymmetry curvature conditions

For an $(0,k)$-tensor $(k \geqslant 1)$ field T on a pseudo-Riemannian manifold (M,g), we define the $(0,k+2)$-tensor field $R \cdot T$ by the condition

$$(R \cdot T)(U,V,X_1,X_2,...,X_k) = (R(U,V) \cdot T)(X_1,X_2,...,X_k) \\ = -T(R(U,V)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,X_2,\ldots,X_{k-1},R(U,V)X_k). \tag{18}$$

A pseudo-Riemannian manifold (M,g) is said to be semisymmetric [16] if $R \cdot R = 0$, Ricci-semisymmetric if $R \cdot S = 0$, Weyl-semisymmetric if $R \cdot C = 0$. Clearly, any semisymmetric manifold is Ricci-semisymmetric as well as Weyl-semisymmetric, and the converse statements do not hold in general [6], [5].

To initiate the definition of pseudosymmetry, we define also an $(0,k+2)$-tensor field $Q(g,T)$ associated with any $(0,k)$-tensor $(k \geqslant 1)$ field T on a pseudo-Riemannian manifold

$$Q(g,T)(U,V,X_1,X_2,...,X_k) = ((U \wedge V) \cdot T)(X_1,X_2,...,X_k) \\ = -T((U \wedge V)X_1,X_2,...,X_k)) - \cdots - T(X_1,X_2,...,X_{k-1},(U \wedge V)X_k). \tag{19}$$

A pseudo-Riemannian manifold (M,g) is called pseudosymmetric [6] if there exists a function $L_R : M \to \mathbb{R}$ such that

$$R \cdot R = L_R \, Q(g,R). \tag{20}$$

Clearly, every semisymmetric manifold is also pseudosymmetric. The converse is not true in general [6]. Examples of semisymmetric Kähler-Norden metrics are found in [15]. In the class of Kähler-Norden metrics, pseudosymmetry reduces to semisymmetry. Indeed, we prove the following:

Theorem 5. *Every pseudosymmetric Kähler-Norden manifold is semisymmetric.*

Proof. Assume that a Kähler-Norden manifold (M,J,g) satisfies the condition (20). Using (1) and (18), we claim that

$$-(R \cdot R)(U,V,X,Y,Z,W) = (R \cdot R)(JU,JV,X,Y,Z,W).$$

Thus, by (20), we have

$$-L_R \, Q(g,R)(U,V,X,Y,Z,W) = L_R \, Q(g,R)(JU,JV,X,Y,Z,W).$$

Suppose that L_R is non-zero at a certain point $p \in M$. Then the above equality gives at the point p

$$-Q(g,R)(U,V,X,Y,Z,W) = Q(g,R)(JU,JV,X,Y,Z,W).$$

Contracting the last identity with respect to V, X, we obtain

$$-\sum_i \varepsilon_i Q(g,R)(U,e_i,e_i,Y,Z,W) = \sum_i \varepsilon_i Q(g,R)(JU,Je_i,e_i,Y,Z,W),$$

which, with the help of (19), can be rewritten in the following form

$$\begin{aligned} & nR(U,Y,Z,W) - R(U,Y,Z,W) + R(Y,U,Z,W) + R(Z,Y,U,W) \\ & + R(W,Y,Z,U) + g(U,Z)S(Y,W) - g(U,W)S(Y,Z) \\ & = -R(U,Y,Z,W) - R(JY,JU,Z,W) - R(JZ,Y,JU,W) - R(JW,Y,Z,JU) \\ & - g(JU,Z)\sum_i \varepsilon_i R(e_i,Y,W,Je_i) + g(JU,W)\sum_i \varepsilon_i R(e_i,Y,Z,Je_i). \end{aligned}$$

Hence, using (1) and the first Bianchi identity, we get

$$\begin{aligned} nR(U,Y,Z,W) &= g(JU,W)S(JY,Z) - g(JU,Z)S(JY,W) \\ & \quad + g(U,W)S(Y,Z) - g(U,Z)S(Y,W). \end{aligned} \tag{21}$$

Contracting (21) with respect to Y, Z, we find

$$S(U,W) = \frac{1}{n}(rg(U,W) + r^*(JU,W)).$$

This leads to $R \cdot S = 0$. Using this fact and (21), we obtain $R \cdot R = 0$, that means the semisymmetry of our manifold. □

A pseudo-Riemannian manifold (M, g) is called Ricci-pseudosymmetric [6] if there exists a function $L_S : M \to \mathbb{R}$ such that

$$R \cdot S = L_S\, Q(g,S). \tag{22}$$

Clearly, every Ricci-semisymmetric manifold is also Ricci-pseudosymmetric. The converse is not true in general [6]. However, we shall prove that the Ricci-pseudosymmetry reduces to the Ricci-semisymmetry in the class of Kähler-Norden metrics.

Theorem 6. *Every Ricci-pseudosymmetric Kähler-Norden manifold is Ricci-semisymmetric.*

Proof. Let us assume then the equality (22) holds on M. Now, in the same way as in the proof of Theorem 5, we have

$$-L_S\, Q(g,S)(U,V,X,Y) = L_S\, Q(g,S)(JU,JV,X,Y). \tag{23}$$

Suppose that the function L_S is non-zero at a certain point of M. Therefore, (23) takes the form

$$-Q(g,S)(U,V,X,Y) = Q(g,S)(JU,JV,X,Y). \tag{24}$$

Contracting the last identity with respect to V, X, we get

$$-\sum_i \varepsilon_i Q(g,S)(U,e_i,e_i,Y) = \sum_i \varepsilon_i Q(g,S)(JU,Je_i,e_i,Y),$$

which in virtue of (19) and (1) can be written as

$$S(X,Y) = \frac{1}{n}(rg(X,Y) + r^* g(JX,Y)).$$

This gives immediately $R \cdot S = 0$, which completes the proof. □

It is obvious that a pseudosymmetric manifold is Ricci-pseudosymmetric. And as it is already known (see [6]), the converse does not hold in the class of pseudo-Riemannian manifolds. The below example shows that the converse is not true even in the class of Kähler-Norden manifolds.

Example. Let $(x^1, ..., x^{4m}, x^{4m+1} = u, x^{4m+2} = v)$ be the Cartesian coordinates in the Cartesian space $\mathbb{R}^n$, with $n = 4m + 2 \geqslant 10$. Let

$$K_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Define a Norden metric g on $\mathbb{R}^n$ by

$$[g_{ij}] = \begin{bmatrix} FI_m & I_m & -GI_m & 0 & 0 \\ I_m & 0 & 0 & 0 & 0 \\ -GI_m & 0 & -FI_m & -I_m & 0 \\ 0 & 0 & -I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & K_1 \end{bmatrix},$$

where I_m is the identity matrix of rank m and functions F, G are defined on $\mathbb{R}^n$, depend on u, v only and satisfy the Cauchy-Riemann equations

$$\frac{\partial F}{\partial u} = \frac{\partial G}{\partial v}, \qquad \frac{\partial F}{\partial v} = -\frac{\partial G}{\partial u}.$$

Moreover, define a complex structure J on $\mathbb{R}^n$ by

$$[J^i_j] = \begin{bmatrix} 0 & 0 & -I_m & 0 & 0 \\ 0 & 0 & 0 & -I_m & 0 \\ I_m & 0 & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_2 \end{bmatrix}.$$

One checks that (J, g) is a Kähler-Norden structure on $\mathbb{R}^n$. Moreover, it can be verified that the condition $R \cdot S = 0$ is fulfilled, which means our manifold is Ricci-semisymmetric. On the other hand, $R \cdot R$ is non-zero, which means that the manifold is not semisymmetric. □

A pseudo-Riemannian manifold (M, g) is called Weyl-pseudosymmetric [6] if there exists a function $L_C : M \to \mathbb{R}$ such that

$$R \cdot C = L_C\, Q(g, C). \tag{25}$$

Clearly, every pseudosymmetric as well as Weyl-semisymmetric manifold is Weyl-pseudosymmetric. In dimensions $\geqslant 5$, for a pseudo-Riemannian manifold, it is proved in [8, Theorem 1] that the condition (25) reduces to (20) with $L_R = L_C$ on the set of points at which the tensor C does not vanish. This is not true in dimension 4 [7]. Below, we prove that a stronger result holds good in the class of Kähler-Norden metrics. Namely, we prove the following:

Theorem 7. *Every Weyl-pseudosymmetric Kähler-Norden manifold is semisymmetric.*

Proof. Let (M, J, g) be a Kähler-Norden manifold which is Weyl-pseudosymmetric. At first, we prove that the Weyl-pseudosymmetric reduces to the Weyl-semisymmetry.

To do it let suppose that $R \cdot C$ is non-zero at a certain point $p \in M$. By the Weyl-pseudosymmetry, in the same manner as in the proof of Theorem 5, we obtain

$$L_C\, Q(g, C)(JU, V, X, Y, Z, W) = L_C\, Q(g, C)(U, JV, X, Y, Z, W).$$

Since L_C is non-zero at p, it must be satisfied at p

$$Q(g, C)(JU, V, X, Y, Z, W) = Q(g, C)(U, JV, X, Y, Z, W).$$

Contracting the last identity with respect to V, X, and next using (19) and the first Bianchi identity for C, we obtain

$$\begin{aligned} nC(JU, Y, Z, W) - C(JY, U, Z, W) + C(JZ, Y, W, U) - C(JW, Y, Z, U) \\ = g(U, Z) \sum_i \varepsilon_i C(e_i, Y, W, Je_i) - g(U, W) \sum_i \varepsilon_i C(e_i, Y, Z, Je_i). \end{aligned} \tag{26}$$

Contracting the above relation with respect to U, Z, we find

$$\sum_i \varepsilon_i C(e_i, Y, W, Je_i) = 0,$$

which turns (26) into

$$nC(JU, Y, Z, W) - C(JY, U, Z, W) + C(JZ, Y, W, U) - C(JW, Y, Z, U) = 0. \tag{27}$$

We write this equation three times, with the vector fields U, Z, W cyclically permuted. Summing all three equations gives

$$C(JU, Y, Z, W) + C(JZ, Y, W, U) + C(JW, Y, U, Z) = 0,$$

which used in (27) leads to

$$(n - 1)C(JU, Y, Z, W) - C(JY, U, Z, W) = 0. \tag{28}$$

If we substitute JU and JY instead of U and Y, respectively, into (28) and compare the obtained relation with (28), we deduce that $C = 0$ at point p. Consequently, $R \cdot C = 0$ at p. This is a contradiction.

Thus, our Kähler-Norden manifold (M, J, g) is Weyl-semisymmetric, that is, satisfies the condition $R \cdot C = 0$. Using this and (3) one can easily see that $R \cdot S = 0$. Consequently, we have $R \cdot R = 0$ on M. $\square$

A pseudo-Riemannian manifold (M, g) will be called holomorphically projective-pseudosymmetric if there exists a function $L_P : M \mapsto \mathbb{R}$ such that

$$R \cdot P = L_P \, Q(g, P). \tag{29}$$

Theorem 8. *Every holomorphically projective-pseudosymmetric Kähler-Norden manifold is semisymmetric.*

Proof. At the first part of the proof, we show that the holomorphically projective-pseudosymmetry implies the holomorphically projective semisymmetry.

To do it, let us assume that (M, J, g) is a holomorphically projective-pseudosymmetric Kähler-Norden manifold and $R \cdot P$ is non-zero at a certain point $p \in M$. Using the holomorphically projective-pseudosymmetry, in the same way as in the proof of Theorem 5, we find

$$-L_P \, Q(g, P)(U, V, X, Y, Z, W) = L_P \, Q(g, P)(JU, JV, X, Y, Z, W).$$

Since L_P is non-zero at p, the following formula

$$-Q(g, P)(U, V, X, Y, Z, W) = Q(g, P)(JU, JV, X, Y, Z, W)$$

must be satisfied at p. Contracting the last identity with respect to V, X, we have

$$\begin{aligned}
&nP(U, Y, Z, W) - P(U, Y, Z, W) + P(Y, U, Z, W) + P(Z, Y, U, W) \\
&+ P(W, Y, Z, U) - g(U, Z) \sum_i \varepsilon_i P(e_i, Y, e_i, W) \\
&= - P(U, Y, Z, W) - P(JY, JU, Z, W) - P(JZ, Y, JU, W) - P(JW, Y, Z, JU) \\
&+ g(U, JZ) \sum_i \varepsilon_i P(e_i, Y, Je_i, W) + g(U, JW) \sum_i \varepsilon_i P(e_i, Y, Z, Je_i).
\end{aligned}$$

Hence, using (5), we obtain

$$\begin{aligned}
&nP(U, Y, Z, W) + P(Z, Y, U, W) + P(W, Y, Z, U) \\
&+ P(JZ, Y, JU, W) + P(JW, Y, Z, JU) \\
&= g(U, JZ) \sum_i \varepsilon_i P(e_i, Y, Je_i, W) + g(U, Z) \sum_i \varepsilon_i P(e_i, Y, e_i, W).
\end{aligned} \tag{30}$$

Contracting the above equality with respect to Z, U, we find

$$\sum_i \varepsilon_i P(JW, Y, e_i, Je_i) = 0. \tag{31}$$

Taking JU instead of U in (30), contracting the resulting equation with respect to Z, U, and making use of (31), we deduce

$$\sum_i \varepsilon_i P(e_i, Y, Je_i, W) = 0. \tag{32}$$

Moreover, contracting (30) with respect to Y, Z and using

$$\sum_i \varepsilon_i P(JW, e_i, e_i, JU) = \sum_i \varepsilon_i P(W, e_i, e_i, U),$$

we obtain

$$n\sum_i \varepsilon_i P(U,e_i,e_i,W) = -\sum_i \varepsilon_i P(W,e_i,e_i,U),$$

consequently,

$$\sum_i \varepsilon_i P(U,e_i,e_i,W) = 0. \tag{33}$$

Now, (30) with the help of (32) and (33) can be rewritten in the following form

$$\begin{aligned} nP(U,Y,Z,W) &= P(Y,Z,U,W) + P(Y,W,Z,U) \\ &\quad -P(JZ,Y,JU,W) - P(JW,Y,Z,JU). \end{aligned} \tag{34}$$

When we put JU, JY instead of U, Y into (34), respectively, and compare the obtained formula with (34), we conclude that $P = 0$ at point p. This gives $R\cdot P = 0$, which is a contradiction.

Thus, the manifold (M, J, g) is holomorphically projective semisymmetric. To have the proof complete it is sufficient to use Theorem 2 of [15], which states that holomorphically projective semisymmetry always implies the semisymmetry. □

Final remarks. The conditions of the semisymmetry and pseudosymmetry type for the Riemann, Ricci and Weyl curvature tensors of Kählerian and para-Kählerian manifolds were studied in the papers [2] - [4], [10] - [11], [12], and others. Some of them have inspired the author in her investigations.

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Institute of Mathematics, Wrocław University of Technology, Wybrzeże Wyspiańskiego, 27, 50–370 Wrocław, Poland
E-mail address: `sluka@im.pwr.wroc.pl`