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ON THE EQUATION $e^{ix} = \left(i - \frac{x}{2}\right) / \left(i + \frac{x}{2}\right)$

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Abstract. Applying the theory of the Friedrichs extensions to some integro–differential operator we show an equivalence of the title equation and the equation

$$\sum_{k\in 1+2\mathrm{N}} \left(1 + \frac{i}{k^2}\right) \frac{1}{(k\pi)^2 - x^2} = \frac{1}{8},$$

where $1 + 2\mathrm{N} := \{1, 3, 5, \ldots\}$. Namely the sets of their non-zero real solutions are identical. These solutions form an infinite sequence without finite accumulation points.

1. The Friedrichs extension of the operator $(-\Delta)\colon \mathcal{D} \cap \{\int = 0\} \longrightarrow \{\int = 0\}$.

Let $\Omega \subset \mathbf{R}^n$ be an open set of finite Lebesgue measure (i.e. $|\Omega| := m(\Omega) < \infty$). For an integer $s \in \mathrm{N}$ we will consider the Sobolev space

$$W^{s,2} = \{u \in L^2 \colon D^\alpha u \in L^2, \ \forall \alpha \in \mathrm{N}^n, \ |\alpha| \le s\},$$

where $L^2 := L^2(\Omega, \mathbf{R})$, $|\alpha| := \alpha_1 + \ldots \alpha_n$ and $D^\alpha u$ means the suitable derivative of the sense of Sobolev. It is a real Hilbert space with the scalar product

$$(u, w) \mapsto \sum_{|\alpha| \le s} (D^\alpha u | D^\alpha W)_{L^2}.$$

Let $W_0^{1,2}$ denotes the closure of the subspace $\mathcal{D}$ of test functions on Ω in the Hilbert space $W^{1,2}$. By the following well known Sobolev inequality

$$||\varphi||_{L^{\frac{np}{n-p}}} \le \frac{(n-1)p}{n-p} \max_{1\le i\le n} ||\frac{\partial \varphi}{\partial x_i}||_{L^p} \quad \text{for all } \varphi \in \mathcal{D}(\mathbf{R}^n) \text{ and } 1 \le p < n,$$

(that, up to a constant, is proved in [2]) we infer that in the case $n \geq 2$ the bilinear form

$$(u, w) \mapsto ((u|w)) := \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} | \frac{\partial w}{\partial x_i} \right)_{L^2}$$

is a scalar product in $W_0^{1,2}$ inducing the topology inherited from $W^{1,2}$. It is also true in the case $n = 1$ since then

$$\forall u \in W_0^{1,2} : |\Omega|^{-\frac{1}{2}} \|u\|_{L^2} \leq \|u\|_{L^\infty} \leq \|\dot{u}\|_{L^1} \leq |\Omega|^{\frac{1}{2}} \cdot \|\dot{u}\|_{L^2}.$$

Let $A \subset L^2 \times L^2$ denote the Friedrichs extension of the operator:

$$(L^2 \supset) \ \mathcal{D} \ni \varphi \mapsto -\Delta\varphi \in L^2$$

– see Remark (2.5) in [1].

THEOREM 1.1.

a) *The operator*

$$P : L^2 \ni q \mapsto q - \frac{1}{|\Omega|} \int_\Omega q \, dm \in L^2$$

is the $(\cdot|\cdot)_{L^2}$ – *orthogonal projection onto the hyperplane*

$$\{\int = 0\} := \{q \in L^2 : \int_\Omega q \, dm = 0\}.$$

If $q_1, \ldots, q_N \in W_0^{1,2}$ *are linearly independent,*
then also $P(q_1), \ldots, P(q_N)$ *are linearly independent.*

b) *At the notation* $\eta := A^{-1}(1)$ *we have:*

$$\int_\Omega \eta \, dm = \|\nabla \eta\|_{L^2}^2 > 0$$

and the operator

$$W_0^{1,2} \ni u \mapsto u - \frac{\int_\Omega u \, dm}{\int_\Omega \eta \, dm} \cdot \eta \in W_0^{1,2}$$

is the $((\cdot|\cdot))$*-orthogonal projection onto* $W_0^{1,2} \cap \{\int = 0\}$.

c) *The subspace* $\operatorname{dom} A_0 = \mathcal{D} \cap \{\int = 0\}$ *is* L^2*-dense in* $\{\int = 0\}$ *and* $W^{1,2}$*-dense in* $W_0^{1,2} \cap \{\int = 0\}$.

d) *The restriction* $A_0 = A\big|_{\operatorname{dom} A_0}$ *is a strictly positive operator of the Hilbert space* $\{\int = 0\}$, $\inf A_0 \geq \inf A$.

e) $W_0^{1,2} \cap \{\int = 0\}$ *with the scalar product* $((\cdot|\cdot))$ *is the Fridrichs space of* A_0.

f) *The Friedrichs extension* $\widetilde{A_0}$ *of the operator* A_0 *is defined on the subspace*

$$\operatorname{dom} \widetilde{A_0} = \left\{ u \in W_0^{1,2} \cap \{ \int = 0 \} : \Delta u \in L^2 \right\}$$

(where Δu *is understood in the sense of Sobolev) and*

$$\forall u \in \operatorname{dom} \widetilde{A_0} : \widetilde{A_0}(u) = \frac{1}{|\Omega|} \int_\Omega \Delta u dm - \Delta u.$$

PROOF. It is clear that $P(L^2) \subset \{\int = 0\}$ and $(\text{const}\,.|\{\int = 0\})_{L^2} = 0$. Let $\lambda_1, \ldots, \lambda_N \in \mathbf{R}$ satisfy the condition

$$\sum_{i=1}^{N} \lambda_i P(q_i) = 0 \quad (\text{equality in } L^2).$$

Then $0 = (\sum_i \lambda_i q_i) - c$, where $c := \frac{1}{|\Omega|} \int_\Omega \sum_i \lambda_i q_i dm = \text{const}$. Therefore $\|\nabla c\|_{L^2} = 0$ and simultaneously $c = \sum_i \lambda_i q_i \in W_0^{1,2}$. Consequently c is the zero-vector of the space $W_0^{1,2}$, or in other words:

$$\sum_{i=1}^{N} \lambda_i q_i = 0 \quad (\text{equality in } W_0^{1,2}).$$

Hence $\lambda_1 = \cdots = \lambda_N = 0$, q.e.d.

The function η is a non-zero element of $W_0^{1,2}$ because $A(\eta) \neq 0$. Then $\|\nabla \eta\|_{L^2} > 0$. Clearly

$$\int_\Omega \eta dm = (1|\eta)_{L^2} = (A(\eta)|\eta)_{L^2} = ((\eta|\eta)) = \|\nabla \eta\|_{L^2}^2.$$

For every $q \in W_0^{1,2} \cap \{\int = 0\}$ we have:

$$((\eta|q)) = (A(\eta)|q)_{L^2} = (1|q)_{L^2} = \int_\Omega q dm = 0.$$

Therefore η is $((\cdot|\cdot))$-orthogonal to the subspace $W_0^{1,2} \cap \{\int = 0\}$. This subspace is a hyperplane of $W_0^{1,2}$, as the kernel of the functional

$$W_0^{1,2} \ni u \mapsto \int_\Omega udm \in \mathbf{R}.$$

Hence

$$W_0^{1,2} = \mathbf{R}\eta + (W_0^{1,2} \cap \{\int = 0\}) \quad \text{(orthogonal sum)}.$$

The $((\cdot|\cdot))$-orthogonal projection of an element $u \in W_0^{1,2}$ onto the line $\mathbf{R}\eta$ is equal to:

$$\begin{aligned}((u|\frac{\eta}{\|\nabla\eta\|_{L^2}})) \cdot \frac{\eta}{\|\nabla\eta\|_{L^2}} &= \|\nabla\eta\|_{L^2}^{-2}((\eta|u))\eta = (\int_\Omega \eta dm)^{-1}(A(\eta)|u)_{L^2}\eta \\ &= (\int_\Omega \eta dm)^{-1}(1|u)_{L^2}\eta = (\int_\Omega \eta dm)^{-1}(\int_\Omega udm) \cdot \eta.\end{aligned}$$

Let us choose a function $h \in \mathcal{D}$ such that $\int_\Omega hdm = 1$. For an arbitrarily given function $u \in \{\int = 0\}$ (respectively $u \in W_0^{1,2} \cap \{\int = 0\}$) there is a sequence $(\varphi_\nu) \in \mathcal{D}^{\mathbf{N}}$ convergent to u in L^2 (respectively in $W_0^{1,2}$). In particular $\varphi_\nu \longrightarrow u$ in L^1, thus $\int_\Omega \varphi_\nu\, dm \longrightarrow \int_\Omega u\, dm = 0$. Then

$$\mathcal{D} \cap \{\int = 0\} \ni \varphi_\nu - (\int_\Omega \varphi_\nu dm) \cdot h \longrightarrow u$$

in L^2 (resp. in $W_0^{1,2}$).

The propositions (d), (e), (f) results directly from Remark (2.5) and Theorem(1.16) of [1]. □

2. The case $n = 1$, $\Omega =]0,1[$.

The Sobolev spaces $W^{s,2}$, $W_0^{s,2}$ of one-variable functions are characterized in Examples (2.1), (2.2) in [1].

Let $u \in W_0^{1,2} \cap \{\int = 0\}$ be a non-zero eigenvector of the operator $\widetilde{A_0}$, i.e. there exists a number $\lambda \in \mathbf{R}$ such that $\widetilde{A_0}(u) = \lambda u$. Clearly $\lambda\|u\|_{L^2}^2 = (\widetilde{A_0}(u)|u)_{L^2} \geq (\inf \widetilde{A_0})\|u\|_{L^2}^2$. Hence $\lambda > 0$. The function u satisfies the boundary condition

$$u(0) = u(1) = 0 \tag{2.1}$$

as an element of $W_0^{1,2}$ ($\subset C[0,1]$). By Theorem (1.1)

$$\ddot{u} = \int_0^1 \ddot{u} dm - \lambda u \tag{2.2}$$

($\in W^{2,2}$). Consequently $u \in W^{4,2} \subset C^3[0,1]$ and u satisfies the equation (2.2) in the usual sense. We differentiate both sides of (2.2) and we obtain:

$$\dddot{w} + \lambda w = 0, \tag{2.3}$$

where $w := \dot{u} \in C^2[0,1]$. Therefore $w = \alpha_0 \cos\sqrt{\lambda}x + \beta_0 \sin\sqrt{\lambda}x$ for some $\alpha_0, \beta_0 \in \mathbf{R}$, and

$$\dot{u} = w = \frac{d}{dx}\left(-\frac{\beta_0}{\sqrt{\lambda}}\cos\sqrt{\lambda}x + \frac{\alpha_0}{\sqrt{\lambda}}\sin\sqrt{\lambda}x\right).$$

Hence

$$\exists C \in \mathbf{R}\colon u = \alpha\cos\sqrt{\lambda}x + \beta\sin\sqrt{\lambda}x - C,$$

where $\alpha := -\frac{\beta_0}{\sqrt{\lambda}}$, $\beta := -\frac{\alpha_0}{\sqrt{\lambda}}$. Through (2.1): $C = \alpha$ and obviously

$$u = \alpha\cos\sqrt{\lambda}x + \beta\sin\sqrt{\lambda}x - \alpha. \tag{2.4}$$

By (2.4) and again by (2.1) we get

$$\alpha(1 - \cos\sqrt{\lambda}) = \beta\sin\sqrt{\lambda}. \tag{2.5}$$

Next, integrating both sides of (2.4) over Ω and remembering that $u \in \{\int = 0\}$, we obtain

$$\beta(1 - \cos\sqrt{\lambda}) = \alpha\sqrt{\lambda} - \alpha\sin\sqrt{\lambda}. \tag{2.6}$$

(2.7) A s s u m e that $\beta = 0$.

Then by (2.4) $\alpha \neq 0$ since $u \neq 0$. Hence and by (2.6) $\sqrt{\lambda} = \sin\sqrt{\lambda}$, but $\forall x > 0\colon x > \sin x$. In this way the hypothesis (2.7) has led to a contradiction.

Thereby $\beta \neq 0$. Consider the case

$$\alpha = 0. \tag{2.8}$$

Then in agreement with (2.6) and (2.5): $e^{i\sqrt{\lambda}} = 1$, i.e. $\exists k \in \mathbf{Z}\colon \sqrt{\lambda} = 2\pi k$. In this situation $k \in \mathbf{N} \setminus \{0\}$ and by (2.4)

$$u = \beta \sin 2\pi k x. \tag{2.9}$$

The case that remains is:

$$\alpha \neq 0. \tag{2.10}$$

This time we multiply (2.5) and (2.6)) double-sidedly by α and β, respectively, and we compare the obtained equalities:

$$\cos\sqrt{\lambda} = 1 - \sqrt{\lambda}\cdot\frac{\frac{\beta}{\alpha}}{1+(\frac{\beta}{\alpha})^2}. \tag{2.11}$$

Next we multiply (2.5) and (2.6) double-sidedly by β and $-\alpha$, respectively, and after the comparison of the obtained equalities we get:

$$\sin\sqrt{\lambda} = \sqrt{\lambda}\cdot\frac{1}{1+(\frac{\beta}{\alpha})^2}. \tag{2.12}$$

By (2.11), (2.12) we have

$$1 = \cos^2\sqrt{\lambda} + \sin^2\sqrt{\lambda} = 1 + \frac{\sqrt{\lambda}}{1+(\frac{\beta}{\alpha})^2}(\sqrt{\lambda} - 2\cdot\frac{\beta}{\alpha}).$$

Hence $\frac{\beta}{\alpha} = \frac{1}{2}\sqrt{\lambda}$ and coming back to (2.11), (2.12) we obtain:

$$\cos\theta = \frac{1-(\frac{1}{2}\theta)^2}{1+(\frac{1}{2}\theta)^2}, \quad \sin\theta = \frac{\theta}{1+(\frac{1}{2}\theta)^2}, \tag{2.13}$$

where $\theta := \sqrt{\lambda}$, or in the equivalent form:

$$e^{i\theta} = \frac{i-\frac{1}{2}\theta}{i+\frac{1}{2}\theta}. \tag{2.14}$$

Finally by (2.4)

$$u = \alpha(\cos\theta x + \frac{\beta}{\alpha}\sin\theta x - 1) = \alpha\cdot u_\theta, \tag{2.15}$$

where

$$u_\theta := \cos\theta x + \frac{1}{2}\theta\sin\theta x - 1. \tag{2.16}$$

In the cases (2.8), (2.10) the relations (2.9) and (2.15), respectively, appeared and we received appropriate formulas for λ.

COROLLARY 2.17. *If u is an eigenvector of the operator $\widetilde{A_0}$, then*

$$\exists k \in \mathrm{N} \setminus \{0\} : u \in \mathbf{R} \cdot \sin 2\pi k x$$

or

$$\exists \theta > 0 : \theta \text{ satisfies } (2.14) \text{ and } u \in \mathbf{R} \cdot u_\theta$$

(see (2.16)). Furthemore, the point spectrum $\sigma_p(\widetilde{A_0})$ of the operator $\widetilde{A_0}$ is contained in the union of the disjoint sets:

$$\{(2\pi k)^2 : 0 \neq k \in \mathrm{N}\}, \quad \{\theta^2 : \theta \in \mathcal{V}\},$$

where $\mathcal{V}$ means the class of all non-zero real roots of the equation (2.14).

Conversely, for every $k \in \mathrm{N} \setminus \{0\}$ the function $\sin 2\pi k x$ is an eigenvector of the operator $\widetilde{A_0}$ corresponding to the eigenvalue $(2\pi k)^2$. Let us consider the number $\theta \in \mathcal{V}$. Then θ satisfies (2.13) and, consequently, the introduced in (2.16) function u_θ belongs to $W_0^{1,2} \cap \{\int = 0\}$. Moreover,

$$\int_0^1 (u_\theta)'' dm = (u_\theta)'(1) - (u_\theta)'(0) = -\theta^2. \tag{2.18}$$

Finally, by Theorem 1.1 and the formula (2.18), we have:

$$\widetilde{A_0}(u_\theta) = \theta^2 \cdot u_\theta.$$

Thus u_θ a is an eigenvector of the operator $\widetilde{A_0}$ corresponding to the eigenvalue θ^2. These facts together with Corollary 2.17 give

COROLLARY 2.19. $\sigma_p(\widetilde{A_0}) = \{(2\pi k)^2 : 0 \neq k \in \mathrm{N}\} \cup \{\theta^2 : \theta \in \mathcal{V}\}$,

$$\forall (k, \theta) \in (\mathrm{N} \setminus \{0\}) \times \mathcal{V} : \ker\left(\widetilde{A_0} - (2\pi k)^2 \,\mathrm{id}\right) = \mathbf{R} \cdot \sin 2\pi k x x,$$
$$\ker\left(\widetilde{A_0} - \theta^2 \,\mathrm{id}\right) = \mathbf{R} \cdot u_\theta.$$

Of course $\mathcal{V}$ is symmetric with regard to zero. Moreover,

REMARK 2.20. *The set $\mathcal{V}$ is countable and has no accumulation points.*

PROOF. By (2.18): $||u_\theta||_{L^2} > 0$, $\forall \theta \in \mathcal{V}$. By the Arzela – Ascoli theorem, the inclusion: $W_0^{1,2} \hookrightarrow L^2$ is completely continuous. Clearly the inclusion: $W_0^{1,2} \cap \{\int = 0\} \hookrightarrow \{\int = 0\}$ has the same property.

In agreement with Theorem 1.25 from [1] and Corollary 2.19,

(2.21) $\sigma_p(\widetilde{A}_0)$ has no accumulation points on $\mathbf{R}$

and

(2.22) all the functions

$$\varphi_k := \sin 2\pi kx / \|\sin 2\pi kx\|_{L^2}, \quad k \in \mathbf{N} \setminus \{0\},$$
$$\psi_\theta := u_\theta / \|u_\theta\|_{L^2}, \quad \theta \in \mathcal{V} \cap]0, \infty[,$$

form orthonormal basis in $\{\int = 0\}$.

Immediately from (2.19) and (2.21) we obtain:

$\mathcal{V}$ has no accumulation points on the line $\mathbf{R}$.

It remains to prove that

(2.23) $\mathcal{V}$ is infinite.

(For the time being we do not know if $\mathcal{V}$ is non-empty!) The function $f := (\sin \pi x) - \frac{2}{\pi}$ is L^2-orthogonal to every φ_k, $k \in \mathbf{N} \setminus \{0\}$. Simultaneously, $f \in \{\int = 0\}$, then in view of (2.22):

$$L^2 \setminus \{0\} \ni f = \sum_{\theta \in \mathcal{V}^+} (f|\psi_\theta)_{L^2} \cdot \psi_\theta$$

(the series convergent in L^2),
where $\mathcal{V}^+ := \mathcal{V} \cap]0, \infty[$. Hence, of course, $\mathcal{V} \neq \emptyset$. The function f is not a finite linear combination of the functions $\{\psi_\theta\}$ because $f(0) \neq 0$ and every ψ_θ satisfies (2.1). Thereby (2.23) holds. □

THEOREM 2.24. *The set $\mathcal{V}$ of all non-zero real solutions of the equation (2.14) is identical with the class of all real roots of the equation*

(2.25)
$$\sum_{k \in 1+2\mathbf{N}} \left(1 + \frac{i}{k^2}\right) \cdot \frac{1}{(k\pi)^2 - \theta^2} = \frac{1}{8}.$$

PROOF. Obviously $\theta = 0$ is not a solution of (2.25). However, it satisfies the equation "Re (2.25)", i.e.

(2.26)
$$\sum_{k \in 1+2\mathbf{N}} \frac{1}{(k\pi)^2} = \frac{1}{8}'$$

since, in agreement with Example (2.25) from [1],

(2.27) the family $\{\sqrt{2}\cdot\sin k\pi x : 0 \neq k \in \mathrm{N}\}$ is an orthonormal basis in L^2 composed of eigenvectors of the operato A;

therefore

$$(2.28)\qquad (L^2 \ni)\ 1 = \sum_{k=1}^{\infty}(1|\sqrt{2}\sin k\pi x)_{L^2}\sqrt{2}\sin k\pi x = \sum_{k\in 1+2\mathrm{N}} \frac{4}{k\pi}\sin k\pi x$$

(the series convergent in L^2);
after integration of both sides of the identity (2.28) over Ω we obtain just (2.26).

Suppose that $\theta \in \mathcal{V}$. Since none of odd multiples of the number π satiefies the equation (2.14), then

$$(2.29)\qquad \theta \notin (1+2\mathrm{Z})\pi.$$

For an arbitrary $k \in 2\mathrm{N}$, by Corollary 2.19, the fuctions u_θ, $\sin k\pi x$ are two eigenvectors of the operator $\widetilde{A}_0$ corresnonding to different eigenvalues; by Lemma 1.15–d from [1], they are L^2-orthogotonal. Thereby

$$(2.30)\qquad (u_\theta|\sin k\pi x)_{L^2} = 0, \quad \forall k \in 2\mathrm{N}.$$

Next, using (2.13), we calculate:

$$(2.31)\qquad (u_\theta|\sin k\pi x)_{L^2} = \frac{2\theta^2}{k\pi((k\pi)^2-\theta^2)}, \quad \forall k \in 1+2\mathrm{N}.$$

Now, by (2.27), (2.30) and (2.31), we obtain:

$$(2.32)\qquad \begin{aligned} u_\theta &= \sum_{k=1}^{\infty}(u_\theta|\sqrt{2}\sin k\pi x)_{L^2}\sqrt{2}\sin k\pi x \\ &= 2\sum_{k\in 1+2\mathrm{N}} \frac{2\theta^2}{k\pi((k\pi)^2-\theta^2)}\sin k\pi x \\ &= 4\theta^2 \sum_{k\in 1+2\mathrm{N}} \frac{1}{k\pi}\cdot\frac{1}{(k\pi)^2-\theta^2}\cdot\sin k\pi x \end{aligned}$$

(the series convert in L^2).

We integrate both sides of the identity (2.32) over Ω and we get the imaginary part of the equation (2.25):

$$0 = 8(\frac{\theta}{\pi})^2 \sum_{k\in 1+2\mathbf{N}} \frac{1}{k^2} \cdot \frac{1}{(k\pi)^2 - \theta^2}. \tag{2.33}$$

$u_\theta \in \operatorname{dom} A$, therefore, by (2.27), the series on the right-hand side of (2.32) is convergent also in $\operatorname{dom} A$ (see Theorem 1.30 in [1]) and, consequently,

$$\begin{aligned} -(u_theta)'' = A(u_\theta) &= 4\theta^2 \sum_{k\in 1+2\mathbf{N}} \frac{1}{k\pi} \cdot \frac{1}{(k\pi)^2 - \theta^2} \cdot A(\sin k\pi x) \\ &= 4\theta^2 \sum_{k\in 1+2\mathbf{N}} \frac{1}{k\pi} \cdot \frac{1}{(k\pi)^2 - \theta^2} \cdot (k\pi)^2 \sin k\pi x \end{aligned} \tag{2.34}$$

(the series convergent in L^2).
We integrate both sides of (2.34) over Ω and, remembering (2.18), we obtain the real part of the equation (2.25):

$$\theta^2 = 8\theta^2 \sum_{k\in 1+2\mathbf{N}} \frac{1}{(k\pi)^2 - \theta^2}. \tag{2.35}$$

(Compare with (2.26).) Clearly, the system $\{(2.35), (2.33)\}$ is equivalent to (2.25).

Conversely, let a number $\theta \in \mathbf{R}$ be a solution of the equation (2.25). Then the condition (2.29) must, of course, be satisfied. The family

$$1 + 2\mathbf{N} \ni k \mapsto \frac{1}{k\pi} \cdot \frac{1}{(k\pi)^2 - \theta^2} \cdot \sin k\pi x \in \operatorname{dom} A$$

is summable because it is orthogonal and

$$\begin{aligned} &\sum_{k\in 1+2\mathbf{N}} \left| \frac{1}{k\pi} \frac{1}{(k\pi)^2 - \theta^2} \sin k\pi x \right|^2_{\operatorname{dom} A} \\ &= \sum_{k\in 1+2\mathbf{N}} \left\| \frac{1}{k\pi} \frac{1}{(k\pi)^2 - \theta^2} A(\sin k\pi x) \right\|^2_{L^2} \\ &= \sum_{k\in 1+2\mathbf{N}} \left\| \frac{1}{\sqrt{2}} \frac{k\pi}{(k\pi)^2 - \theta^2} \sqrt{2}(\sin k\pi x \right\|^2_{L^2} \\ &= \sum_{k\in 1+2\mathbf{N}} \left(\frac{1}{\sqrt{2}} \frac{k\pi}{(k\pi)^2 - \theta^2} \right)^2 < \infty \end{aligned}$$

(see Lemma(1.20) in [1]). Therefore we are allowed to put

$$(\operatorname{dom} A \ni)\quad w := 4\theta^2 \sum_{k\in 1+2\mathrm{N}} \frac{1}{k\pi}\frac{1}{(k\pi)^2-\theta^2}\sin k\pi x$$

(the series convergent in dom A)
- compare with (2.32). Using (2.33) we obtain: $w \in \{\int = 0\}$. Thus $w \in \{\int = 0\} \cap \operatorname{dom} A \subset \operatorname{dom} \widetilde{A}_0$ (see Theorem 1.1) and, remembering (2.35) and (2.28), we may calculate:

(2.36)

$$\begin{aligned}
\widetilde{A}_0(w) &= \int_0^1 \ddot{w}dm - \ddot{w} = A(w) - \int_0^1 A(w)dm \\
&=4\theta^2 \sum_{k\in 1+2\mathrm{N}} \frac{1}{k\pi}\frac{1}{(k\pi)^2-\theta^2} A(\sin k\pi x) \\
&\quad - 4\theta^2 \sum_{k\in 1+2\mathrm{N}} \frac{1}{k\pi}\frac{1}{(k\pi)^2-\theta^2}\cdot \int_0^1 A(\sin k\pi x)dm \\
&=4\theta^2 \sum_{k\in 1+2\mathrm{N}} \frac{k\pi}{(k\pi)^2-\theta^2}\sin k\pi x - 8\theta^2 \sum_{k\in 1+2\mathrm{N}} \frac{1}{(k\pi)^2-\theta^2} \\
&=4\theta^2 \sum_{k\in 1+2\mathrm{N}} \frac{k\pi}{(k\pi)^2-\theta^2}\sin k\pi x - \theta^2 \\
&=4\theta^2 \sum_{k\in 1+2\mathrm{N}} \frac{k\pi}{(k\pi)^2-\theta^2}\sin k\pi x - \theta^2 \cdot \sum_{k\in 1+2\mathrm{N}} \frac{4}{k\pi}\sin k\pi x = \theta^2 w.
\end{aligned}$$

At the opportunity we have calculated:

$$\int_\Omega A(w)dm = \theta^2; \tag{2.37}$$

in particular $||w||_{L^2} > 0$. This fact together with the following relation

$$\widetilde{A}_0(w) = \theta^2 w \tag{2.38}$$

obtained from (2.36), spells that θ^2 is an eigenvalue of the operator $\widetilde{A}_0$. By Corollary 2.19,

$$\theta^2 \in \{(2\pi k)^2 : 0 \neq k \in \mathrm{N}\} \cup \{\zeta^2 : \zeta \in \mathcal{V}\}. \tag{2.39}$$

(2.40) Assume that $\theta^2 = (2\pi k)^2$ for some $k \in \mathbf{N} \setminus \{0\}$.

Then, in agreement with (2.38) and Corollary 2.19,

$$w \in \ker(\widetilde{A}_0 - (2\pi k)^2 \,\mathrm{id}) = \mathbf{R} \cdot sin 2\pi kx.$$

In particular, $A(w) \in \mathbf{R} \cdot A(\sin 2\pi kx) = \mathbf{R} \cdot \sin 2\pi kx \subset \{\int = 0\}$, but it is impossible in view of (2.37). In this manner the hypothesis (2.40) has led to a contradiction.

Thereby $\theta^2 \notin \{(2\pi k)^2 : 0 \neq k \in \mathbf{N}\}$ and, coming back to (2.39), we get:

$$\theta^2 \in \{\zeta^2 : \zeta \in \mathcal{V}\}.$$

Thus $\theta \in \mathcal{V}$ since $\mathcal{V} = -\mathcal{V}$.

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