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ON TOTALLY UMBILICAL HYPERSURFACE WITH CONHARMONIC CURVATURE TENSOR

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ABSTRACT. The purpose of this paper is to study conharmonically recurrent Weyl spaces corresponding to the tensor K_{hijk}. In Section II, some relations which are needed in Section III are obtained. In Section III, it is shown that while the totally umbilical hypersurface W_n of the recurrent Weyl space is conharmonically Ricci recurrent, W_n is recurrent. After then, it is proved that conharmonically recurrent Weyl space is also conformally recurrent, but the converse is true if and only if the condition $\dot{\nabla}_l R = \lambda_l R$ holds.

1. INTRODUCTION

The geometrical features of Weyl's theory consists of a space-time manifold W_n on which is defined a symmetric (torsion free) linear connection Γ and, in the first instance, a Lorentz metric g. The manifold W_n and all structures on W_n are assumed smooth. The connection Γ is not assumed to be a metric connection with respect to g or any other metric on W_n. Rather, Γ and g are related in such a way as to recreate Weyl's original idea that parallel transport, with respect to Γ, of a tangent vector k at $p \in W_n$ along a curve c to a point $q \in W_n$ may result in change of the length of k (with respect to g). However the ratio of the lengths of k at p and q, where this makes sense (i.e., if k is non-null), depends only on p, q and c and not on k . Let W_n be a manifold of dimension n ($n > 2$) and let Γ be a symmetric linear connection on W_n. Then Γ is called a Weyl connection if there exists a metric g on W_n such that $\nabla g = g \otimes T$ for some 1-form T on W_n, where ∇ denotes covariant differentiaton with respect to Γ. If W_n admits a Weyl connection, it is called a Weyl manifold.

In local coordinates this reads $\nabla_k g_{ij} = 2T_k g_{ij}$ where in coordinate notation ∇_k denotes the covariant derivative with respect to Γ, and is just means that the tensor g is recurrent with respect to Γ with recurrence 1-form T . With g_{ij}, Γ, and the complementary vector T_k, this is equivalent to the following expression for the connection associated with Γ:

$$(1.1) \qquad \Gamma^h_{kl} = \frac{1}{2} g^{hm}(\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl}) - (\delta^h_k T_l + \delta^h_l T_k - g^{hm} g_{kl} T_m),$$

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Now suppose that Γ is fixed but g and T are changed to $\breve{g} = \lambda^p g$ and $\breve{T} = T + \partial(\ln \lambda)$ where λ is real valued function on W_n. Then $\nabla \breve{g} = \breve{g} \otimes \breve{T}$ still holds as does (1.1) for $\Gamma, \breve{g}$ and $\breve{T}$. Such changes $(g, T) \to (\lambda^p g, T + \partial(\ln \lambda))$ are the gauge transformations introduced by Weyl [1], [2].

Suppose that the metrics of W_n and W_{n+1} are elliptic and that they are given by $g_{ij} du^i du^j$ and $g_{ab} dx^a dx^b$, respectively, which are connected by the relation

$$g_{ij} = g_{ab} x_i^a x_j^b \ (i, j = 1, 2, ..., n; a, b = 1, 2, ..., n+1) \tag{1.2}$$

where x_i^a denotes the covariant derivative of x^a with respect to u^i. On the basis of (1.1) [3] and [4], using T_k as a normalizer Zlatanov introduced in [5] a prolonged covariant differentiation of the satellites A of g_{ij} with weight $\{p\}$ by the law

$$\dot{\nabla}_k A = \nabla_k A - pT_k A. \tag{1.3}$$

One can show that the prolonged covariant derivative of A, relative to W_n and W_{n+1}, is related by

$$\dot{\nabla}_k A = x_k^c \dot{\nabla}_c A. \tag{1.4}$$

By [5] we have $\dot{\nabla}_k g_{ij} = 0$ and $\dot{\nabla}_k g^{ij} = 0$ where g^{ij} is the reciprocal tensor of g_{ij}.

Let n^a be the contravariant components of vector field in W_{n+1} normal to W_n and let it normalized by the condition $g_{ab} n^a n^b = 1$. The moving frame $\{x_a^i, n_a\}$ in W_n, reciprocal to moving frame $\{x_i^a, n^a\}$ is defined by the relations [2]

$$n^a n_a = 1, \ n_a x_i^a = 0, \ n^a x_a^i = 0, \ x_i^a x_a^j = \delta_i^j. \tag{1.5}$$

Differentiating covariantly of each side of $(1.5)_4$ with respect to u^k and remembering that the weight of x_i^a is $\{0\}$, the following form

$$\dot{\nabla}_k x_i^a = \nabla_k x_i^a = w_{ik} n^a \tag{1.6}$$

holds.

The curvature tensor of the hypersurface R^h_{ijk} is given by

$$R^h_{ijk} = \frac{\partial \Gamma^h_{ik}}{\partial x^j} - \frac{\partial \Gamma^h_{ij}}{\partial x^k} + \Gamma^h_{mj} \Gamma^m_{ik} - \Gamma^h_{mk} \Gamma^m_{ij} \tag{1.7}$$

where $R^h_{ijk} = g^{hm} R_{mijk}$.

2. Totally umbilical hypersurface immersed in a recurrent Weyl space

If W_n admits of a tensor field $T_{...}$ such that

$$\dot{\nabla}_k T... = \lambda_k T_{...} \tag{2.1}$$

where λ_k is a non-zero vector field of W_n, then W_n is called a T-recurrent Weyl space and is denoted by $T_n - W$.

We note that, since the prolonged covariant derivative preserves the weight, ϕ_s is a satellite of g_{ij} with weight $\{0\}$.

A hypersurface of a Weyl space is called totally umbilical if $w_{ij} = \rho g_{ij}$ where ρ is a satellite of g_{ij} with weight $\{-1\}$. From this definition it follows that $\rho = \frac{M}{n}$ where M is the mean curvature of the hypersurface defined by $M = w_{ij} g^{ij}$. A hypersurface of a Weyl space is called totally geodesic if $w_{ij} = 0$.

The generalization of Gauss and Mainardi-Codazzi equations have the following forms [6]

$$R_{hijk} = \Omega_{hijk} + \bar{R}_{abcd} x_h^a x_i^b x_j^c x_k^d \tag{2.2}$$

$$\dot{\nabla}_k w_{ij} - \dot{\nabla}_j w_{ik} + \bar{R}_{abcd} x_i^b x_j^c x_k^d n^a = 0 \tag{2.3}$$

where $\bar{R}_{abcd}$ is the covariant curvature tensor of W_{n+1} and Ω_{hijk} is the Sylvestrian of w_{ij} defined by $\Omega_{hijk} = w_{hj} w_{ik} - w_{hk} w_{ij}$. These formulae have also been obtained in [7].

Let W_n be a hypersurface of recurrent Weyl space W_{n+1} with recurrence vector ϕ_a which is not orthogonal to the hypersurface W_n. If we denote the tangential component of ϕ_a by ϕ_r, then we have

$$\phi_k = x_k^a \phi_a. \tag{2.4}$$

Since W_{n+1} is recurrent-Weyl space, we can write

$$\lambda_r \bar{R}_{abcd} = \dot{\nabla}_r \bar{R}_{abcd} = x_r^e \dot{\nabla}_e \bar{R}_{abcd}. \tag{2.5}$$

Using (2.2), we get

$$\dot{\nabla}_r R_{hijk} = \dot{\nabla}_r \Omega_{hijk} + \dot{\nabla}_r (\bar{R}_{abcd} x_h^a x_i^b x_j^c x_k^d). \tag{2.6}$$

With the help of the equations (1.6) and (2.5), the formula (2.6) can be brought in the following form [6]

$$\begin{aligned} \dot{\nabla}_r R_{hijk} = {} & \dot{\nabla}_r \Omega_{hijk} + \phi_e \bar{R}_{abcd} x_h^a x_i^b x_j^c x_k^d x_r^e + \bar{R}_{abcd} x_i^b x_j^c x_k^d w_{hr} n^a + \\ & + \bar{R}_{abcd} x_h^a x_j^c x_k^d w_{ir} n^b + \bar{R}_{abcd} x_h^a x_i^b x_k^d w_{jr} n^c + \\ & + \bar{R}_{abcd} x_h^a x_i^b x_j^c w_{kr} n^d \ . \end{aligned} \tag{2.7}$$

If we use the equations (2.2),(2.3),(2.4),(2.7) and remembering that $w_{ij} = \frac{M}{n} g_{ij}$ and M is scalar invariant, then we find

$$\begin{aligned} \dot{\nabla}_r R_{hijk} = {} & \phi_r R_{hijk} + \frac{M}{n^2}[(\dot{\nabla}_j M) G_{hirk} + (\dot{\nabla}_k M) G_{hijr} + (\dot{\nabla}_i M) G_{kjrh} \\ & + (\dot{\nabla}_h M) G_{kjir}] + \frac{2M}{n^2}(\dot{\nabla}_r M) G_{hijk} - \frac{M^2}{n^2} \phi_r G_{hijk} \end{aligned} \tag{2.8}$$

where $G_{hijk} = g_{hj} g_{ik} - g_{hk} g_{ij}$. Multiplying (2.8) by g^{hk} and g^{ij}, we obtain, respectively

$$\begin{aligned} \dot{\nabla}_r R_{ij} = {} & \phi_r R_{ij} + \frac{M}{n^2}[(2-n)(\dot{\nabla}_j M) g_{ir} - 2n(\dot{\nabla}_r M) g_{ij} + \\ & + (2-n)(\dot{\nabla}_i M) g_{jr}] + \frac{M^2}{n^2}(n-1) \phi_r g_{ij} \end{aligned} \tag{2.9}$$

$$\dot{\nabla}_r R = \phi_r R + \frac{2M}{n^2}(\dot{\nabla}_r M)(-n^2-n+2) + \frac{M^2}{n}(n-1)\phi_r. \tag{2.10}$$

3. Conharmonic curvature tensor of a Weyl space

Let $W_n(g_{ij}, T_k)$ and $\bar{W}_n(\bar{g}_{ij}, \bar{T}_k)$ be two Weyl spaces with connections ∇_k and $\bar{\nabla}_k$, respectively, and let the map $\tau : W_n \to \bar{W}_n$ be a conformal mapping. As a special case, let the transformed expressions of the fundamental metric tensor g_{ij} and the coefficients of Weyl connection Γ^i_{kl} be the following forms [8]

$$\bar{g}_{ij} = g_{ij}\ ,\ \bar{g}^{ij} = g^{ij}, \tag{3.1}$$

$$\bar{\Gamma}^i_{kl} = \Gamma^i_{kl} + \delta^i_k P_l + \delta^i_l P_k - g_{kl} g^{im} P_m, \tag{3.2}$$

where the vector P_k is called the vector of conformal mapping such as

$$P_k = T_k - \bar{T}_k. \tag{3.3}$$

Let us seek the differentiable harmonic function A with weight $\{p\}$ defined by [9]

$$\bar{A} = e^{c\int P_j du^j} A,\ c = \frac{2-n-2p}{2}. \tag{3.4}$$

and then, we have the following expression

$$g^{kl}\nabla_k P_l + \frac{1}{2}(n-2)P^k P_k = 0. \tag{3.5}$$

Since a conformal transformation with P_k satisfying (3.5) transforms a harmonic function into a harmonic one in above sense: (3.4), we call it conharmonic transformation.

The conharmonic curvature tensor is in the following form [10]

$$\begin{aligned} K^h_{ijk} = R^h_{ijk} &- \frac{1}{n}(\delta^h_k R_{[ij]} - \delta^h_j R_{[ik]} + g_{ij} g^{hm} R_{[mk]} - g_{ik} g^{hm} R_{[mj]} + \\ &+ 2\delta^h_i R_{[kj]}) - \frac{1}{(n-2)}(\delta^h_k R_{(ij)} - \delta^h_j R_{(ik)} + g_{ij} g^{hm} R_{(mk)} - \\ &- g_{ik} g^{hm} R_{(mj)})\ . \end{aligned} \tag{3.6}$$

The conharmonic curvature tensor K^h_{ijk} of a Weyl space satisfies the following condition [10]

$$K_{ij} = \frac{1}{2-n} g_{ij} R \tag{3.7}$$

where K_{ij} is conharmonic Ricci tensor.

If a Weyl hypersurface W_n immersed in a recurrent Weyl space W_{n+1} is totally geodesic, then the hypersurface is recurrent Weyl with recurrence vector λ_r [6].

A totally geodesic hypersurface W_n immersed in a recurrent Weyl space W_{n+1} is conharmonically recurrent $(n > 2)$.

Proposition 3.1. If W_n is conharmonically Ricci-recurrent (may not be Ricci-recurrent), then the expression $\phi_r - 2T_r$ is locally gradient $(n > 2)$.

Proof. From (3.7) and (2.1), we get $\dot{\nabla}_r R = \phi_r R$. Thus, remembering that the scalar curvature R is scalar invariant with weight $\{-2\}$, using (1.3), we have

$$\phi_s - 2T_s = \frac{\nabla_s R}{R} \ (R = c_1 \bar{R};\ c_1 \neq 0,\ \text{const.})$$

where $\bar{R}$ is the scalar curvature of Weyl space W_{n+1}. Then, we say that $\phi_s - 2T_s$ is locally gradient.

Theorem 3.1. If a totally umbilical Weyl hypersurface W_n immersed in a recurrent Weyl space W_{n+1} is a conharmonically Ricci-recurrent, then W_n is a conharmonically recurrent Weyl space $(n > 2)$.

Proof. Let W_n be a totally umbilical hypersurface of a recurrent Weyl space W_{n+1}. Let W_n be also conharmonically Ricci-recurrent. Multiplying (2.10) by g_{ij}, we get

$$\dot{\nabla}_r(g_{ij}R) = \phi_r[Rg_{ij} + \frac{M^2}{n}(n-1)g_{ij}] + \frac{2M}{n^2} g_{ij}(\dot{\nabla}_r M)(-n^2 - n + 2). \tag{3.8}$$

Using the equation (3.7) in the form (3.8), we find

$$\frac{1}{n^2} g_{ij}(n-1)[nM^2\phi_r - 2(n+2)M(\dot{\nabla}_r M)] = 0, \tag{3.9}$$

then, we obtain

$$\frac{\dot{\nabla}_r M}{M} = \frac{n}{2(n+2)}\phi_r. \tag{3.10}$$

On the other hand, from the equation (2.9),

$$\dot{\nabla}_r R_{[jk]} = \phi_r R_{[jk]} \tag{3.11}$$

and

$$\begin{aligned} \dot{\nabla}_r R_{(jk)} &= \phi_r R_{(jk)} + \frac{M^2}{n^2}(n-1)\phi_r g_{jk} \\ &+ \frac{M}{n^2}[(2-n)(\dot{\nabla}_k M)g_{jr} - 2n(\dot{\nabla}_r M)g_{jk} + (2-n)(\dot{\nabla}_j M)g_{kr}]. \end{aligned} \tag{3.12}$$

Taking the prolonged covariant derivative of (3.6) with respect to u^r and putting the equations (3.11) and (3.12) in this expression, then we get

$$\begin{aligned} \dot{\nabla}_r K_{hijk} &= \dot{\nabla}_r R_{hijk} + \phi_r(K_{hijk} - R_{hijk}) - \\ &- \frac{2M^2}{n^2(n-2)}(n-1)\phi_r G_{ihjk} - \frac{M}{n^2(n-2)}[(2-n)(\dot{\nabla}_i M)G_{kjhr} + \\ &+ (2-n)(\dot{\nabla}_j M)G_{hikr} - 4n(\dot{\nabla}_r M)G_{hikj} + \\ &+ (2-n)(\dot{\nabla}_h M)G_{irjk} - (2-n)(\dot{\nabla}_k M)G_{hijr}]\ . \end{aligned} \tag{3.13}$$

Using (2.8) in (3.13), we obtain

$$\dot{\nabla}_r K_{hijk} = \phi_r K_{hijk} - \frac{1}{n(n-2)} G_{hijk} M[(\dot{\nabla}_r M)\frac{2(n+2)}{n} - \phi_r M]. \tag{3.14}$$

Using the expression (3.10), we get

$$\dot{\nabla}_r K_{hijk} = \phi_r K_{hijk}. \tag{3.15}$$

Corollary 3.1. If a totally umbilical Weyl hypersurface W_n immersed in a recurrent Weyl space W_{n+1} is a conharmonically Ricci-recurrent, then W_n is a recurrent Weyl space $(n > 2)$.

Proof. Multiplying (2.8) by g^{hr} and g^{ik} and using the equation (3.10), we obtain

$$g^{hr} g^{ik} (\dot{\nabla}_r R_{hijk} - \phi_r R_{hijk}) = \frac{(n-1)(n-2)}{2n^2} M^2 \phi_j. \tag{3.16}$$

If we multiply the expression (3.13) by g^{hr} and g^{ik} and use the equations (3.10), (3.15) and (3.16), we get $M^2 \phi_j = 0$. From this, since $\phi_j \neq 0$,$(n > 2)$, we find $M = 0$. In this case, using $M = 0$ and the expression (3.13), the proof is completed.

Corollary 3.2. If a totally umbilical Weyl hypersurface W_n immersed in a recurrent Weyl space W_{n+1} is a conharmonically recurrent, then W_n is a recurrent Weyl space $(n > 2)$.

Proof. Conharmonically recurrent Weyl space is also conharmonically Ricci recurrent. From Corollary 3.1, the result is clear.

Corollary 3.3. If a totally umbilical Weyl hypersurface W_n immersed in a recurrent Weyl space W_{n+1} is a Ricci recurrent, then W_n is a recurrent Weyl space $(n > 2)$.

Proof. Since Ricci recurrent Weyl space is also conharmonically Ricci recurrent, from Corollary 3.1, the proof is clear.

Theorem 3.2. A conharmonically recurrent Weyl space is also a conformally recurrent Weyl space. Conversely, a conformally recurrent Weyl space with its recurrence vector field ϕ_r is conharmonically recurrent if its scalar curvature satisfies $\dot{\nabla}_r R = \lambda_r R$.

Proof. Suppose that W_n be a conharmonically recurrent Weyl space. The so-called conformal curvature tensor introduced by F. Özen and S.A. Uysal [12], is in the following form

$$
\begin{aligned}
C_{hijk} &= R_{hijk} + \frac{2}{n(n-2)}[g_{hk}R_{[ij]} - g_{hj}R_{[ik]} + g_{ij}R_{[hk]} - g_{ik}R_{[hj]} - (n-2)g_{hi}R_{[kj]}] \\
&\quad - \frac{1}{n-2}(g_{hk}R_{ij} - g_{hj}R_{ik} + g_{ij}R_{hk} - g_{ik}R_{hj}) \\
&\quad + \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{hj}g_{ik}).
\end{aligned}
$$

The conformal tensor C_{hijk} and conharmonic tensor K_{hijk} are related by the following condition [12]

$$C_{hijk} = K_{hijk} - \frac{R}{(n-1)(2-n)}(g_{hk}g_{ij} - g_{hj}g_{ik}). \tag{3.17}$$

Transvecting (3.17) with g^{hk} and g^{ij} and using (3.7), we have $\dot{\nabla}_r R = \phi_r R$. Consequently, from (3.17), we find

$$\dot{\nabla}_r C_{hijk} = \phi_r C_{hijk}. \tag{3.18}$$

Hence, every conharmonically recurrent Weyl space is conformally recurrent.

Conversely, let W_n be a conformally recurrent Weyl space with the recurrence vector ϕ_r. In this case, the equation (3.18) holds. Thus from (3.17), we get

$$\dot{\nabla}_r C_{hijk} - \phi_r C_{hijk} = \dot{\nabla}_r K_{hijk} - \phi_r K_{hijk} - \frac{(g_{hk}g_{ij} - g_{hj}g_{ik}}{(n-1)(2-n)}(\dot{\nabla}_r R - \phi_r R).$$

Hence, $\dot{\nabla}_r K_{hijk} = \phi_r K_{hijk}$ if $\dot{\nabla}_r R = \phi_r R$ is satisfied.

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