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DISTALITY VERSUS SHADOWING AND EXPANSIVENESS

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The aim of this note is to show that certain two classes of homeomorphisms important in the topological dynamics, namely distal homeomorphisms and expansive homeomorphisms having the pseudo orbits tracing property, are basically disjoint when the phase space is metric and compact. This result is not a big surprise. In fact, distality was introduced by Hilbert, see [5], in attempt to generalize the notion of isometry. On the other hand, expansiveness and the pseudo orbits tracing property corresponds to the concept of the hyperbolic set for a diffeomorphism. Moreover, Aoki, Dateyama and Komuro in [2] proved that if the phase space is connected then the class of distal homeomorphisms and the class of those having the pseudo orbits tracing property are disjoint. Also, if the space is self-dense then the expansiveness itself excludes the distality, see Remark 2. In this note we assume nothing on the space, we still assume that the homeomorphism admits non-periodic points what is an externally weak assumption. Besides, we consider a larger class than distal homeomorphisms, namely semisimple homeomorpisms.

Let (X, d) be a compact metric space with a distance d. Let $f : X \to X$ be a homeomorphism onto X. Recall [1] that f is semisimple iff the space X admits the decomposition, $X = \bigcup E_\alpha$, where E_α are minimal sets (i.e. each E_α is closed, invariant and there is not a proper subset of E_α with the above properties. f is distal iff for all points $x, y \in X, x \neq y$ there exists $\epsilon > 0$ such that for all $n \in Z$ we have $d(f^n x, f^n y) \geq \epsilon$. It is known [1] that any distal homeomorphism is semisimple. On the other hand, there exist semisimple homeomorphisms that are not distal,

Recall that f is expansive if there exists $e > 0$ such that

$$d(f^n x, f^n y) \leq e, \text{for all } n \in Z \Rightarrow x = y.$$

Recall that a sequence $\{x_n\}_{n\in Z}$ is a δ-pseudo orbit if $d(fx_n, x_{n+1}) \leq \delta$, for all $n \in Z$. We say that a homeomorphism f has the pseudo orbits tracing property, abbr. POTP, or shadowing property if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that any δ-pseudo orbit is ϵ-traced by some point $x \in X$, i. e. $d(f^n x, x_n) \leq \epsilon$, for all $n \in Z$.

Expansive homeomorphisms having the POTP have been studied as a topological version of the theory of diffeomorphisms of compact manifolds when one considers their hyperbolic sets, see [1], [5] and references therein for more information.

Denote by $Per(f)$ the set of periodic points of f, i. e. $Per(f) = \{x \in X : f^n x = x$, for some $n \in Z\}$. By $\Omega(f)$ we denote the set of nonwandering points, where a point $x \in X$ is nonwandering if for every neighborhood U of x there is $n \in Z, n \neq 0$ such that $f^n U \cap U \neq \emptyset$.

THEOREM. *The set of homeomorphisms having the following properties*

(a) $X \setminus Per(f) \neq \emptyset$,
(b) f *is semisimple,*
(c) f *is expansive and has the POTP,*

is empty.

COROLLARY. *The set of homeomorphisms having properties*

(a') X *is uncountable,*
or
(a'') X *is self-dense,*
or
(a''') X *is connected and not a single point,*

and

(b) f *is semisimple,*
(c) f *is expansive and has the POTP,*

is empty.

REMARK 1. The third option in the Corollary resembles a theorem of Aoki [1,2] that a distal homeomorphisms on a compact connected metric space different from a singleton do not have the POTP. In fact, Aoki's result may look better than the Corollary in a sense that there is only a slight difference between distal and semisimple homeomorphisms; on the other hand, expansiveness is a quite restrictive assumption, it is, however, not the case when the homeomorphism has already the POTP.

REMARK 2. If the space is self-dense then expansiveness excludes distality. In fact, if f is expansive then there exist points $x, y, x \neq y$ such that $d(f^n x, f^n y) \longrightarrow 0$, as $n \longrightarrow \infty$, Theorem 10.36 in [4], hence f is not distal.

First, we recall certain results concerning expansiveness and the POTP, see [1], [5]. Assume that a homeomorphism f is expansive and has the POTP. Then, the set of nonwandering points $\Omega(f)$ has the spectral decomposition into basic sets,

$$\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_r,$$

where each Ω_i is closed invariant, contains a dense orbit and is disjoint with the others Ω_j. Moreover, the set of periodic points $Per(f)$ is dense in $\Omega(f)$. The restriction $f \mid \Omega(f)$ is expansive and has the POTP.

We denote by $W^s(x)$ the stable "manifold" at a point x, $W^s(x) = \{x \in X : d(f^n x, f^n y) \longrightarrow 0, \text{as } n \longrightarrow \infty\}$. By expansiveness, if $d(f^n x, f^n y) \leq e$ for all positive n large enough then $y \in W^s(x)$. Expansiveness also implies that the set of periodic points with a fixed period is finite, hence the set $Per(f)$ is countable.

We will also need the following:

PROPOSITION. *Let f be an expansive homeomorphism with the POTP. Then, for every $x \in X$ there exists $a \in \Omega(f)$ such that $x \in W^s(a)$.*

PROOF. Let $x \in X$. By the result mentioned above, there exists $\delta > 0$ such that any δ-pseudo orbit from $\Omega(f)$ is $e/2$-traced by a point from $\Omega(f)$, where e is an expansive constant. Let $c(\delta/2)$ be a number such that $d(u,v) \leq c(\delta/2)$ implies $d(fu, fv) \leq \delta/2$. As $dist(f^n x, \Omega(f)) \longrightarrow 0$, as $n \longrightarrow \infty$, which is a standard observation, we have a number N such that for $n \geq N$ we have $d(f^n x, y_n) \leq min(e/2, \delta/2, c(\delta/2))$, with some points $y_n \in \Omega(f)$. Let x' be a point such that $f^N x' = y_N$. As $\Omega(f)$ is invariant, $x' \in \Omega(f)$.

Define

$$x_n = \begin{cases} f^n x' & \text{for } n < N \\ y_n & \text{for } n \geq N \end{cases}$$

These points form a δ-pseudo orbit in $\Omega(f)$ as

$$d(fy_n, y_{n+1}) \leq d(fy_n, ff^n x') + d(f^{n+1} x', y_{n+1}) \leq \delta, \quad \text{for } n \geq N.$$

This orbit is $\epsilon/2$-traced by a point $a \in \Omega(f)$. Hence for $n \geq N$ we have $d(f^n x, f^n a) \leq d(f^n x, y_n) + d(y_n, f^n a) \leq e$. So by expansiveness $x \in W^s(a)$.

PROOF OF THE THEOREM. Assume that a homeomorphism f satisfies conditions (a), (b) and (c). We will get a contradiction by examining the two following cases.

CASE 1. $\Omega(f) \setminus Per(f) \neq \emptyset$. There is a basic set, say Ω_i such that $\Omega_i \setminus Per(f) \neq \emptyset$. This set contains a dense orbit, say $o(b)$ which is not periodic. As f is semisimple we may choose a minimal set E_α such that $b \in E_\alpha$. So,

$\Omega_i = cl\ (o(b)) \subset E_\alpha$. On the other hand, $Per(f)$ is dense in $\Omega(f)$ and we find a periodic point $p \in \Omega_i \subset E_\alpha$. So, the periodic orbit $o(p)$ is a proper subset of the minimal set E_α, which is an absurd.

CASE 2. $\Omega(f) = Per(f)$. Condition (a) provides a point $x \in X \setminus \Omega(f)$. Let $a \in \Omega(f)$ be such that $x \in W^s(a)$, see Proposition. Then, the ω-limit sets of x and a are equal and as a is periodic point, $\omega(x)$ is a periodic orbit. A minimal set containing x must also contain its ω-limit set, which is a contradiction.

PROOF OF THE COROLLARY. It follows from the observation that if f is expansive then $Per(f)$ is countable. Next, a self-dense compact metric space must be uncountable, by the Baire Category Theorem. Finally, condition (a") implies (a"').

Added in proof. Proof of the Theorem can be simplified as follows. (1) If $\Omega(f) = Per(f)$, then, by condition (a) we have a point $x \in X \setminus \Omega(f)$. Let E_α be a minimal set containing x. Now, the set $\omega(x) \subset E_\alpha$ is invariant and closed. On the other hand, it is contained in $\Omega(f)$, so does not contain x, hence, E_α is not minimal. (2) If $\Omega(f) \neq Per(f)$, then, by the spectral decomposition, we can find a non-periodic point x and a basic set Ω_i, such that $cl\ (o(x)) = \Omega_i$. Let E_α be a minimal set containing x. Now, this set is invariant and closed, hence is equal to Ω_i. By its minimality, it does not contain periodic points. We get a contradiction because periodic points are dense in $\Omega(f)$, hence in $\Omega_i = cl\ (o(x))$.

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