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APPLICATIONS OF BOTT CONNECTION TO FINSLER GEOMETRY

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1. Introduction

In the present paper, we shall investigate connections theory in complex Finsler geometry. The basic tool in this paper is the so-called *Bott connection* which is a partial connection defined by a splitting on the fundamental sequence of vector bundles (see the definition below).

Let $\pi : E \to M$ be a holomorphic vector bundle over a complex manifold. We denote by T_E and T_M the holomorphic tangent bundles of E and M respectively. Moreover we denote by $T_{E/M}$ the relative tangent bundle of the holomorphic projection π. Then we get the fundamental sequence of vector bundles:

$$\mathbb{O} \to T_{E/M} \xrightarrow{i} T_E \xrightarrow{d\pi} \pi^* T_M \to \mathbb{O}. \tag{1.1}$$

We also denote by $\Omega^1_\bullet$ the corresponding holomorphic cotangent bundle.

We take an open covering $\mathcal{U} = \{U, V, \cdots\}$ of M with a local frame field $s_U = (s_1, \cdots, s_r)$ of E on each U. The covering $\{(U, s_U)\}_{U\in\mathcal{U}}$ induces a complex coordinate system (z_U, ξ_U) on each $\pi^{-1}(U)$, where $z_U = (z_U^1, \cdots, z_U^n)$ is a coordinate on U and $\xi_U = (\xi_U^1, \cdots, \xi_U^r)$ is the fibre coordinate on $\pi^{-1}(z) = E_z$ $(z \in U)$. If we denote by

$$g_{UV} = \left({g_{UV}}^i_j\right) : U \cap V \to GL(r, \mathbb{C})$$

the transition functions relative to the covering $\{(U, s_U)\}_{U\in\mathcal{U}}$, the coordinate transformation law is given by the form

$$\begin{cases} z_U^\alpha = z_U^\alpha(z_V) \\ \xi_U^i = \sum {g_{UV}}^i_j(z_V)\xi_V^j. \end{cases}$$

We define a local section $\{\sigma_{UV}\}$ of $\pi^*\Omega^1_M \otimes T_{E/M}$ over $\pi^{-1}(U \cap V)$ by

$$\sigma_{UV} = \sum_{i,\alpha} \left(\sum_j \frac{\partial {g_{UV}}^i_j}{\partial z_V^\alpha} \xi_V^j \right) dz_V^\alpha \otimes \frac{\partial}{\partial \xi_U^i}.$$

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Then we can easily verify that the family $\{\sigma_{UV}\}$ satisfies $\sigma_{UV}+\sigma_{VW}+\sigma_{WU}=0$ on $U\cap V\cap W\neq\phi$. A splitting h of the sequence (1.1) is defined by a local sections $\{N_U\}$ of $\pi^*\Omega^1_M\otimes T_{E/M}$ over $\pi^{-1}(U)$ satisfying

$$N_V - N_U = \sigma_{UV}. \tag{1.2}$$

If a splitting $h:\pi^*T_M\to T_E$ is given in this sequence, we have a natural connection $\nabla:\Gamma(T_{E/M})\to\Gamma(T_{E/M}\otimes\Omega^1_E)$ on the bundle $\varpi:T_{E/M}\to E$ from the given splitting h. In Finsler geometry, such a connection ∇ plays an important role.

2. Bott connections

2.1. Ehresmann connections. Let $\pi:E\to M$ be a holomorphic vector bundle over a complex manifold M. An *Ehresmann connection* H for π is a smooth distribution $H\subset T_E$ for which the morphism $d\pi$ in the sequence (1.1) induces an isomorphism $H\cong\pi^*T_M$. If an Ehresmann connection H is given for π, we get a smooth splitting

$$T_E = H\oplus T_{E/M}. \tag{2.1}$$

For a fixed Ehresmann connection H, the smooth splitting (2.1) induces the dual splitting $\Omega^1_{\mathcal{X}}=H^*\oplus\pi^*\Omega^1_M$, and thus the differential operators $d:\mathcal{O}_E\to\Omega^1_E$ is decomposed as $d=d^v+d^h$ by the differential $d^h:\mathcal{O}_{\mathcal{X}}\to\pi^*\Omega^1_M$ along H and the differential $d^v:\mathcal{O}_E\to\Omega^1_{E/M}$ along vertical direction. We also decompose the operators ∂ and $\bar\partial$ as $\partial=\partial^v+\partial^h$ and $\bar\partial=\bar\partial^v+\bar\partial^h$ respectively.

We denote by $\mathcal{S}$ the sheaf of germs of linear functionals along the fibres of π. A splitting h is defined by the action of ∂^h on $\mathcal{S}$. If we put $\partial^h\xi^i=-\sum N^i_\alpha(z,\xi)dz^\alpha$ on each $\pi^{-1}(U)$, the functions $\{N^i_\alpha\}$ satisfy the homogeneity

$$N^i_\alpha(z,\lambda\xi)=\lambda N^i_\alpha(z,\xi) \tag{2.2}$$

for all $\lambda\in\mathbb{C}$. For such functions $\{N^i_\alpha\}$, we define a local section N_U of $\pi^*\Omega^1_M\otimes T_{E/M}$ by $N_U=\sum N^i_\alpha dz^\alpha\otimes(\partial/\partial\xi^i)$, then we can easily verify that $\{N_U\}$ satisfies (1.2). The splitting $h:\pi^*T_M\to T_{E/M}$ is defined by the *lift* X_α of natural frame fields $\{\partial/\partial z^\alpha\}$:

$$X_\alpha=\frac{\partial}{\partial z^\alpha}-\sum_i N^i_\alpha(z,\xi)\frac{\partial}{\partial\xi^i}. \tag{2.3}$$

A splitting h is said to be *linear* if $\partial^h\mathcal{S}\subset\mathcal{S}\otimes H^*$. In this case, since the functions $N^i_\alpha(z,\xi)$ is linear in (ξ^i) along the fibre E_z, there exist functions $\gamma^i_{j\alpha}(z)$ satisfying $N^i_\alpha=\sum\gamma^i_{j\alpha}(z)\xi^j$. Then we see that the $(1,0)$-form $\theta^i_j(z)=\sum\gamma^i_{j\alpha}(z)dz^\alpha$ defines a connection $\underline{\nabla}:\Gamma(E)\to\Gamma(E\otimes T^*_M)$ of E. In the present paper, we shall consider the case where h is a *non-linear* case.

An Ehresmann connection H of π is said to be *integrable* if it the distribution H is closed under the Lie bracket operator: $[\Gamma(H),\Gamma(H)]\subset\Gamma(H)$. The obstruction for H to be integrable is given by the torsion form $T = (T^i)$ defined by

$$T^i = -\frac{1}{2}\sum_{\alpha,\beta} R^i_{\alpha\beta} dz^\alpha \wedge dz^\beta - \sum_{\alpha,\beta} R^i_{\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta,$$

where the *inetgrability tensors* $R^i_{\alpha\beta}$ and $R^i_{\alpha\bar\beta}$ are defined by

$$R^i_{\alpha\beta} = X_\beta N^i_\alpha - X_\alpha N^i_\beta, \;\; R^i_{\alpha\bar\beta} = X_{\bar\beta} N^i_\alpha$$

respectively with the conjugate $X_{\bar\alpha}$ of X_α. Moreover, we define $\Phi = (\Phi^i_{\bar j})$ by

$$\Phi^i_{\bar j} = \sum_\alpha \frac{\partial N^i_\alpha}{\partial \bar\xi^j} dz^\alpha. \tag{2.4}$$

It is easily seen that, if $T = 0$ and $\Phi = 0$, then H is integrable and holomorphic.

Remark 2.1. If the bundle $\pi : E \to M$ admits a holomorphically integrable Ehresmann connection H, we have the holomorphic splitting $T_E = \pi^* T_M \oplus T_{E/M}$. Furthermore the fundamental group $\pi_1(M)$ acts on the canonical fibre X by complex automorphism, and so $\pi : E \to M$ is the fibre bundle associated with the universal covering $\tilde M$, that is, $E = (\tilde M \times X)/\pi_1(M)$.

2.2. **Bott connections.** A morphism $D : \Gamma(T_{E/M}) \to \Gamma(T_{E/M} \otimes \Omega^1_E)$ is called a *partial connection* on the relative tangent bundle $\varpi : T_{E/M} \to E$ if it satisfies the Leibniz condition $D(fs) = d^h f \otimes s + fDs$ for $^\forall s \in \Gamma(T_{E/M})$ and $^\forall f \in C^\infty(E)$. An Ehresmann connection H for π induces a partial connection D on $T_{E/M}$ as follows.

Definition 2.1. ([Ai5]) A partial connection D of $(1,0)$-type on $T_{E/M}$ defined by

$$D_X Y = P([X, Y])$$

for $^\forall X \in \Gamma(H)$ and $^\forall Y \in \Gamma(T_{E/M})$ is called a *complex Bott connection*, where $P : T_E \to T_{E/M}$ is the natural projection.

The connection form $\omega = (\omega^i_j)$ of D is given by the $(1,0)$-form $\omega^i_j = \sum \Gamma^i_{j\alpha} dz^\alpha$ with

$$\Gamma^i_{j\alpha}(z,\xi) = \frac{\partial N^i_\alpha}{\partial \xi^j}. \tag{2.5}$$

By the homogeneity (2.2) of N^i_α, we have

$$\sum \Gamma^i_{j\alpha}(z,\xi)\xi^j = N^i_\alpha. \tag{2.6}$$

The curvature form $\Omega^D = D^2$ of D is given by $\Omega^D = d^h\omega + \omega \wedge \omega$. Then we have

Proposition 2.1. *The curvature Ω^D of the Bott connection D is given by*

$$\Omega^D = \Pi - \Phi \wedge \overline{\Phi}, \tag{2.7}$$

where $\Pi = (\Pi^i_j)$ is defined by

$$\Pi^i_j = \frac{\partial T^i}{\partial \xi^j} = -\frac{1}{2}\sum_{\alpha,\beta} \frac{\partial R^i_{\alpha\beta}}{\partial \xi^j} dz^\alpha \wedge dz^\beta - \sum_{\alpha,\beta} \frac{\partial R^i_{\alpha\bar{\beta}}}{\partial \xi^j} dz^\alpha \wedge d\bar{z}^\beta.$$

Proof. Since $X_\beta \Gamma^i_{j\alpha} = \partial(X_\beta N^i_\alpha)/\partial \xi^j + \sum_l \Gamma^i_{l\alpha}\Gamma^l_{j\beta}$, we have

$$\partial^h \omega^i_j + \sum_l \omega^i_l \wedge \omega^l_j = -\frac{1}{2}\sum_{\alpha,\beta} \frac{\partial R^i_{\alpha\beta}}{\partial z^j} dz^\alpha \wedge dz^\beta. \tag{2.8}$$

Similarly we have

$$\bar{\partial}^h \omega^i_j = -\sum_{\alpha,\beta} \frac{\partial R^i_{\alpha\bar{\beta}}}{\partial \xi^j} dz^\alpha \wedge d\bar{z}^\beta - \sum_l \Phi^i_{\bar{l}} \wedge \overline{\Phi^l_{\bar{j}}}. \tag{2.9}$$

Hence we have $\Omega = \Pi - \Phi \wedge \bar{\Phi}$.

Q.E.D.

If the curvature Ω^D of D vanishes identically, then D is said to be *flat*.

3. Finsler geometry

3.1. **Finsler metrics.** Let $\pi : E \to M$ be a complex vector bundle with a complex structure $J_E \in \mathrm{End}(E)$ over a smooth manifold M.

Definition 3.1. A *Finsler metric* on E is a smooth assignment of a norm $\|\cdot\|_x$ to each fibre $E_x = \pi^{-1}(x)$ $(x \in M)$. A *complex Finsler metric* of a complex vector bundle (E, J_E) is a Finsler metric on E satisfying

$$\|(aI_E(x) + bJ_E(x))\,\xi\|_x = \sqrt{a^2+b^2}\,\|\xi\|_x \tag{3.1}$$

for $\forall \xi \in E_x$ and $\forall a, b \in \mathbb{R}$, where I_E is the identity morphism of E. The triplet $(E, J_E, \|\cdot\|)$ is called a *complex Finsler vector bundle.*

Let $E \otimes \mathbb{C}$ be the complexfication of E, and let $E \otimes \mathbb{C} = E^{1,0} \oplus \overline{E^{1,0}}$ be the canonical decomposition. The condition (3.1) is equivalent to

$$\left\|(a + \sqrt{-1}b)\xi\right\|_x = \sqrt{a^2+b^2}\,\|\xi\|_x\,, \tag{3.2}$$

where $a, b \in \mathbb{R}$ and $\xi \in E^{1,0}_x \cong \mathbb{C}^r$.

In the sequel, we assume that E is a holomorphic vector bundle of $\mathrm{rank}(E) = r$ (≥ 2) over a complex manifold M. For explicit expressions of Finsler metrics, we use the natural coordinate system on E induced from an open covering $\{(U, s_U)\}_{U \in \mathcal{U}}$.

Let $F : E \to \mathbb{R}$ be the fundamental function relative to the covering $\{(U, s_U)\}_{U\in\mathcal{U}}$ of $(E, \|\cdot\|)$ defined by $\|v\|_z^2 = F \circ \varphi_U(v)$, where $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$ the local trivialization adapted to $\{(U, s_U)\}_{U\in\mathcal{U}}$. The function F satisfies the following conditions:

1. $F(z, \xi) \geq 0$ and $F(z, \xi) = 0$ if and only if $\varphi_U^{-1}(z, \xi) = 0$,
2. $F(z, \lambda\xi) = |\lambda|^2 F(z, \xi)$ for $^\forall\lambda \in \mathbb{C}$,
3. F is smooth on $E^\times = E - \{0\}$.

Then $\|\cdot\|$ is said to be *convex* if the Hermitian matrix $(F_{i\bar{j}})$ defined by

$$F_{i\bar{j}} = \frac{\partial^2 F}{\partial \xi^i \bar{\xi}^j} \tag{3.3}$$

is positive-definite. It is easily shown that the definition of convexity is independent on the choice of the open cover $\{\mathcal{U}, (s_U)\}$ of E. If $\|\cdot\|$ is convex, then $(F_{i\bar{j}})$ defines a Hermitian metric $\langle\cdot,\cdot\rangle$ on $\varpi : T_{E/M} \to E$ by

$$\left\langle \frac{\partial}{\partial \xi^i}, \frac{\partial}{\partial \xi^j} \right\rangle := F_{i\bar{j}}. \tag{3.4}$$

Remark 3.1. The complex structure J_E of E is naturally lifted to a complex structure $J_{\tilde{E}}$ on the bundle $T_{E/M} \cong \pi^* E = \tilde{E}$ by $J_{\tilde{E}} := \pi^* J_E$. If we denote by $g(X, Y)$ the real part of $\langle X, Y\rangle$, that is, $\langle X, Y\rangle = g(X, Y) + \sqrt{-1}g(J_{\tilde{E}}X, Y)$, the real metric g satisfies $g(J_{\tilde{E}}X, J_{\tilde{E}}Y) = g(X, Y)$, and thus the g is a so-called *generalized Finsler metric* on E by a theorem due to Ichijyō[Ic3] and Fukui[Fu].

The Hermitian metric defined by (3.4) induces a Kähler metric on each fibre $E_z \cong \mathbb{C}^r$. Hence the $\partial\bar{\partial}$-exact real $(1,1)$-form $\omega_E = \sqrt{-1}\partial\bar{\partial}F$ defines a pseudo-Kähler metric on the total space E whose restriction ω_z to each fibre E_z is a Kähler metric on E_z.

Example 3.1. Let $h = (h_{i\bar{j}}(z))$ be an arbitrary Hermitian metric on E. The function $L : E \to \mathbb{R}$ defined by

$$F(z, \xi) = \sum h_{i\bar{j}}(z)\xi^i\bar{\xi}^j \tag{3.5}$$

defines a convex Finsler metric on E. □

Example 3.2. Let $\mathcal{D}$ be a strictly convex domain of $\mathbb{C}^{n+1}$ with smooth boundary. The *Kobayashi-Royden metric* $K_{\mathcal{D}}$ is defined by $K_{\mathcal{D}}(z, \xi) := \inf\{1/R\}$, where the infimum is taken all holomorphic maps $\varphi : \Delta(R) \to M$ satisfying $\varphi(0) = z$ and $d\varphi(d/dt)_0 = \xi$ for $^\forall(z, \xi) \in T_{\mathcal{D}}$. By the early work due to Lempert[Le], the function $F = K_{\mathcal{D}}^2$ defines a convex Finsler metric on the tangent bundle $T_{\mathcal{D}}$. □

3.2. **Bott connections of Finsler bundles.** Let $(E, \|\cdot\|)$ be a convex Finsler vector bundle. We define a splitting $h : \pi^*T_M \to T_E$ of the sequence (1.1) so that its Bott connection D is metrical with respect to the Hermitian metric $\langle\cdot,\cdot\rangle$:

$$d^h\langle X, Y\rangle = \langle DX, Y\rangle + \langle X, DY\rangle \tag{3.6}$$

for all $X, Y \in \Gamma(T_{E/M})$. Since this condition can be written as $\partial^h F_{i\bar{j}} = \sum F_{m\bar{j}}\omega_i^m$, we have

$$\omega_j^i = \sum_m F^{i\bar{m}}\partial^h F_{j\bar{m}}. \tag{3.7}$$

Then, from (2.6) we have

Proposition 3.1. *For the fundamental function F, the functions*

$$N_\alpha^i := \sum_m F^{\bar{m}i}\frac{\partial^2 F}{\partial z^\alpha \partial\bar{\xi}^m} = \sum_{j,m} F^{\bar{m}i}\frac{\partial F_{j\bar{m}}}{\partial z^\alpha}\xi^j \tag{3.8}$$

define an Ehresmann connection H for π, where $(F^{\bar{m}i})$ is the inverse of $(F_{i\bar{j}})$.

We define a splitting h of the sequence (1.1) by the coefficients N_α^i in (3.8), from which we get a canonical Bott connection D of $\{T_{E/M}, \langle\cdot,\cdot\rangle\}$ by the formula (2.5). If we define a section $\mathcal{E}$ of $T_{E/M}$ by

$$\mathcal{E} = \sum_j \xi^j \frac{\partial}{\partial\xi^j},$$

the relations (2.6) is equivalent to

$$D\mathcal{E} \equiv 0, \tag{3.9}$$

and, since $\langle\mathcal{E}, \mathcal{E}\rangle = F(z, \xi)$ and D satisfies the metrical condition (3.6), we have

$$d^h F = \partial^h F + \bar{\partial}^h F \equiv 0. \tag{3.10}$$

For the Bott connection D of $(E, \|\cdot\|)$, we have proved the following proposition.

Proposition 3.2. ([Ai3]) *Let D be the complex Bott connection of $(E, \|\cdot\|)$ defined by the connection H of (4.11). Then we have*

1. $R_{\alpha\beta}^i \equiv 0$,
2. $\partial^h\omega + \omega\wedge\omega \equiv 0$.

By this proposition, some quantities are simplified. By the identity $\partial^h\omega + \omega\wedge\omega \equiv 0$, the curvature Ω^D of D is given by

$$\Omega^D = \bar{\partial}^h\omega, \tag{3.11}$$

and its components R^i_j are given in the form $R^i_j = \sum R^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$, where

$$R^i_{j\alpha\bar{\beta}}(z,\xi) = -X_{\bar{\beta}}\Gamma^i_{j\alpha} \tag{3.12}$$

is the curvature tensor of Ω^D. ¿From (2.5) and the homogeneity (2.2) of N^i_α, we have

$$\sum_j \xi^j R^i_{j\alpha\bar{\beta}} \equiv R^i_{\alpha\bar{\beta}}. \tag{3.13}$$

We consider a Finsler bundle $(E, \|\cdot\|)$ whose Bott connection D is flat. By definitions and the identity (3.13), we have $R^i_{\alpha\bar{\beta}} = 0$ and thus $\Pi = 0$. Then, by (2.7), we have $\Phi = 0$. Hence we get

$$\frac{\partial N^i_\alpha}{\partial \bar{\xi}^j} = 0,$$

and

$$\frac{\partial N^i_\alpha}{\partial \bar{z}^\beta} = \frac{\partial N^i_\alpha}{\partial \bar{z}^\beta} - \sum_l \overline{N^l_\beta} \frac{\partial N^i_\alpha}{\partial \bar{\xi}^l} = R^i_{\alpha\bar{\beta}} = 0.$$

We see that, if $\Omega^D = 0$, then the coefficients N^i_α are holomorphic. Hence we have

Proposition 3.3. *If the Bott connection D of $(E, \|\cdot\|)$ is flat, then the sequence* (1.1) *splits holomorphically.*

If D is flat, since $R^i_{\alpha\beta} = R^i_{\alpha\bar{\beta}} = 0$, the PDE

$$\frac{\partial \Psi^i}{\partial z^\alpha} = -N^i_\alpha(z, \Psi(z)) \tag{3.14}$$

is completely integrable and has a holomorphic solution $\zeta^i(z) = \Psi^i(z, (z_0, \xi_0))$ for an arbitrary initial point $(z_0, \xi_0) \in E$. By the identity (2.6), this PDE is also written as

$$\frac{\partial \zeta^i}{\partial z^\alpha} + \sum_j \Gamma^i_{j\alpha}(z, \zeta(z))\zeta^j = 0. \tag{3.15}$$

For a solution $\zeta^i(z)$ of (3.14), we define a holomorphic section $\tilde{\zeta}$ of $T_{E/M}$ by

$$\tilde{\zeta} = \sum \zeta^i(z) \left(\frac{\partial}{\partial \xi^j}\right)_{(z,\zeta(z))}.$$

Then, by definition, we have $\|\zeta(z)\|^2 = \langle\tilde{\zeta}, \tilde{\zeta}\rangle = F(z, \zeta(z))$ and

$$\begin{aligned}
\frac{\partial}{\partial z^\alpha}\|\zeta(z)\|^2 &= \left(\frac{\partial F}{\partial z^\alpha}\right)_{(z,\zeta(z))} + \sum \frac{\partial \zeta^l}{\partial z^\alpha}\left(\frac{\partial F}{\partial \xi^l}\right)_{(z,\zeta(z))} \\
&= \left(\frac{\partial F}{\partial z^\alpha} - \sum_l N^l_\alpha \frac{\partial F}{\partial \xi^l}\right)_{(z,\zeta(z))} \\
&= (X_\alpha F)_{(z,\zeta(z))} \\
&= 0.
\end{aligned}$$

Hence the norm of ζ is constant, and so ζ is a non-vanishing holomorphic section of E. On the other hand, since the condition (3.15) can be written as $D\tilde{\zeta} = 0$, we see that the flatness of D implies the existence of parallel section of $T_{E/M}$.

Theorem 3.1. *If the Bott connection D of $(E, \|\cdot\|)$ is flat, then there exists a flat Hermitian metric on E.*

Proof. For the solution $\zeta \in H^0(M, \mathcal{O}(E))$ of (3.14), we define a holomorphic map $f_\zeta : M \to E$ by $f_\zeta(z) = (z, \zeta(z))$. We introduce a Hermitian metric $g = (g_{i\bar{j}}(z))$ on E by

$$g_{i\bar{j}}(z) = F_{i\bar{j}}(z, \zeta(z)).$$

The Hermitian connection θ^i_j of (E, g) is given by

$$\theta^i_j = \sum_m g^{i\bar{m}} \partial g_{j\bar{m}} = \sum_m \left(F^{i\bar{m}} \partial^h F_{j\bar{m}}\right)_{(z,\zeta)} = f^*_\zeta \omega^i_j.$$

Hence the curvature $\Omega^g = f^*_\zeta \Omega^D$ of (E, g) vanishes identically.

Q.E.D.

3.3. **Complex Finsler connections and flat Finsler metrics.** The complex Bott connection D of $(E, \|\cdot\|)$ is extended to an ordinary connection ∇ on $T_{E/M}$. In fact, since $T_{E/M} \cong \pi^* E$, the relative tangent bundle $T_{E/M}$ admits a canonical relative flat connection $\nabla^0 : \Gamma(T_{E/M}) \to \Gamma(T_{E/M} \otimes \Omega^1_E)$ characterized by the property $\nabla^0(\pi^{-1}s) = 0$ for every $s \in \Gamma(E)$. The connection $\nabla : \Gamma(T_{E/M}) \to \Gamma(T_{E/M} \otimes \Omega^1_E)$ is given by

$$\nabla = D \oplus \nabla^0.$$

It is noted that ∇ is not compatible with the Hermitian metric $\langle\cdot,\cdot\rangle$ on $T_{E/M}$.

Definition 3.2. The connection ∇ is called the *Finsler connection* of $(E, \|\cdot\|)$.

Since $\partial^h \omega + \omega \wedge \omega \equiv 0$, the curvature form $\Omega^\nabla = d\omega + \omega \wedge \omega$ of ∇ is given by

$$\Omega^\nabla = \Omega^D + d^v \omega. \tag{3.16}$$

In the case of $d^v\omega \equiv 0$, then we have $\Omega^\nabla \equiv \Omega^D$. In this case, such a $(E, \|\cdot\|)$ is said to be *modeled on a complex Minkowski space*, and it is proved that there exists a Hermitian metric h_F on E such that the Finsler connection ∇ is given by pull-back $\pi^*\nabla^{h_F}$ of the Hermitian connection ∇^{h_F} of h_F (cf. [Ai1]). If the curvature Ω^∇ of ∇ vanishes identically, then ∇ is said to be *flat*. In this case, $(E, \|\cdot\|)$ is modeled on a complex Minkowski space and its associated Hermitian metric h_F is flat.

A Hermitian bundle (E, h) is flat if and only if there exists an open cover $\{(U, s_U)\}_{U\in\mathcal{U}}$ of E with a parallel orthonormal holomorphic frame s_U. Then its norm function F_h with respect to $\{(U, s_U)\}_{U\in\mathcal{U}}$ is independent on the base point $z \in M$. In the case of Finsler metrics, the flatness of Finsler metrics is defined as follows:

Definition 3.3. A complex Finsler vector bundle $(E, \|\cdot\|)$ is said to be *flat* if there exists an open cover $\{(U, s_U)\}$ of E such that the pseudo-Kähler potential F of the Kähler morphism $E \to M$ relative to $\{(U, s_U)\}_{U\in\mathcal{U}}$ is independent on the base point $z \in M$.

Unlike the case of Hermitian metrics, by Theorem 4.2, the flatness of D does not implies the flatness of Finsler metrics. In [Ai3] and [Ai4], we have discussed the flatness of the connection ∇ with relation to the flatness of Finsler metrics. The following theorem is given in [Ai3]. The proof is essentially the same as the one in [Ai3], however, we shall reproduce here for the convenience.

Theorem 3.2. *A complex Finsler vector bundle $(E, \|\cdot\|)$ is flat if and only if the curvature R^∇ of ∇ vanishes identically.*

Proof. We denote by F the fundamental function of $\|\cdot\|$ relative to a fixed covering $\{\mathcal{U}, (s_U)\}$. We suppose that $(E, \|\cdot\|)$ is flat. The fundamental function F relative to the adapted covering $\{(U, s_U)\}_{U\in\mathcal{U}}$ is independent on $z \in M$. Then, by (3.8) the coefficients N^i_α vanish identically, and so by (2.5) the connection forms ω vanish on each $U \in \mathcal{U}$. Hence, from (3.11) and (3.16) the curvature R^∇ of ∇ vanishes identically.

Conversely we shall prove that the flatness of ∇ implies the flatness of $(E, \|\cdot\|)$. For this purpose, we shall fix an open covering $\{(U, s_U)\}_{U\in\mathcal{U}}$. Then it defines a local trivialization $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$ by $\varphi_U(v) = (z^\alpha, \xi^i)$ for ${}^\forall v = \sum \xi^i s_i(z) \in \pi^{-1}(U)$. Since ∇ is flat, $(E, \|\cdot\|)$ is modeled on a complex Minkowski space and its associated Hermitian metric h_F is flat. Thus ∇ is the Hermitian connection of this flat Hermitian metric h_F. Hence we can introduce a holomorphic frame field $\tilde{s}_U = (\tilde{s}_1, \cdots, \tilde{s}_r)$ on each $U \in \mathcal{U}$ with respect to which the connection form $\tilde{\omega}$ of

∇ vanishes identically on each $U \in \mathcal{U}$. Hence we have another local trivialization $\tilde{\varphi}_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$ by $\tilde{\varphi}_U(v) = (z^\alpha, \tilde{\xi}^i)$ for ${}^\forall v = \sum \tilde{\xi}^i \tilde{s}_i(z) \in \pi^{-1}(U)$. We denote by $A_U = \left(A^i_j\right)$ the transition function between the frame fields s_U and $\tilde{s}_U$, that is, $\tilde{s}_j(z) = \sum s_i(z) A^i_j(z)$. The coordinate transition $\Psi_U = \varphi_U \circ \tilde{\varphi}_U^{-1}$ is given by

$$\Psi_U(z, \tilde{\xi}) = \left(z^\alpha, \sum_j A^i_j(z) \tilde{\xi}^j \right).$$

We note that, since $\tilde{s}_U$ is a parallel frame filed on U, the functions $A^i_j(z)$ satisfy

$$\frac{\partial A^i_j}{\partial z^\alpha} + \sum_l \Gamma^i_{l\alpha}(z) A^l_j(z) = 0 \tag{3.17}$$

on each $U \in \mathcal{U}$. The fundamental function $\tilde{F}$ of $\|\cdot\|$ relative to $\{(U, \tilde{s}_U)\}$ is given by $\tilde{F}(z, \tilde{\xi}) = (\Psi_U^* F)(z, \tilde{\xi})$.

$$\begin{tikzcd}
 & U \times \mathbb{C}^r \arrow[dd, "\Psi_U"] \arrow[dr, "\tilde{F}"] & \\
\pi^{-1}(U) \arrow[ur, "\tilde{\varphi}_U"] \arrow[dr, "\varphi_U"'] & & \mathbb{R}. \\
 & U \times \mathbb{C}^r \arrow[ur, "F"'] &
\end{tikzcd}$$

Then, since $A^i_j(z)$ satisfy (3.17) we have

$$\Psi_{U*}\left(\frac{\partial}{\partial z^\alpha}\right) = \frac{\partial}{\partial z^\alpha} - \sum_{j,l} \Gamma^i_{j\alpha}(z) A^j_l(z) \tilde{\xi}^l \frac{\partial}{\partial \xi^i} = \frac{\partial}{\partial z^\alpha} - \sum_i N^i_\alpha(z, \xi) \frac{\partial}{\partial \xi^i} = X_\alpha$$

and $X_\alpha F = 0$ by (3.10), we have

$$\frac{\partial \tilde{F}}{\partial z^\alpha} = \left(\frac{\partial}{\partial z^\alpha}\right)(\Psi_U^* F) = \Psi_{U*}\left(\frac{\partial}{\partial z^\alpha}\right) F = X_\alpha F = 0,$$

which shows $\tilde{F} = \tilde{F}(\tilde{\xi})$. Hence $(E, \|\cdot\|)$ is flat.

Q.E.D.

By the proof of Theorem 4.3, we have

Corollary 3.1. *A complex Finsler bundle $(E, \|\cdot\|)$ is flat if and only if it is modeled on a complex Minkowski space and its associated Hermitian bundle (E, h_F) is flat.*

4. Some remarks

Let $\pi_{P_E} : P_E \to M$ be the projective bundle associated with E. We denote by $\pi_{L_E} : L_E \to P_E$ the tautological line bundle, and we also denote by $L_E^\times$ the open submanifold of L_E consisting of the non-zero elements. The holomorphic map $\tau : E^\times \to P_E \times E$ defined by $\tau(v) = ([v], v)$ maps $E^\times$ to $L_E^\times$ bi-holomorphically. Then it is shown that any Finsler metric on E is naturally identified as a Hermitian metric on L_E as follows(cf. [Ko1]).

For the projective bundle P_E associated with E, we introduce a standard open covering $\{U_j; U \in \mathcal{U}, 1 \leq j \leq r\}$ of P_E from an open cover $\{(U, ts_U)\}$ of E by putting $U_j = \pi^{-1}(U) \cap \{\xi^j \neq 0\} = \{(z, [\xi]) \in P_E; z \in U,\ \xi^i \neq 0\}$. On each open set U_j, we define a holomorphic section $t_j : U_j \to L_E$ by

$$t_j(z, [\xi]) = \left((z, [\xi]), \left(\frac{\xi^1}{\xi^j}, \cdots, \frac{\xi^r}{\xi^j}\right)\right).$$

Then $\{t_j\}$ defines a local trivialization $\varphi_j : U_j \times \mathbb{C} \to \pi_{L_E}^{-1}(U_j)$ by $\varphi_j((z, [\xi]), \lambda) = \lambda t_j(z, [\xi])$. Hence the bi-holomorphism τ can be written as

$$\tau(z, \xi) = ((z, [\xi]), \xi) = \varphi_j\left((z, [\xi]), \xi^j\right) \cong \left((z, [\xi]), \xi^j\right).$$

On the other hand, any Hermitian metric h_{L_E} on L_E is defined by the family of positive functions $\{h_{L_E,j}\}$ on each U_j satisfying

$$h_{L_E,i} = \left|\frac{\xi^j}{\xi^i}\right|^2 h_{L_E,j} \tag{4.1}$$

on $U_i \cap U_j \neq \phi$. Hence, by using the map τ and identification

$$L^2(z, \xi) = h_{L_E}(\tau(z, \xi)) = |\xi^j|^2 h_{L_E,j}(z, [\xi]), \tag{4.2}$$

any Finsler metric $\|\cdot\|$ on E is identified with a Hermitian metric h_{L_E} on the tautological line bundle L_E.

Since $\mathcal{E}$ is generated by the action μ_λ and $\mathcal{L}_{\mathcal{E}}\partial\bar{\partial}\log F = 0$, the real $(1,1)$-form

$$\omega_{P_E,F} = \sqrt{-1}\partial\bar{\partial}\log F \tag{4.3}$$

is invariant by the action μ_λ for $^\forall\lambda \in \mathbb{C}$, and so $\omega_{P_E,F}$ may be considered as a real $(1,1)$-form on P_E. We suppose that $(E, \|\cdot\|)$ is convex. Since

$$\sqrt{-1}\partial\bar{\partial}F_z = \sqrt{-1}F_z(\partial\bar{\partial}\log F_z + \partial\log F_z \wedge \bar{\partial}\log F_z), \tag{4.4}$$

the function $\log F_z$ is strictly subharmonic on each fibre $P(E_z)$. Thus we have obtained a Kähler morphism $\pi_{P_E} : P_E \to M$ with a pseudo-Kähler metric $\omega_{P_E,F}$ from an arbitrary $(E, \|\cdot\|)$.

Proposition 4.1. *Let $(E, \|\cdot\|)$ be a convex Finsler vector bundle. Then the projective bundle $\pi_{P_E} : P_E \to M$ is a Kähler morphism with the pseudo-Kähler metric $\omega_{P_E,F}$.*

Conversely, from an arbitrary pseudo-Kähler metric ω_{P_E} of the projective bundle P_E, it induces a convex Finsler metric $\|\cdot\|$ on E, that is, we have

Theorem 4.1. *A holomorphic vector bundle admits a convex Finsler metric if and only if the projective bundle $\pi_{P_E} : P_E \to M$ associated with E is a Kähler morphism.*

Proof. We express locally $\omega_{P_E} = \sqrt{-1}\partial\bar{\partial}G_j$ on U_j for a C^∞-function G_j on U_j. Since $G_j - G_i$ is pluri-harmonic, there exists a 1-cocycle $K_{ij} \in Z^1(\mathcal{U}_{P_E}, \mathcal{O}_{P_E})$ satisfying $G_j - G_i = K_{ij} + \overline{K_{ij}}$ on $U_i \cap U_j \neq \phi$. Restricting to each fibre $P(E_z)$, we have $\omega_z = \sqrt{-1}\partial\bar{\partial}G_{j,z}$ and $G_{j,z} - G_{i,z} = K_{ij,z} + \overline{K_{ij,z}}$. Then $\{K_{ij,z}\}$ is a 1-cocycle on $P(E_z) \cong \mathbb{P}^{r-1}$, and since $H^1(\mathbb{P}^{r-1}, \mathcal{O}) = 0$, we may put

$$K_{ij,z} = (K_{j,z} - \log \xi^j) - (K_{i,z} - \log \xi^i)$$

for a 0-cochain $\{K_{j,z}\}$ on $P(E_z)$. Hence we have

$$G_{j,z} - (K_{j,z} + \overline{K_{j,z}}) + \log |\xi^j|^2 = G_{i,z} - (K_{i,z} + \overline{K_{i,z}}) + \log |\xi^i|^2$$

If we put $F_{j,z}([\xi]) = \exp\{G_{j,z} - (K_{j,z} + \overline{K_{j,z}})\}$ on $U_{j,z}$, we have $|\xi^j|^2 F_{j,z}([\xi]) = |\xi^i|^2 F_{i,z}([\xi])$. Since each $|\xi^j|^2 F_{j,z}([\xi]) := F_z(\xi)$ depends on $z \in M$ smoothly, the function $F_z(\xi)$ defines a complex Finsler metric $F(z,\xi)$ on E. Consequently we have

$$F(z,\xi) = |\xi^j|^2 \exp\{G_j - (K_j + \overline{K_j})\} \tag{4.5}$$

for a family of local functions $\{K_j\}$ on U_j which are holomorphic in $[\xi]$ and smooth in z. Moreover, because of $\sqrt{-1}\partial\bar{\partial}\log F_z = \sqrt{-1}\partial\bar{\partial}\log F_{j,z} = \sqrt{-1}\partial\bar{\partial}G_{j,z} > 0$ and (4.4), F defines a convex Finsler metric on E.

Q.E.D.

The convex Finsler metric $\|\cdot\|$ determined from ω_{P_E} should be satisfy the relation $\omega_z = \sqrt{-1}\partial\bar{\partial}\log F_z(\xi)$ on each fibre P_{E_z}. Hence we have

Proposition 4.2. *A pseudo-Kähler metric ω_{P_E} on P_E determines a unique convex Finsler metric $\|\cdot\|$ on E up to a locally conformal equivalence.*

We say that the pseudo-Kähler metric Π_{P_E} is flat if there exists an open covering $\{(U, s_U)\}_{U\in\mathcal{U}}$ of E such that the pseudo-Kähler potential $\{G_j\}$ of Π_{P_E} is independent on the base point $z \in M$. We also say the Finsler metric induced from Π_{P_E} is *projectively flat* if Π_{P_E} is flat. The projective flatness of a Finsler metric $\|\cdot\|$ is characterized in terms of curvature of the Finsler connection ∇ of $(E, \|\cdot\|)$ (cf. [Ai5]), and it is easily shown that the projective flatness of $\|\cdot\|$ is equivalent to the locally-conformal flatness of $\|\cdot\|$ in the sense of [Ai3].

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