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VARIATIONAL FIRST-ORDER QUASILINEAR EQUATIONS

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Abstract. The systems of first-order quasilinear partial differential equations defined on the 1-jet bundle of a fibred manifold which come from a variational problem defined by an affine Lagrangian are characterized by means of the Hamilton–Cartan formalism and the theory of formal integrability.

1. Introduction

The goal of this paper is to analyse the role that Hamiltonian formalism and formal integrability play in studying the variational character of first-order quasilinear partial differential systems.

Let $p\colon M \to N$ be a fibred manifold over an orientable connected C^∞ manifold N. Set $\dim N = n$, $\dim M = m + n$. Let $p_1: J^1 \to N$ be the 1-jet bundle of local sections of p, and let $p_{10}\colon J^1M \to J^0M = M$ denote the canonical projection: $p_{10}(j^1_x s) = s(x)$. Throughout this paper Greek indices run from 1 to m, and Latin indices run from 1 to n.

If (x^i, y^α) is a fibred coordinate system for the submersion p defined on an open subset $U \subseteq M$, we denote by $(x^i, y^\alpha; y^\alpha_i)$ the coordinate system induced on J^1U in a natural way; *i.e.*, $y^\alpha_i(j^1_x s) = \partial(y^\alpha \circ s)/\partial x^i(x)$.

The starting point is the following basic fact:

Proposition 1. *Let Lv be a Lagrangian density on J^1M, where v is a volume form on N. The Poincaré–Cartan form Θ of Lv is p_{10}-projectable if and only if L is an affine function, or in other words, locally there exist functions $A^0, A^i_\alpha \in C^\infty(M)$ such that $L = A^0 + A^i_\alpha y^\alpha_i$. In this case, the Euler–Lagrange equations of Lv are a system of first-order quasilinear equations on J^1M.*

Proof. It is an easy consequence of the local expression of the Poincaré–Cartan form

$$\Theta = (-1)^{i-1}\frac{\partial L}{\partial y^\alpha_i}\theta^\alpha \wedge v_i + Lv, \tag{1}$$

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associated with Lv, where we have chosen the fibred coordinate system (x^i, y^α) such that,

$$v = dx^1 \wedge \ldots \wedge dx^n, \quad v_i = dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n,$$

and θ^α are the standard 1-contact forms on $J^1 M$; *i.e.*, $\theta^\alpha = dy^\alpha - y_i^\alpha dx^i$. Then, as a simple calculation shows, the Euler–Lagrange equations are:

$$\frac{\partial A^0}{\partial y^\alpha} - \frac{\partial A_\alpha^i}{\partial x^i} = \Big(\frac{\partial A_\alpha^i}{\partial y^\beta} - \frac{\partial A_\beta^i}{\partial y^\alpha}\Big)\frac{\partial y^\beta}{\partial x^i},$$

where the index α is free. ■

This result poses the problem of determining which systems of first-order quasi-linear equations on $J^1 M$ come from an affine Lagrangian as above. The characterization is as follows:

Theorem 2. *With the previous hypotheses and assumptions, the system of C^ω equations*

$$F_\alpha = F_{\alpha\beta}^i \frac{\partial y^\beta}{\partial x^i}, \qquad F_{\alpha\beta}^i = F_{\beta\alpha}^i, \qquad F_\alpha, F_{\alpha\beta}^i \in C^\infty(M), \tag{2}$$

is variational with respect to an affine Lagrangian if and only if the differential form $\xi \in \Omega^{n+1}(M)$ defined by

$$\xi = F_\alpha dy^\alpha \wedge v + (-1)^i F_{\alpha\beta}^i dy^\alpha \wedge dy^\beta \wedge v_i \tag{3}$$

is closed; that is, if and only if

$$\left.\begin{aligned} F_{\alpha\beta}^i + F_{\beta\alpha}^i &= 0,\\ \frac{\partial F_{\alpha\beta}^i}{\partial x^i} - \frac{\partial F_\beta}{\partial y^\alpha} + \frac{\partial F_\alpha}{\partial y^\beta} &= 0,\\ \frac{\partial F_{\alpha\beta}^i}{\partial y^\gamma} + \frac{\partial F_{\beta\gamma}^i}{\partial y^\alpha} + \frac{\partial F_{\gamma\alpha}^i}{\partial y^\beta} &= 0. \end{aligned}\right\} \tag{4}$$

In this case, ξ is the exterior differential of the Poincaré–Cartan form associated with the corresponding Lagrangian density Lv.

The result can also be formulated by saying that a $(n-1)$-horizontal (over N) differential $(n+1)$-form on M is variational if and only if it is closed.

In the particular case $n = \dim N = 1$, that is, for ordinary differential equations, the result of Theorem 2 was stated in [3]. Moreover, the conditions (4) agree with those obtained in [2, I-VII] in the case of an affine Lagrangian although in the present work the result is obtained by a completely different method.

2. Proof of Theorem 2

Let us first consider the Lagrangian $L = A^0 + A_\alpha^i y_i^\alpha$. The associated Euler–Lagrange equations are

$$\frac{\partial A^0}{\partial y^\alpha} - \frac{\partial A_\alpha^i}{\partial x^i} = \Big(\frac{\partial A_\alpha^i}{\partial y^\beta} - \frac{\partial A_\beta^i}{\partial y^\alpha}\Big)\frac{\partial y^\beta}{\partial x^i},$$

and hence the associated differential form ξ reads

$$\xi = \Big(\frac{\partial A^0}{\partial y^\alpha} - \frac{\partial A^i_\alpha}{\partial x^i}\Big) dy^\alpha \wedge v + (-1)^i \Big(\frac{\partial A^i_\alpha}{\partial y^\beta} - \frac{\partial A^i_\beta}{\partial y^\alpha}\Big) dy^\alpha \wedge dy^\beta \wedge v_i,$$

which coincides with the exterior differential of the Poincaré–Cartan form and ξ is obviously closed. In fact, in this case from the formula (1) we obtain $\Theta = A^0 v + (-1)^{i-1} A^i_\alpha dy^\alpha \wedge v_i$, and hence $\xi = d\Theta$.

Conversely, let us suppose that the equations (4) hold, so that the differential form ξ in (3) is closed. Then, the crucial point of the result is to prove that there exists a local primitive form $\zeta \in \Omega^n(M)$, $\xi = d\zeta$, which, in addition, is $(n-1)$-horizontal with respect to p; *i.e.*, $i_{Y_0} i_{Y_1} \omega = 0$ for all p-vertical tangent vectors Y_0, Y_1 on M. This means that ζ is written as

$$\zeta = A^0 v + (-1)^{i-1} A^i_\alpha dy^\alpha \wedge v_i, \quad A^0, A^i_\alpha \in C^\omega(M). \tag{5}$$

Therefore, the equation $\xi = d\zeta$ has a local solution if and only if the following system of PDEs is integrable:

$$F_\alpha = \frac{\partial A^0}{\partial y^\alpha} - \frac{\partial A^i_\alpha}{\partial x^i}, \tag{6}$$

$$F^i_{\alpha\beta} = \frac{\partial A^i_\alpha}{\partial y^\beta} - \frac{\partial A^i_\beta}{\partial y^\alpha}. \tag{7}$$

Theorem 3. *If the conditions* (4) *hold, then the system* (6,7) *is involutive and, hence, formally integrable.*

Proof. Let us denote by $\wedge^n_2 T^*M$ (resp. $\wedge^{n+1}_3 T^*M$) the subbundle of $\wedge^n T^*M$ (resp. $\wedge^{n+1}_3 T^*M$) defined by $i_{Y_0} i_{Y_1} \omega = 0$ (resp. $i_{Y_0} i_{Y_1} i_{Y_2} \omega = 0$) for all p-vertical tangent vectors Y_0, Y_1 (resp. Y_0, Y_1, Y_2) in M. Let

$$\Phi \colon J^1(\wedge^n_2 T^*M) \to \wedge^{n+1}_3 T^*M$$

be the affine bundle morphism given by

$$\Phi(j^1_y \zeta) = (d\zeta)_y - \xi_y.$$

Set $R_1 = \ker(\Phi, 0)$.

We introduce coordinates $(x^i, y^\alpha, z^0, z^i_\alpha)$ (resp. $(x^i, y^\alpha, w_\alpha, w^i_{\alpha\beta})$) in $\wedge^n_2 T^*M$ (resp. $\wedge^{n+1}_3 T^*M$) as follows

$$\begin{aligned} \zeta &= z^0(\zeta) v + z^i_\alpha(\zeta) dy^\alpha \wedge v_i, \\ \eta &= w_\alpha(\eta) dy^\alpha \wedge v + w^i_{\alpha\beta}(\eta) dy^\alpha \wedge dy^\beta \wedge v_i. \end{aligned}$$

Let $(x^i, y^\alpha, z^0, z^i_\alpha; z^0_i, z^0_\alpha, z^i_{\alpha,j}, z^i_{\alpha,\beta})$ denote the system of coordinates induced in $J^1(\wedge^n_2 T^*M)$; precisely,

$$z^0_i(j^1_y\zeta) = \frac{\partial(z^0 \circ \zeta)}{\partial x^i}(y),\ z^0_\alpha(j^1_y\zeta) = \frac{\partial(z^0 \circ \zeta)}{\partial y^\alpha}(y),$$

$$z^i_{\alpha,j}(j^1_y\zeta) = \frac{\partial(z^i_\alpha \circ \zeta)}{\partial x^j}(y),\ z^i_{\alpha,\beta}(j^1_y\zeta) = \frac{\partial(z^i_\alpha \circ \zeta)}{\partial y^\beta}(y).$$

Then, the equations of Φ are

$$w_\alpha \circ \Phi = z^0_\alpha - z^i_{\alpha,i} - F_\alpha, \quad w^i_{\alpha\beta} \circ \Phi = z^i_{\alpha,\beta} - z^i_{\beta,\alpha} - F_{\alpha\beta}. \tag{8}$$

Hence Φ has constant rank and R_1 is a fibred submanifold of $J^1(\wedge^n_2 T^*M)$.

Moreover, a section ζ of the vector bundle $\wedge^n_2 T^*M$ satisfies the equations (6,7) if and only if $j^1_y\zeta \in R_1$ at every point $y \in M$; that is, ζ is a solution of R_1.

Lemma 4. *With the above notations, the vectors*

$$u^1 = \partial/\partial x^1, \dots, u^n = \partial/\partial x^n, v^1 = \partial/\partial y^1, \dots, v^m = \partial/\partial y^m$$

constitute a quasiregular basis for R_1 at each point of M.

Proof (of Lemma). The symbol of Φ,

$$\sigma_1 = \sigma_1(\Phi) \colon T^*M \otimes \wedge^n_2 T^*M \to \wedge^{n+1}_3 T^*M$$

is given by $\sigma_1(\omega \otimes \zeta) = \omega \wedge \zeta$, for every $\omega \in T^*M$, $\zeta \in \wedge^n_2 T^*M$, or in local coordinates

$$\sigma_1(dx^j \otimes v) = 0,$$
$$\sigma_1(dy^\beta \otimes v) = dy^\beta \wedge v,$$
$$\sigma_1(dx^j \otimes (dy^\alpha \wedge v_i)) = (-1)^j \delta^j_i dy^\alpha \wedge v,$$
$$\sigma_1(dy^\beta \otimes (dy^\alpha \wedge v_i)) = dy^\beta \wedge dy^\alpha \wedge v_i.$$

Set $g_1 = \ker \sigma_1$. In order to calculate $\dim g_1$ we first notice that an element

$$\chi = \lambda_j dx^j \otimes v + \lambda^i_{j\alpha} dx^j \otimes dy^\alpha \wedge v_i + \mu_\beta dy^\beta \otimes v + \mu^i_{\beta\alpha} dy^\beta \otimes dy^\alpha \wedge v_i$$

belongs to g_1 if and only if $(-1)^i \lambda^i_{i\alpha} + \mu_\alpha = 0$, $\mu^i_{\alpha\beta} + \mu^i_{\beta\alpha} = 0$. Hence

$$\dim g_1 = n + n^2 m + \tfrac{1}{2} nm(m+1) = n(1+nm) + \tfrac{1}{2} nm(m+1).$$

Now we must count the dimension of the spaces

$$g_{1,u^1,\dots,u^k} = \{\chi \in g_1 : i_{u^1}\chi = \cdots = i_{u^k}\chi = 0\}.$$

We observe that $g_{1,u^1} = \{\chi \in g_1 : \lambda_1 = \lambda^i_{1\alpha} = 0\}$, and hence

$$\dim g_{1,u^1} = \dim g_1 - 1 - nm = (n-1)(1+nm) + \tfrac{1}{2} nm(m+1).$$

In a similar way, $g_{1,u^1,\dots,u^k} = \{\chi \in g_{1,u^1,\dots,u^{k-1}} : \lambda_k = \lambda^i_{k\alpha} = 0\}$, so that

$$\dim g_{1,u1,\dots,u^k} = \dim g_{1,\dots,u^{k-1}} - 1 - nm = (n-k)(1+nm) + \tfrac{1}{2} nm(m+1),$$

and $\dim g_{1,u^1,\dots,u^n} = \frac{1}{2} nm(m+1)$.

Repeating the same operation with

$$g_{1,u^1,\dots,u^n,v^1} = \left\{ \chi \in g_{1,u^1,\dots,u^n} : \mu_1 = \mu^i_{1\alpha} = 0 \right\},$$

we notice that all the μ_β vanish automatically as every $\lambda^i_{j\alpha}$ vanishes in $g_{1,u^1,\dots,u^n}$. Hence, we must only add the condition that the mn coefficients $\mu^i_{\beta\alpha}$ vanish for $\beta = 1$:

$$\dim g_{1,u^1,\dots,u^n,v^1} = \dim g_{1,u^1,\dots,u^n} - nm = \tfrac{1}{2}nm(m+1) - nm = \tfrac{1}{2}nm(m-1).$$

In an analogous way, when passing from $g_{1,u^1,\dots,u^n,v^1,\dots,v^{\gamma-1}}$ to $g_{1,u^1,\dots,u^n,v^1,\dots,v^\gamma}$ we must eliminate the $\mu^i_{\gamma\alpha}$ with $\alpha \geq \gamma$. Hence

$$\begin{aligned}
\dim g_{1,u^1,\dots,u^n,v^1,\dots,v^\gamma} &= \dim g_{1,u^1,\dots,u^n,v^1,\dots,v^{\gamma-1}} - n(m-\gamma+1) \\
&= \tfrac{1}{2}n(m-(\gamma-2))(m-(\gamma-1)) - n(m-\gamma+1) \\
&= \tfrac{1}{2}n(m-\gamma+1)(m-\gamma),
\end{aligned}$$

and so $\dim g_{1,u^1,\dots,u^n,v^1,\dots,v^m} = 0$.

We have still to calculate the dimension of $g_2 = \ker \sigma_2(\Phi)$, where

$$\sigma_2 = \sigma_2(\Phi) \colon S^2T^*M \otimes \wedge^n_2 T^*M \to T^*M \otimes \wedge^{n+1}_3 T^*M,$$

is the prolongation of the symbol. It is defined as follows: Let $[f]$ be the equivalence class of a function $f \in C^\infty(M)$, modulo sums with functions with vanishing second-order derivatives. The prolongation of the symbol is obtained by applying

$$\begin{gathered}
S^2T^*M \otimes \wedge^n_2 T^*M \to J^1(\wedge^{n+1}_3 T^*M) \\
[f]_y \otimes \zeta_y \mapsto j^1_y(df \wedge \zeta),
\end{gathered}$$

and then restricting to the associated vector bundle of $J^1(\wedge^{n+1}_3 T^*M)$. Hence, the expression of σ_2 in local coordinates is the following (where $\odot$ stands for the symmetric product):

$$\begin{aligned}
\sigma_2(dx^j \odot dx^k \otimes v) &= 0, \\
\sigma_2(dx^j \odot dy^\beta \otimes v) &= dx^j \otimes (dy^\beta \wedge v), \\
\sigma_2(dy^\beta \odot dy^\gamma \otimes v) &= dy^\beta \otimes (dy^\gamma \wedge v) + dy^\gamma \otimes (dy^\beta \wedge v), \\
\sigma_2(dx^j \odot dx^k \otimes (dy^\alpha \wedge v_i)) &= (-1)^i \left(\delta^k_i dx^j \otimes (dy^\alpha \wedge v) + \delta^j_i dx^k \otimes (dy^\alpha \wedge v) \right) \\
\sigma_2(dx^j \odot dy^\beta \otimes (dy^\alpha \wedge v_i)) &= dx^j \otimes (dy^\beta \wedge dy^\alpha \wedge v_i) + (-1)^i \delta^j_i dy^\beta \otimes (dy^\alpha \wedge v), \\
\sigma_2(dy^\beta \odot dy^\gamma \otimes (dy^\alpha \wedge v_i)) &= dy^\beta \otimes (dy^\gamma \wedge dy^\alpha \wedge v_i) + dy^\gamma \otimes (dy^\beta \wedge dy^\alpha \wedge v_i).
\end{aligned}$$

To calculate $\dim g_2$ we notice that an element

$$\begin{aligned}
\bar{\chi} = \bar{\lambda}_{(jk)} dx^j \odot dx^k \otimes v &+ \bar{\lambda}^i_{(j\kappa)\alpha} dx^j \odot dx^k \otimes (dy^\alpha \wedge v_i) \\
+ \bar{\mu}_{j\beta} dx^j \odot dy^\beta \otimes v &+ \bar{\mu}^i_{j\beta\alpha} dx^j \odot dy^\beta \otimes (dy^\alpha \wedge v_i) \\
+ \bar{\nu}_{(\beta\gamma)} dy^\beta \odot dy^\gamma \otimes v &+ \bar{\nu}^i_{(\beta\gamma)\alpha} dy^\beta \odot dy^\gamma \otimes (dy^\alpha \wedge v_i)
\end{aligned}$$

(where $(nm) = (nm)$ are symmetric subindices) belongs to g_2 if and only if

$$\begin{aligned}
\bar{\mu}_{j\beta} + \sum_i (-1)^i \bar{\lambda}^i_{(ij)} &= 0 \\
\bar{\nu}_{(\beta\gamma)} + \sum_i (-1)^i \bar{\mu}^i_{i\alpha\beta} &= 0 \\
\bar{\mu}^i_{j\beta\alpha} + \bar{\mu}^i_{j\alpha\beta} &= 0 \\
\bar{\nu}^i_{(\beta\gamma)\alpha} - \bar{\nu}^i_{(\alpha\beta)\gamma} &= 0 \\
\bar{\nu}^i_{(\beta\gamma)\alpha} - \bar{\nu}^i_{(\alpha\gamma)\beta} &= 0.
\end{aligned}$$

Hence the dimension of g_2 can be counted as follows: The $n(n+1)/2$ coefficients $\bar{\lambda}_{(jk)}$ and the $\frac{1}{2}n^2(n+1)m$ coefficients $\bar{\lambda}^i_{(,jk)\alpha}$ can be chosen freely. The latter completely determine the coefficients $\bar{\mu}_{j\beta}$. The $n^2m(m+1)/2$ coefficients $\bar{\mu}^i_{j\beta\alpha}$ with $\alpha \geq \beta$ can be freely chosen, and they determine the remaining $\bar{\mu}^i_{j\beta\alpha}$ and thus also the $\bar{\nu}_{(\beta\gamma)}$ Finally, the $nm(m+1)(m+2)/6$ coefficients $\bar{\nu}^i_{(\beta\gamma)\alpha}$ with $\alpha \geq \beta \geq \gamma$ determine all the remaining $\bar{\nu}^i_{(\beta\gamma)\alpha}$. Hence,

$$\dim g_2 = \tfrac{1}{2}(nm+1)n(n+1) + \tfrac{1}{2}n^2m(m+1) + \tfrac{1}{6}nm(m+1)(m+2).$$

For the basis to be quasiregular (and hence, for the system to be involutive), the following dimension equality must hold:

$$\dim g_2 = \dim g_1 + \sum_{j=1}^{n} \dim g_{1,u^1,\dots,u^j} + \sum_{\beta=1}^{m-1} \dim g_{1,u^1,\dots,u^n,v^1,\dots,v^\beta}. \tag{9}$$

Moreover, we have

$$\begin{aligned}
\dim g_1 + \sum_{j=1}^{n-1} \dim g_{1,u^1,\dots,u^j} &= \sum_{j=0}^{n-1} \Big((n-j)(1+nm) + \tfrac{1}{2}nm(m+1) \Big) \\
&= \tfrac{1}{2}n^2m(m+1) + \tfrac{1}{2}(1+mn)n(n+1),
\end{aligned}$$

and

$$\begin{aligned}
\dim g_{1,u^1,\dots,u^n} + \sum_{\gamma=1}^{m-1} \dim g_{1,u^1,\dots,u^n,v^1,\dots,v^\gamma} &= \sum_{\gamma=0}^{m-1} \tfrac{1}{2}n(m-\gamma+1)(m-\gamma) \\
&= \tfrac{1}{6}nm(m+1)(m+2),
\end{aligned}$$

as it can be easily shown by induction on m. Hence, the condition (9) is satisfied, and the differential system in question is involutive. ■(Lemma)

Now, all the obstructions for integrability must lie in the first prolongation.

The first prolongation of the system is

$$\frac{\partial F_\alpha}{\partial x^j} = \frac{\partial^2 A^0}{\partial x^j \partial y^\alpha} - \frac{\partial^2 A^i_\alpha}{\partial x^i \partial x^j}, \tag{10}$$

$$\frac{\partial F_\alpha}{\partial y^\gamma} = \frac{\partial^2 A^0}{\partial y^\alpha \partial y^\gamma} - \frac{\partial^2 A^i_\alpha}{\partial x^i \partial y^\gamma}, \tag{11}$$

$$\frac{\partial F^i_{\alpha\beta}}{\partial x^j} = \frac{\partial^2 A^i_\alpha}{\partial x^j \partial y^\beta} - \frac{\partial^2 A^i_\beta}{\partial x^j \partial y^\alpha}, \tag{12}$$

$$\frac{\partial F^i_{\alpha\beta}}{\partial y^\gamma} = \frac{\partial^2 A^i_\alpha}{\partial y^\beta \partial y^\gamma} - \frac{\partial^2 A^i_\beta}{\partial y^\alpha \partial y^\gamma}. \tag{13}$$

Let us thus look for every possible integrability condition, by checking all the linear relations that can be satisfied by the equations (6) (7), (10), (11), (12) and (13). The equations (6) and (10) cannot be related to any other equation because $\partial A^0/\partial y^\alpha$ only appears in the α-th component of (6) and, in a similar way, $\partial^2 A^0/\partial x^j \partial y^\alpha$ only appears only in the (j,α)-th component of (10). Equations (7) can be related among themselves in order to obtain

$$F^i_{\alpha\beta} = -F^i_{\beta\alpha}. \tag{14}$$

In an analogous way, equations (13) can be combined among themselves to give the conditions

$$\frac{\partial F^i_{\alpha\beta}}{\partial y^\gamma} = -\frac{\partial F^i_{\beta\alpha}}{\partial y^\gamma},$$

(which is a direct consequence of (14)), and also the conditions

$$\frac{\partial F^i_{\alpha\beta}}{\partial y^\gamma} + \frac{\partial F^i_{\beta\gamma}}{\partial y^\alpha} + \frac{\partial F^i_{\gamma\alpha}}{\partial y^\beta} = 0. \tag{15}$$

Finally, the only possible way to eliminate the A's in the equations (11) and (12) produce the condition

$$\frac{\partial F_\beta}{\partial y^\alpha} - \frac{\partial F_\alpha}{\partial y^\beta} = \frac{\partial F^i_{\alpha\beta}}{\partial x^i}. \tag{16}$$

Since we realise that this compatibility conditions are precisely the conditions (4) assuring that ξ is closed, we have obtained the formal integrability of the system of equations (6,7). ∎

Finally, by using the Cartan-Kähler Theorem, we can assure the local integrability and hence the existence of a differential n-form ω with the local expression (5) such that $d\omega = \xi$. Furthermore ω is the Poincaré–Cartan form associated to the Lagrangian $L = A^0 + A^i_\alpha y^\alpha_i$, whose Euler–Lagrange equations are precisely $F_\alpha = F^i_{\alpha\beta}(\partial y^\beta/\partial x^i)$. Thus, the proof is complete.

Remark 5. *In fact, we can drop the analiticity hypothesis, and obtain Theorem 2 for the case in which the equations (2) are just C^∞. The reason for that is that the equation R_1 is associated to a differential operator with constant coefficients (as can be seen in (8)). Hence the Ehrenpreis–Malgrange Theorem (see* [1, Chapter

X,1.2], *and the references therein) can be used to state that formal integrability assures integrability in the* C^∞ *case.*

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