МАТЕМАТИЧЕСКАЯ ЖИЗНЬ

MATH-SELFIE

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This is a short overview of some sections of applied functional analysis, convexity, optimization, and nonstandard models.

1. Introduction

Mathematics is the logic of natural sciences, the unique science of the provable forms of reasoning quantitatively and qualitatively.

Functional analysis had emerged at the junctions of geometry, algebra and the classical calculus, while turning rather rapidly into the natural language of many traditional areas of continuous mathematics and approximate methods of analysis. Also, it has brought about the principally new technologies of theoretical physics and social sciences (primarily, economics and control).

Of most interest for the author are some contiguous sections of the constituents of functional analysis and model theory that are promising in search for modernization of the theoretical techniques of socializing the problems with many solutions.

The traditions of functional analysis were implanted in Siberia by S. L. Sobolev and L. V. Kantorovich.

Their thesis of the unity of functional analysis and applied mathematics was, is, and should be the branding mark of the Russian mathematical school. This is the author's deep belief.

F. Bacon distinquished between pure and mixed mathematics; see [1]. L. Euler used the attribute "pure," which has been in common parlance since then; see [2]. S. Feferman introduced the concept of "scientifically applicable mathematics" as "that part of everyday mathematics which finds its applications in the other sciences"; see [3, p. 30]. In this article applied mathematics is understood according to the Feferman definition.

The main areas dwelt upon below are functional analysis, nonstandard methods of analysis, convex geometry, and optimization. These are listed according to their significance. Each of them remains in the sphere of the author's interests from the moment of the first appearance, but the time spent and the efforts allotted have been changing every now and then. Here these areas are addressed in chronological order.

2. Optimal Location of Convex Bodies

Using the ideas of linear programming invented by L. V. Kantorovich, it turned out possible to distinguish some classes of extremal problems of optimal location of convex surfaces that could not be treated by the classical methods in principle. The decisive step forward was to address such a problem by the standard approach of programming which consists in transition to the dual problem. The latter turns out solvable by the technique of mixed volumes, abstraction of the duality ideas of H. Minkowski, and modification of one construction in measure theory that belongs to Yu. G. Reshetnyak [4]. The revealed descriptions of new classes of inequalities over convex surfaces in combination with the technique of surface area measures by A. D. Alexandrov [5] had led to reducing to linear programs the isoperimetric-type problems with however many constraints; i.e., the problems that fall beyond the possibility of symmetrizations. In fact, the extensive class was discovered of the problems whose solutions can be written down explicitly by translating the problems into convex programs in appropriate function spaces. A few of these results are presented in the survey article [6]. The conception of H-convexity from this paper is considered now as definitive in the numerous studies in generalized convexity and search for schemes of global optimization; in particular, see [7].

The most visual and essential progress is connected with studying some abstractions of the Urysohn problem of maximizing the volume of a convex surface given the integral of the breadth of the surface. By the classical result of P. S. Urysohn [8] which was published in the year of his death—1924, this is a ball as follows from the suitable symmetry argument. In the 1970s the functional-analytical approach was illustrated with the example of the *internal Urysohn problem*: Granted the integral breadth, maximize the volume of a convex surface that lies within an a priori given convex body, e.g., a simplex in $\mathbb{R}^N$. The principal new obstacle in the problem is that no symmetry argument is applicable in analogous internal or external problems. It turned out that we may solve the problem in some generalized sense—"modulo" the celebrated Alexandrov Theorem on reconstruction of a convex surface from its surface area measure. For the Urysohn problem within a polyhedron the solution will be given by the Lebesgue measure on the unit sphere with extra point loads at the outer normals to the facets of the polyhedron. The internal isoperimetric problem falls beyond the general scheme even within a tetrahedron.

Considering the case of $N = 3$ in 1995, A. V. Pogorelov found in one of his last papers [9] the shape of the "soup bubble" within a tetrahedron in the same generalized sense—this happens to be the vector sum of a ball and the solution of the internal Urysohn problem. In the recent years quite a few papers has been written about the double bubbles. These studies are also close to the above ideas.

3. Ordered Vector Spaces

Of most importance in this area of functional analysis are the problems stemming from the Kantorovich heuristic principle.

In his first paper of 1935 on the brand-new topic L. V. Kantorovich [10] wrote:

> In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in these spaces) in the same way as linear functionals.

It is worth noting that his definition of semiordered linear space contains the axiom of Dedekind completeness which was denoted by I_6. L. V. Kantorovich demonstrated the role of

K-spaces by widening the scope of the Hahn–Banach Theorem. The heuristic principle turned out applicable to this fundamental Dominated Extension Theorem; i.e., we may abstract the Hahn–Banach Theorem on substituting the elements of an arbitrary K-space for reals and replacing linear functionals with operators acting into the space.

The Kantorovich heuristic principle has found compelling juctifications in his own research as well as in the articles by his students and followers. Attempts at formalizing the heuristic ideas by Kantorovich started at the initial stages of K-space theory and yielded the so-called identity preservation theorems. They asserted that if some algebraic proposition with finitely many function variables is satisfied by the assignment of all real values then it remains valid after replacement of reals with members of an arbitrary K-space. To explain the nature of the Kantorovich transfer principle became possible only after half a century by using the technique of nonstandard models of set theory.

The abstract ideas of L. V. Kantorovich in the area of K-spaces are tied with linear programming and approximate methods of analysis. He wrote about the still-unrevealed possibilities and underestimation of his theory for economics and remarked:

> But the comparison and correspondence relations play an extraordinary role in economics and it was definitely clear even at the cradle of K-spaces that they will find their place in economic analysis and yield luscious fruits.

The problem of the scope of the Hahn–Banach Theorem, tantamount to describing the possible extensions of linear programming, was rather popular in the decade past mid-1970s. Everyone knows that linear programs lose their effectiveness if only integer solutions are sought. S. N. Chernikov abstracted linear programming from the reals to some rings similar to the rationals in [11]. Rather topical in the world mathematical literature was the problem of finding the algebraic systems that admit the full strength of the ideas of L. V. Kantorovich. The appropriate answer was given by describing the abstract modules that allow for the tools equivalent to the Hahn–Banach Theorem; see [12]. These are K-spaces viewed as modules over rather "voluminous" algebras of their orthomorphisms. This result was resonated to some extent in the theoretical background of mathematical economics as relevant to the hypothesis of "divisible goods."

One of the rather simple particular cases of these results is a theorem characterizing a lattice homomorphism. The latter miraculously attracted attention of vector-lattice theorists who founded new proofs and included the theorem in monographs as *Kutateladze's Theorem*; e.g., [13, p. 114]. Many years had elapsed before Boolean valued analysis explained that the modules found are in fact dense subfields of the reals in an appropriate nonstandard model of set theory.

In this area some unexpected generalizations of the Kreĭn–Milman Theorem to noncompact sets had been found that stimulated a few articles on the abstraction of Choquet theory to vector lattices; see [14]. Ordered vector spaces have opened opportunities to advance applications of Choquet theory to several problems of modern potential theory such as describing interconnections of the Dirichlet problem with Bauer's geometric simplices in infinite dimensions and introducing the new objects—supremal generators of function spaces which are convenient in approximation by positive operators. Note that the conception of supremal generation which bases on the computational simplicity of calculating the join of two reals had turned out close to some ideas of idempotent analysis that emerged somewhat later in the research by V. P. Maslov and his students; see [15].

4. Nonsmooth Analysis and Optimization

Note the rather numerous papers on convex analysis, one of the basic sections of nonlinear analysis. Convex analysis is the calculus of linear inequalities. The concept of convex set does not reach the age of 150 years, and convex analysis as a branch of mathematics exists a bit longer than half a century. The solution sets of simultaneous linear inequalities are the same as convex sets which can be characterized by their gauges, support functions or distributions of curvature. Functional analysis is impossible without convexity since the existence of a nonzero continuous linear functional is provided if and only if the ambient space has nonempty proper open convex subsets.

Convex surfaces have rather simple contingencies, and convex functions are directionally differentiable in the natural sense and their derivatives are nonlinear usually in quite a few points. But these points extreme in the direct and indirect senses are most important. Study of the local behavior of possible fractures at extreme points is the subject of subdifferential calculus.

The most general and complete formulas were found for recalculating the values and solutions of rather general convex extremal problems under the changes of variables that preserve convexity. The key to these formulas is the new trick of presenting an arbitrary convex operator as the result of an affine change of variables in a particular sublinear operator, a member of some family enumerated by cardinals. The basic results in this area were published in [16]. The literature uses the term *Kutateladze's canonical operator* (cp. [17, pp. 123–125] and [18, p. 92]). These formulas led to the Lagrange principle for new classes of vector optimization problems and the theory of convex ε-programming. The problem of approximate programming consists in the searching of a point at which the value of a (possibly vector valued) function differs from the extremum by at most some positive error vector ε. The constraints are also given to within some accuracy of the order of ε. The standard differential calculus is inapplicable here, but the new methods of subdifferential calculus solve many problems of the sort. These results became rather topical, entered textbooks, and were redemonstrated with reference to the Russian priority. The literature uses the term *Kutateladze's approximate solutions* (for instance, cp. [19]). Many years later the help of infinitesimal analysis made it possible to propose the tricks that are not connected with the bulky recalculations of errors. To this end, the error should be considered as an infinitesimal, which is impossible within the classical set-theoretic stance.

Applications to nonsmooth analysis are connected primarily with inspecting the behavior of the contingencies of general rather than only convex correspondences. In this area there were found some new rules for calculating various types of tangents and one-sided directional derivatives. The advances in these areas base on using the technique of model theory as well.

Many extremal problems are studied in various branches of mathematics, but they use only scalar target functions. Multiple criteria problems have appeared rather recently and beyond the realm of mathematics. This explains the essential gap between the complexity and efficiency of the mathematical tools which divides single and multiple criteria problems. So it stands to reason to enrich the stock of purely mathematical problems of vector optimization. The author happened to distinguish some class of geometrically reasonable problems of vector optimization whose solutions can be presented in a relatively lucid form of conditions for surface area measures. As model examples, the Urysohn problems were considered with extra targets like flattening in a given direction, symmetry, or optimization of the volume of the convex hulls of several surfaces; see [20] and [21].

5. New Models for Mathematical Analysis

In the recent decades much research is done into the nonstandard methods located at the junctions of analysis and logic. This area requires the study of some new opportunities of modeling that open broad vistas for consideration and solution of various theoretical and applied problems.

A model of a mathematical theory is usually called nonstandard if the membership within the model has interpretation different from that of set theory. This understanding is due to L. Henkin. The simplest example of nonstandard modeling is the classical trick of presenting reals as points of an axis.

The new methods of analysis are the adaptation of nonstandard set theoretic models to the problems of analysis. The two technologies are most popular: infinitesimal analysis also known as Robinsonian nonstandard analysis and Boolean valued analysis.

Infinitesimal analysis by A. Robinson appeared in 1960 and is characterized by legitimizing the usage of actual infinities and infinitesimals which were forbidden for the span of about thirty years in the mathematics of the twentieth century. In a sense, nonstandard analysis implements a partial modern return to the classical infinitesimal analysis. The recent publications in this area can be partitioned into the two groups: The one that is most proliferous uses infinitesimal analysis for "killing quantifiers," i.e., simplifying definitions and proofs of the classical results. The other has less instances but contributes much more to mathematics, searching the opportunities unavailable to the standard methods; i.e., it develops the technologies whose description is impossible without the new syntax based on the predicate of standardness. We should list here the development of the new schemes for replacing the infinite objects as parts of finite sets: nonstandard hulls, Loeb measures, hyperapproximation, etc. Part of this research is done in Novosibirsk. In particular, the author's results on infinitesimal programming [22] belong to the second group.

Boolean valued analysis is characterized by the terms like Boolean valued universe, descents and ascents, cyclic envelopes and mixings, Boolean sets and mappings, etc. The technique here is much more complicated that of infinitesimal analysis and just a few analysis are accustomed to it. The rise of this branch of mathematical logic was connected with the famous P.-J. Cohen's results of 1961 on the independence of the continuum hypothesis, whose understanding drove D. Scott, R. Solovay, and P. Vopěnka to the construction of the Boolean valued models of set theory.

D. Scott foresaw the role of Boolean valued models in mathematics and wrote as far back as in 1969 (see [23, p. 91]):

> We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument.

G. Takeuti was one of the first who pointed out the role of these models for functional analysis (in Hilbert space) and minted the term *Boolean valued analysis* in [24]. The models of infinitesimal analysis can be viewed among the simplest instances of Boolean valued universes.

The progress of Boolean valued analysis in the recent decades has led to a profusion of principally new ideas and results in many areas of functional analysis, primary, in the theory of Dedekind complete vector lattices and the theory of von Neumann algebras as well as in convex analysis and the theory of vector measures. Most of these advances are connected with Novosibirsk and Vladikavkaz; see [25]–[27]. It is not an exaggeration to say that Boolean valued analysis left the realm of logic and has become a section of order analysis.

The new possibilities reveal the exceptional role of universally complete vector lattices—extended K-spaces in the Russian literature. It was completely unexpected that each of them turns out to be a legitimate model of the real axis, so serving the same fundamental role in mathematics as the reals. Kantorovich spaces are indeed instances of the models of the reals, which corroborated the heuristic ideas of L. V. Kantorovich. This remarkable result was discovered by E. I. Gordon [28].

Adaptation of nonstandard models to the problems of analysis occupies the central place in the research of the author and his closest colleagues. In this area we have developed the special technique of ascending and descending, gave the criteria of extensional algebraic systems, suggested the theory of cyclic monads, and indicated some approaches to combining infinitesimal and Boolean valued models.

These ideas lie behind solutions of various problems of geometric and applied functional analysis among which we list the drastically new classification of the Clarke type one-sided approximations to arbitrary sets and the corresponding rules for calculating infinitesimal tangents, the nonstandard approach to approximate solutions of convex programs in the form of *infinitesimal programming*, the new formulas for projecting to the principal bands of the space of regular operators which are free from the usual limitations on the order dual, etc.

We can also mention the new method of studying some classes of bounded operators by the properties of the kernels of their strata. This method bases on applying the Kantorovich heuristic principle to the folklore fact that a linear functional can be restored from each of its hyperplanes to within a scalar multiplier. In 2005 this led to the description of the operator annihilators of Grothendieck spaces; see [29]. In 2010 the method made it possible to suggest the operator forms of the classical Farkas Lemma in the theory of linear inequalities, so returning to the origins of linear programming; see [30] and [31].

Of great importance in this area are not only applications but also inspections of the combined methods that involve Boolean valued and infinitesimal techniques. At least the two approaches are viable: One consists in studying a standard Boolean valued model within the universes of Nelson's or Kawai's theory. Infinitesimals descend there from some external universe. The other bases on distinguishing infinitesimals within Boolean valued models. These approaches were elaborated to some extent, but the synthesis of the tools of various versions of nonstandard analysis still remain a rather open problem.

Adaptation of the modern ideas of model theory to functional analysis projects among the most important directions of developing the synthetic methods of pure and applied mathematics. This approach yields new models of numbers, spaces, and types of equations. The content expands of all available theorems and algorithms. The whole methodology of mathematical research is enriched and renewed, opening up absolutely fantastic opportunities. We can now use actual infinities and infinitesimals, transform matrices into numbers, spaces into straight lines, and noncompact spaces into compact spaces, yet having still uncharted vast territories of new knowledge.

Quite a long time had passed until the classical functional analysis occupied its present position of the language of continuous mathematics. Now the time has come of the new powerful technologies of model theory in mathematical analysis. Not all theoretical and applied mathematicians have already gained the importance of modern tools and learned how to use them. However, there is no backward traffic in science. The modern methods are doomed to reside in the realm of mathematics for ever, and they will shortly become as elementary and omnipresent in calculuses and calculations as Banach spaces and linear operators.

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МАТСЕЛФИ

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Краткое обсуждение некоторых разделов выпуклой геометрии, функционального анализа, оптимизации и нестандартных моделей в сфере интересов автора.

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