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ON F-PLANAR MAPPINGS ONTO RIEMANNIAN SPACES

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ABSTRACT. In this paper we consider F-planar mappings from affine-connected spaces onto (pseudo-) Riemannian spaces. We found the equations of these mappings in the form of the system of Cauchy equations under some very general conditions. These results generalize the results obtained for geodetic, holomorphically projective and special F-planar mappings of Riemannian and Kählerian spaces, by N.S. Sinyukov, J. Mikeš, V.V. Domashev, I.N. Kurbatova, V.E. Berezovsky, M. Shiha. We continue the investigations of the F-planar mappings for covariantly constant structures.

1. INTRODUCTION

This paper is concerned with certain questions of F-planar mapping from affine-connected spaces onto (pseudo-) Riemannian spaces. The analysis is carried out in tensor form, locally in a class of sufficiently smooth real functions.

Let us consider the space A_n with an affine connection without torsion equipped with a coordinate system x in which, the affine connection $\Gamma^h_{ij}(x)$, the affinor structure $F^h_i(x)$ is defined.

A curve L: $x^h = x^h(t)$ is said to be *F-planar* (J. Mikeš, N.S. Sinykov [13], [11]) if, under the parallel translation along it, the tangent vector $\lambda^h \stackrel{\text{def}}{=} dx^h/dt$ lies in the tangent 2-plane formed by the tangent vector λ^h and its conjugate $F^h_\alpha \lambda^\alpha$, i.e.

$$\nabla_t \lambda^h \equiv d\lambda^h/dt - \Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \rho_1 \lambda^h + \rho_2 F^h_\alpha \lambda^\alpha,$$

where ρ_1 and ρ_2 are functions of the parameter t.

F-planar curves generalize, in a natural way, geodesic, analytically planar ([14], [17], [18], [19], [20]), and quasigeodesic curves ([15]).

Let in the spaces A_n and $\bar{A}_n$, together with the objects of affine connections Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$, the affinor structures F^h_i and $\bar{F}^h_i$ be defined.

A diffeomorfism γ: $A_n \to \bar{A}_n$ is said to be an *F-planar mapping* [13] if, under this mapping, any F-planar curve A_n passes into the $\bar{F}$-planar curve $\bar{A}_n$.

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Under the condition $\operatorname{Rank}\|F_i^h - \rho\delta_i^h\| > 1$ the mapping of A_n onto $\bar{A}_n$ is F-planar if and only if the conditions

$$\begin{aligned} &(\mathrm{a}) \quad \bar{\Gamma}_{ij}^h(x) &&= \Gamma_{ij}^h(x) + \delta_i^h\psi_j + \delta_j^h\psi_i + F_i^h\varphi_j + F_j^h\varphi_i; \\ &(\mathrm{b}) \quad \bar{F}_i^h(x) &&= \alpha F_i^h(x) + \beta\delta_i^h, \end{aligned} \tag{1}$$

holds ([13], [8], [11]), where $\psi_i(x)$, $\varphi_i(x)$ are covectors, $\alpha(x)$, $\beta(x)$ are functions in the coordinate system x which is general with respect to the mapping.

F-planar mappings generalize geodesic (if $\varphi_i \equiv 0$ or $F_i^h = \alpha\delta_i^h$), quasigeodesic, holomorphically projective, planar, and almost geodesic of the type of π_2 mappings ([1], [11], [14], [15], [17], [18], [20]).

If space A_n with affine connection admits an F-planar mapping onto a Riemannian space $\bar{V}_n$, then equation (1a) are equivalent to the equation

$$\bar{g}_{ij,k} = 2\psi_k\bar{g}_{ij} + \psi_{(i}\bar{g}_{j)k} + \varphi_k\bar{F}_{(ij)} + \varphi_{(i}\bar{F}_{j)k}\,, \tag{2}$$

where $\psi_i(x)$, $\varphi_i(x)$ are covectors, $\alpha(x)$, $\beta(x)$ are functions, $\bar{F}_{ij} \overset{\text{def}}{=} \bar{g}_{i\alpha}F_j^\alpha$, and $\bar{g}_{ij}$ is the metric tensor of $\bar{V}_n$. Here and in what follows comma denotes the covariant derivative in A_n and $(i\,j)$ denotes a symmetrization of indices.

The necessity of condition (2) follows from (1a) and from investigation of covariant derivative of the metric tensor $\bar{g}_{ij}$ of the space A_n with the affine connections and its sufficiency follows from the complementary investigation of this derivative.

2. Fundamental equation of F-planar mapping in Cauchy form

In the space A_n equations (2) form a system of differential equations with covariant derivative relative to the components of the unknown tensors $\bar{g}_{ij}$, ψ_i and φ_i. Under the condition $|\bar{g}_{ij}| \neq 0$ the solution of (2) generate a Riemannian space $\bar{V}_n$ with the metric tensor $\bar{g}_{ij}$, on which the space A_n admits an F-planar mapping, where the structure $\bar{F}_i^h$ in V_n is (non-uniquely) defined by formulas (1b) .

We shall prove that the general solution of the system (2) in the given space A_n depends on a finite number of parameters. ¿From this follows that from equations (2) we can find a fundamental system describing the F-planar mappings in the Cauchy form. It holds

Theorem 1. *Let A_n be a space with affine connection and let be defined an affinor $F_i^h(x)$ such that* $\operatorname{Rank}\|F_i^h - \rho\delta_i^h\| > 5$. *Then A_n admits an F-planar mapping onto a Riemannian space $\bar{V}_n$ if and only if the system of differential equations of* *the Cauchy type:*

$$\begin{aligned}
&\text{(a)} \quad \bar{g}_{ij,k} &&= 2\psi_k \bar{g}_{ij} + \psi_{(i}\bar{g}_{j)k} + \varphi_k \bar{F}_{(ij)} + \varphi_{(i}\bar{F}_{j)k};\\
&\text{(b)} \quad \psi_{i,j} &&= \alpha \bar{g}_{ij} + \beta \bar{F}_{ij} + \overset{1}{Q}_{ij}(\bar{g}, \psi, \varphi);\\
&\text{(c)} \quad \varphi_{i,j} &&= \beta \bar{g}_{ij} + \gamma \bar{F}_{ij} + \overset{2}{Q}_{ij}(\bar{g}, \psi, \varphi);\\
&\text{(d)} \quad \alpha_{,i} &&= \overset{3}{Q}_{i}(\bar{g}, \psi, \varphi, \alpha, \beta, \gamma);\\
&\text{(e)} \quad \beta_{,i} &&= \overset{4}{Q}_{i}(\bar{g}, \psi, \varphi, \alpha, \beta, \gamma);\\
&\text{(f)} \quad \gamma_{,i} &&= \overset{5}{Q}_{i}(\bar{g}, \psi, \varphi, \alpha, \beta, \gamma);
\end{aligned} \tag{3}$$

has a solution in A_n for the unknown tensors $\bar{g}_{ij}(x)$ ($\bar{g}_{ij} = \bar{g}_{ji}$, $\|\bar{g}_{ij}\| \neq 0$), covectors $\psi_i(x)$, $\varphi_i(x)$ and functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$.

Here $\overset{\sigma}{Q}$ ($\sigma = \overline{1,5}$) are tensors which are expressed as the functions of the shown arguments, and also of the objects defined in A_n, i.e. affine connection and affinor F_i^h.

Proof. Let A_n be a space with any affine connection and let be there define affinor $F_i^h(x)$ following relation satisfying

$$\operatorname{Rank} \|F_i^h - \rho \delta_i^h\| > 5, \tag{4}$$

where ρ is a function. Let space A_n admits of an F-planar mappings onto a Riemannian space $\bar{V}_n$. Then in A_n the equation (2) holds.

We shall investigate the integrability conditions of these equations. Let them differentiate covariantly by x^l and then alternate by indices k and l. With respect to Ricci identity and equations (2) we find the following:

$$\begin{aligned}
2\psi_{[kl]}\bar{g}_{ij} + \psi_{il}\bar{g}_{jk} + \psi_{jl}\bar{g}_{ik} - \psi_{ik}\bar{g}_{jl} - \psi_{jk}\bar{g}_{il} +\\
+\varphi_{[kl]}\bar{F}_{(ij)} + \varphi_{il}\bar{F}_{jk} + \varphi_{jl}\bar{F}_{ik} - \varphi_{ik}\bar{F}_{jl} - \varphi_{jk}\bar{F}_{il} = \overset{6}{Q}_{ijkl}(\bar{g}, \psi, \varphi),
\end{aligned} \tag{5}$$

where $[kl]$ is the alternation by k and l without division, $\psi_{ij} \overset{\text{def}}{=} \psi_{i,j}$; $\varphi_{ij} \overset{\text{def}}{=} \varphi_{i,j}$. The tensor $\overset{6}{Q}$ has a form analogical to previous tensors $\overset{\sigma}{Q}$, where $\sigma = \overline{1,5}$. Its concrete form is the following:

$$\overset{6}{Q} \overset{\text{def}}{=} \bar{g}_{i\alpha} Q^{\alpha}_{jkl} + \bar{g}_{j\alpha} Q^{\alpha}_{ikl},$$

where

$$Q^h_{ikl} \stackrel{\text{def}}{=} R^h_{ikl} + F^h_\alpha F^\alpha_l \varphi_i \varphi_k - F^h_\alpha F^\alpha_k \varphi_i \varphi_l + F^h_{i,[k}\varphi_{l]} - F^h_{[k,l]}\varphi_i +$$
$$+\delta^h_k(\psi_i\psi_l + \psi_\alpha F^\alpha_i \varphi_l + \psi_\alpha F^\alpha_l \varphi_i) + F^h_k(\varphi_\alpha F^\alpha_i \varphi_l + \varphi_\alpha F^\alpha_l \varphi_i) -$$
$$-\delta^h_l(\psi_i\psi_k + \psi_\alpha F^\alpha_i \varphi_k - \psi_\alpha F^\alpha_k \varphi_i) - F^h_l(\varphi_\alpha F^\alpha_i \varphi_k + \varphi_\alpha F^\alpha_k \varphi_i) +$$

and R^h_{ijk} is the Riemannian tensor.

We shall investigate the homogeneous equation in the form

$$\begin{gathered} 2\,\overset{*}{\psi}_{[kl]}\bar g_{ij} + \overset{*}{\psi}_{il}\bar g_{jk} + \overset{*}{\psi}_{jl}\bar g_{ik} - \overset{*}{\psi}_{ik}\bar g_{jl} - \overset{*}{\psi}_{jk}\bar g_{il} + \\ + \overset{*}{\varphi}_{[kl]}\bar F_{(ij)} + \overset{*}{\varphi}_{il}\bar F_{jk} + \overset{*}{\varphi}_{jl}\bar F_{ik} - \overset{*}{\varphi}_{ik}\bar F_{jl} - \overset{*}{\varphi}_{jk}\bar F_{il} = 0. \end{gathered} \tag{6}$$

with unknowns $\overset{*}{\psi}_{ij}$ and $\overset{*}{\varphi}_{ij}$. We shall prove that this equation has, by the condition (4), the solution in the form

$$\text{(a)}\quad \overset{*}{\psi}_{ij} = \alpha\bar g_{ij} + \beta\bar F_{ij}; \quad \text{(b)}\quad \overset{*}{\varphi}_{ij} = \beta\bar g_{ij} + \gamma\bar F_{ij}, \tag{7}$$

where α, β, γ are numbers.

a) Let us assume that there exists a vector ε^h such that the vectors $\varepsilon^\alpha\ \overset{*}{\varphi}_{\alpha i}$, $\varepsilon^\alpha\bar g_{\alpha i}$ and $\varepsilon^\alpha\bar F_{\alpha i}$ are linearly independent.

Then there exists a vector η^i such that holds

$$\varepsilon^\alpha\eta^\beta\ \overset{*}{\varphi}_{\alpha\beta} = 1, \quad \varepsilon^\alpha\eta^\beta\bar g_{\alpha\beta} = 0, \quad \varepsilon^\alpha\eta^\beta\bar F_{\alpha\beta} = 0.$$

Contracting (6) with $\varepsilon^i\varepsilon^j\eta^l$ we see that the vector $\varepsilon^\alpha\bar F_{\alpha i}$ is a linear combination of the following vectors

$$\eta^\alpha\ \overset{*}{\psi}_{[k\alpha]}, \quad \eta^\alpha\ \overset{*}{\varphi}_{[k\alpha]}, \quad \varepsilon^\alpha\bar g_{k\alpha}.$$

After the contraction (6) with $\varepsilon^j\eta^l$ and the elimination of vector $\varepsilon^\alpha\bar F_{\alpha i}$ with $\varepsilon^j\eta^l$, we see that Rank $\|\bar F_{ij} - \alpha g_{ij}\| \le 5$, which is a contradiction with (4).

Therefore the vectors $\varepsilon^\alpha\ \overset{*}{\varphi}_{\alpha i}$, $\varepsilon^\alpha\bar g_{\alpha i}$ and $\varepsilon^\alpha\bar F_{\alpha i}$ are linearly dependent for any vector ε^h. It follows from this fact that for any ε^h the equation $\overset{*}{\varphi}{}^{[i}_\alpha\delta^j_\beta F^{k]}_\gamma\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma = 0$ holds, where $\overset{*}{\varphi}{}^h_i \stackrel{\text{def}}{=} \bar g^{h\alpha}\ \overset{*}{\varphi}_{\alpha i}$. This condition is equivalent to

$$\overset{*}{\varphi}{}^{[i}_{(\alpha}\delta^j_\beta F^{k]}_{\gamma)} = 0, \tag{8}$$

where $[ijk]$ and $(\alpha\beta\gamma)$ denote the alternation and the symmetrisation by mentioned indices, respectively.

Since $F^h_i \ne \alpha\delta^h_i$, there exists a vector ε^i such that ε^i and $\xi^i \stackrel{\text{def}}{=} \varepsilon^\alpha F^i_\alpha$ are lineary independent. Contracting (8) with $\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma$, we see that the vector $\varepsilon^\alpha\ \overset{*}{\varphi}{}^i_\alpha$ is a linear combination of the vectors ε^i and ξ. Then, after the contraction, (8) with $\varepsilon^\beta\varepsilon^\gamma$ we obtain that $\overset{*}{\varphi}{}^i_\alpha = \beta\delta^i_\alpha + \gamma F^i_\alpha + a_\alpha\varepsilon^i + b_\alpha\xi^i$, where a_α, b_α are covectors and β, γ are functions. Under the assumption that a_α or b_α is non-zero, after the substitution of $\overset{*}{\varphi}{}^i_\alpha$ into (8) we get a contradiction with (4).

Hence $\overset{*}{\varphi}{}^i_\alpha = \beta\delta^i_\alpha + \gamma F^i_\alpha$. From this formulas (7b) follow easily.

b) Analogously, let us suppose the existence of a vector ε^h such that the vectors $\varepsilon^\alpha \overset{*}{\psi}_{\alpha i}$, $\varepsilon^\alpha \bar{g}_{\alpha i}$ and $\varepsilon^\alpha \bar{F}_{\alpha i}$ are linearly independent. However, this assumption is in contradiction with (4) and the regularity of the metric tensor $\bar{g}_{ij}$. That is why the vectors $\varepsilon^\alpha \overset{*}{\psi}_{\alpha i}$, $\varepsilon^\alpha \bar{g}_{\alpha i}$ and $\varepsilon^\alpha \bar{F}_{\alpha i}$ are linearly dependent for any vector ε^h. ¿From this follows that $\overset{*}{\psi}_{ij} = \alpha\bar{g}_{ij} + \bar{\beta}\bar{F}_{ij}$, where α, $\bar{\beta}$ are numbers. Substituting this relation and (7b) into (6), we see that $\bar{\beta} = \beta$.

In this way we proved that the general solution of the homogeneous system of equations (6) is of the form (7). Therefore the conditions (5) imply the equations (3b) and (3c).

Further we shall investigate the integrability conditions of equations (3b). Differentiating the equations (3b) covariantly by x^k and then alternating by j and k, by the Ricci identity and (3a, b, c), we obtain

$$\bar{g}_{ij}\alpha_{,k} - \bar{g}_{ik}\alpha_{,j} + \bar{F}_{ij}\beta_{,k} - \bar{F}_{ik}\beta_{,j} = \overset{7}{Q}_{ijk}(\bar{g}, \psi, \varphi, \alpha, \beta, \gamma). \tag{9}$$

The homogeneous equation

$$\bar{g}_{ij}\overset{*}{\alpha}_k - \bar{g}_{ik}\overset{*}{\alpha}_j + \bar{F}_{ij}\overset{*}{\beta}_k - \bar{F}_{ik}\overset{*}{\beta}_j = 0$$

with unknowns $\overset{*}{\alpha}_i$ and $\overset{*}{\beta}_i$ has only trivial solution $\overset{*}{\alpha}_i = 0$, $\overset{*}{\beta}_i = 0$ if the conditions (4) are satisfied. That is why the equations (3d, e) follow from the condition (9).

Similarly, the last equation (3f) of the system (3) can be obtained using the integrability conditions of equations (3c).

Evidently, the system (3) is closed with respect to unknown tensors $\bar{g}_{ij}$, ψ_i, φ_i, α, β, γ. The Theorem 1 is proved.

We know from the theory of differential equations that the initial value problem (3) with initial conditions

$$\bar{g}_{ij}(x_o) = \overset{o}{\bar{g}}_{ij};\ \psi_i(x_o) = \overset{o}{\psi}_i;\ \varphi_i(x_o) = \overset{o}{\varphi}_i;\ \alpha(x_o) = \overset{o}{\alpha};\ \beta(x_o) = \overset{o}{\beta};\ \gamma(x_o) = \overset{o}{\gamma},$$

has at most one solution. As the tensor $\bar{g}_{ij}$ is symmetric, the general solution of this system depends on

$$r \leq \tfrac{1}{2}n(n+5) + 3$$

real parameters.

¿From this the following theorem follows.

Theorem 2. *Let A_n be a space with affine connection, where an affinor $F^h_i(x)$ is defined such that* Rank $\|F^h_i - \rho\delta^h_i\| > 5$. *The set of all Riemannian spaces $\bar{V}_n$,* *for which A_n admits F-planar mappings, depends on at most $\frac{1}{2}n(n+5)+3$ real parameters.*

This theorem was proved in v [7] under more restrictive conditions: $\operatorname{Rank}\|F_i^h - \rho\delta_i^h\| > 18$. By a detailed analysis of the proof we can see in both theorems 1 and 2 that the condition $\operatorname{Rank}\|F_i^h - \rho\delta_i^h\| > 5$ can be substituted by the assumptions $n > 8$ and $\operatorname{Rank}\|F_i^h - \rho\delta_i^h\| > 4$.

Theorems 1 and 2 generalize similar results obtained by N.S. Sinyukov [17] for geodesic mappings of Riemannian spaces, J. Mikeš and V.E. Berezovski [12] for geodesic mappings of spaces with affine connection onto Riemannian spaces, V.V. Domashev and J. Mikeš [2], [6], for holomorphically projective mappings of Kählerian spaces, I.N. Kurbatova [5] for holomorphically projective mappings of hyperbolic Kählerian spaces, K- and H-spaces and M. Shiha [16] for holomorphically projective mappings of m-parabolic Kählerian spaces (see [17], [10], [11]).

3. F-planar mappings with covariantly constant conditions of affinor structures F

As we said before, F-planar mappings generalize a whole series of previously studied mappings.We list below some conditions under which the F-planar mapping will be one of the mappings studied earlier by authors.

Let us recall that an affinor F_i^h is said to be an *e-structure* if the relation [17], [18]

$$F_\alpha^h F_i^\alpha = e\,\delta_i^h, \quad \text{where} \quad e = \pm 1, 0, \tag{10}$$

is satisfied.

The affinor $\overset{*}{F}{}_i^h$ is *equivalent to e-structure* if there exist an e-structure F_i^h and numbers α, β such that

$$\overset{*}{F}{}_i^h = \alpha F_i^h + \beta\delta_i^h. \tag{11}$$

holds.

We have a following theorem.

Theorem 3. *Let a diffeomorphism $A_n \rightarrow \bar{A}_n$ be a non-affine F-planar mapping. If the structures F_i^h and $\bar{F}_i^h$ are covariantly constant and* $\operatorname{Rank}\|\bar{F}_i^h - \rho\delta_i^h\| \geq 4$, *then this mapping is semigeodesic of type $\pi_2(e)$ and the structures are covariantly constant equivalent e-structures.*

Proof. Let A_n admits non-affine F-planar mapping onto $\bar{A}_n$, and the structures F_i^h and $\bar{F}_i^h$ are covariantly constant in A_n and $\bar{A}_n$, respectively, and $\operatorname{Rank}\|\bar{F}_i^h - \rho\delta_i^h\| \geq 4$. Then the formulas (1) hold.

We express covariant derivative F_i^h in the space $\bar{A}_n$: $F_{i|j}^h \equiv \partial_j F_i^h + \bar{\Gamma}_{\alpha j}^h F_i^\alpha - \bar{\Gamma}_{ij}^\alpha F_\alpha^h$, where $\partial_i \overset{\text{def}}{=} \partial/\partial x^i$. Using formula (1a) we obtain:

$$F_{i|j}^h = F_{i,j}^h + F_i^\alpha \psi_\alpha \delta_j^h + (F_i^\alpha \varphi_\alpha - \psi_i) F_j^h - \varphi_i F_\alpha^h F_j^\alpha\,, \tag{12}$$

where "," and "|" are covariant differentiate in A_n and $\bar{A}_n$, respectively.

We differentiate formulas (1b) in $\bar{A}_n$ covariantly. As we have assumed $F^h_{i,j} = 0$ and $\bar{F}^h_{i|j} = 0$, by substitution (12) we obtain

$$\partial_j a F^h_i + \partial_j b \delta^h_i + a(F^\alpha_i \psi_\alpha \delta^h_j + (F^\alpha_i \varphi_\alpha - \psi_i) F^h_j - \varphi_i F^h_\alpha F^\alpha_j) = 0. \qquad (13)$$

By $\partial_j a \neq 0$ we come to contradiction with Rank $\|\bar{F}^h_i - \rho\delta^h_i\| \geq 4$. Thus $a \equiv$ const. Analogously, for $n > 3$ formulas (13) imply that $b \equiv$ const.

Since $a \neq 0$, formula (13) can be simplified:

$$F^\alpha_i \psi_\alpha \delta^h_j + (F^\alpha_i \varphi_\alpha - \psi_i) F^h_j - \varphi_i F^h_\alpha F^\alpha_j = 0. \qquad (14)$$

The mapping $f\colon A_n \to \bar{A}_n$is not affine and hence $\psi_i \neq 0$ or $\varphi_i \neq 0$. If $\varphi_i = 0$, then for $\psi_i \neq 0$ it follows from (14) that $F^h_i = \rho\delta^h_i$, which is a contradiction. So we have $\varphi_i \neq 0$. Than from the relation (14) we obtain

$$F^h_\alpha F^\alpha_i = \alpha\delta^h_i + \beta F^h_i, \qquad (15)$$

where α, β are functions.

We can show that α, β are constants by covariant derivations of the relations (15) in A_n . Then we can easily see that we can choose numbers c a d such that for affinor structure $\overset{*}{F}{}^h_i \overset{\text{def}}{=} cF^h_i + d\delta^h_i$ holds $\overset{*}{F}{}^h_\alpha \overset{*}{F}{}^\alpha_i = e\delta^h_i$, where $e = \pm 1, 0$. This means that the affinor F^h_i is eqvivalent to e-structure.

Since in our case a and b in (1b) are constant, we can prove analogously that the structure $\bar{F}^h_i$ is also equivalent to e-structure. Moreover, both structures F^h_i and $\bar{F}^h_i$ are simultaneously covariantly constant in A_n and in $\bar{A}_n$.

It follows from the facts mentioned above that in formulas (1) the original structures can be substitute by equivalent covariantly constant e-structures. That is why for F-planar mapping $f\colon A_n \to \bar{A}_n$ the formulas (1a), $F^h_{i,j} = 0$ and $F^h_\alpha F^\alpha_i = e\delta^h_i$ are satisfied. These conditions show that the mapping f is almost geodesic mapping of type $\pi_2(e)$ in the sence of N.S. Sinyukov [17], [18]. The proof of Theorem 3 is now complete.

A.Z. Petrov investigated *quasigeodesic mappings* of 4-dimensional pseudo-Riemannian spaces $V_4 \to \bar{V}_4$, which are in fact special F-planar mappings, under the condition of preserving the structure $\bar{F}^h_i \equiv F^h_i$ and the skew-symmetry of tensors $F^\alpha_i g_{\alpha j}$ and $\bar{F}^\alpha_i \bar{g}_{\alpha j}$, where g_{ij} and $\bar{g}_{ij}$ are metric tensors of V_4 and $\bar{V}_4$.

The following result holds:

Theorem 4. *Let a diffeomorphism of Riemannian spaces $f\colon V_n \to \bar{V}_n$ be a non-affine F-planar mapping. If* Rank $\|F^h_i - \rho\delta^h_i\| \geq 4$, *$F^\alpha_i g_{\alpha j}$ and $\bar{F}^\alpha_i \bar{g}_{\alpha j}$ are skew-symmetric and covariantly constant in V_n and $\bar{V}_n$, then V_n and $\bar{V}_n$ are Kähler spaces and this mapping is holomorphically-projective.*

Proof. Let $f\colon V_n \to \bar{V}_n$ be non-affine F-planar mapping, Rank $\|F^h_i - \rho\delta^h_i\| \geq 4$, and $\bar{F}^\alpha_i \bar{g}_{\alpha j}$ be skew-symmetric and covariantly constant in V_n and $\bar{V}_n$, respectively.

By Theorem 3, the affinor structures are connected by the formula (1b) with a, b constant and we have $F^h_\alpha F^\alpha_i = \alpha\delta^h_i + \beta F^h_i$, α, β are constant.

As $F_i^\alpha g_{\alpha j} + F_j^\alpha g_{\alpha i} = 0$ holds, contracting this relation with F_k^j, we obtain

$$g_{\alpha\beta} F_i^\alpha F_j^\beta + \alpha g_{ik} + \beta F_k^\alpha g_{\alpha i} = 0.$$

Alternating this expression, we see that $\beta F_k^\alpha g_{\alpha i} = 0$, which implies $\beta = 0$. This means that $F_\alpha^h F_i^\alpha = \alpha \delta_i^h$. Analogously we can see that $\bar{F}_\alpha^h \bar{F}_i^\alpha = \bar{\alpha}\delta_i^h$ holds, too.

Substituting (1b), we easily obtain $2ab\, F_i^h = \delta_i^h(\bar{\alpha} - \alpha a^2 - b^2)$. Hence $b = 0$, $\bar{\alpha} = \alpha a^2$, $a \neq 0$.

If $\bar{\alpha} = \alpha = 0$ then V_n and $\bar{V}_n$ are parabolic Kählerian spaces (see [10], [16]) and $f\colon V_n \to \bar{V}_n$ is a holomorphically projective mapping.

Let us suppose $\bar{\alpha} \neq 0$ and $\alpha \neq 0$. If we put $e = \operatorname{sign}\alpha \equiv \operatorname{sign}\bar{\alpha}$ and

$$\overset{*}{F}{}_i^h \overset{\text{def}}{=} \frac{1}{\sqrt{|\alpha|}} F_i^h \quad \text{and} \quad \overset{*}{\bar{F}}{}_i^h \overset{\text{def}}{=} \frac{1}{\sqrt{|\bar{\alpha}|}} \bar{F}_i^h,$$

then $\overset{*}{F}{}_i^h$ and $\overset{*}{\bar{F}}{}_i^h$ are Kählerian structures and V_n and $\bar{V}_n$ are Kählerian spaces ("classical" Kählerian for $e = -1$ and hyperbolic Kählerian for $e = 1$). It is easy to see that the mapping $f\colon V_n \to \bar{V}_n$ is holomorphically projective (see [10], [17], [18], [20]).

Further, we studied holomorphically projective mappings of almost Hermitian spaces. Results for this type of spaces are interesting from the point of view of their classification (Gray-Hervella [4]).

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