UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXVI

1987

Intersections of Trajectories and Transversals in Dynamical Systems

by Krzysztof Ciesielski

Introduction. The purpose of this paper is to present some properties of dynamical systems in so-called dichotomic manifolds. In fact, we will investigate these systems in dependence upon the properties of intersections of the trajectories and transversals (see Hajek [4]). We define three types of such systems; those in which there exist a transversal and a trajectory having infinite number of intersection points (we say that such systems satisfy the condition (TR_∞)); those in which every transversals and trajectories intersect in at most one point (satisfying (TR_0)) and those which do not fulfil (TR_∞) and in which there exist a trajectory and a transversal which intersect more than once (satisfying (TR_2)).

First we consider systems in the plane. We show that some results proved in [5] and [3] for closed dynamical systems hold for those which fulfil (TR_0) or (TR_2). It is also proved that such systems are locally closed in small neighbourhoods of periodic trajectories.

Next we characterize systems satisfying the condition (TR_0), using transversals homeomorphic to the circle. It is possible to construct such transversal if and only if the dynamical system does not satisfy (TR_0).

The last part of the paper describes the systems fulfilling (TR_2). In particular, a theorem is proved which shows that for any such system and any natural number n there exist a transversal and a trajectory with exactly n intersection points.

I. Throughout this paper R denotes the set of real numbers, R^n the Euclidean n-space,

$$I = \{x \in R^n\colon\ 0 \leqslant x \leqslant 1\},\quad S^n = \{x \in R^{n+1}\colon\ ||x|| = 1\} \text{ and } B^n = \{x \in R^n\colon\ ||x|| \leqslant 1\}.$$

Let us put $B(A, \varepsilon) = \{x \in R^n\colon \inf\{||x-y||, y \in A\} < \varepsilon\}$; $||\cdot||$ denotes the Euclidean norm. For a given set $A \subset R^n$, $\bar{A}$ denotes the closure of A. By a simple arc or a Jordan curve we mean a subset C of a metric space X homeomorphic to I or S^1, respectively.

An orientable 2-manifold X is termed *dichotomic* if every Jordan curve C decomposes X into two open connected sets D_1 and D_2 with a common boundary C. By Jordan Theorem, R^2 is a dichotomic 2-manifold; in this case we denote a bounded component of $R^2 \setminus C$ by $\operatorname{In} C$.

Throughout this paper M will denote a given dichotomic 2-manifold.

By (X, π) we will denote the *dynamical system*, i.e. a metric space X and a continuous map $\pi\colon R\times X \to X$ satisfying two axioms: $\pi(0, x) = x$ and $\pi(t, \pi(s, x)) = \pi(t+s, x)$ for any $x \in X$ and $t, s \in R$. The set X is called the *phase space*. If $X = R^2$, then the dynamical system is called the *planar system*.

For every $x \in X$ we define:

$\pi^+(x) = \pi(R_+ \times \{x\})$

$L^+(x) = \{y \in X$: there exists sequence $\{t_n\} \subset R$ such that $t_n \to \infty$, $\pi(t_n, x) \to y\}$.

By obvious modifications of the above we get symmetric sets $\pi^-(x)$ and $L^-(x)$. We define the *trajectory* through x: $\pi(x) = \pi^+(x) \cup \pi^-(x)$ and the *limit set* of x: $L(x) = L^+(x) \cup L^-(x)$ (see [2], [4], [8], [9]).

For a given dynamical system (X, π) we define:

(1) The *positive region of attraction* for a set $D \subset X$ as

$$A^+(D) = \{y \in X\colon \emptyset \neq L^+(y) \subset D\}.$$

(2) The set $D \subset X$ is called *invariant*, if $\pi(x) \subset D$ for every $x \in D$.
The set D is called *positively (negatively) invariant*, if $\pi^+(x) \subset D$ $(\pi^-(x) \subset D)$ for every $x \in D$.

(3) A point $x \in X$ is called *critical*, if $\pi(x) = \{x\}$. The set of critical points of (X, π) is denoted by $S(\pi)$.

(4) A point $x \in X$ is called *periodic*, if there exists $t \neq 0$ such that $\pi(t, x) = x$ and $\pi(x) \neq \{x\}$. The set of periodic points of (X, π) is denoted by $P(\pi)$.

(5) We put $T(\pi) = X\setminus(S(\pi) \cup P(\pi))$ and we call *regular* every point belonging to $T(\pi)$.

(6) A subset D of X is said to be *positively (negatively) stable* if every neighbourhood U of M contains a positively (negatively) invariant neighbourhood of D.

(7) The set $D \subset X$ is said to be *asymptotically positively (negatively) stable* if it is positively (negatively) stable and $A^+(D)(A^-(D))$ is a neighbourhood of D.

(Remark: the stability defined above does not coincide in general with the so-called Lyapunov stability — in the sense of [2], [8], [9].)

(8) If X is a 2-manifold, then a simple arc or a Jordan curve C is called a *transversal* if there exists $\varepsilon > 0$ such that $\pi(t_1, C) \cap \pi(t_2, C) = \emptyset$ for every $0 \leqslant t_1 < t_2 \leqslant \varepsilon$.

A homeomorphism from $I(S^1)$ onto C is called a *parametrization* of C. The number ε from the definition is termed the t-extend of the transversal.

Remark: we do not require the transversal C not to be tangent to any trajectory intersecting C.

Now we will remind some propositions (compare [1], [2], [4], [8], [9]).

Assume that a dynamical system (X, π) is given.

1.1. Schönflies THEOREM. *Every homeomorphism from S^1 into R^2 can be extended to a homeomorphism from R^2 onto itself.*

1.2. PROPOSITION. (V. 3.8. in [2]). *If $X = R^n$ and M is a compact positively invariant subset of X homeomorphic to B^n, then M contains a critical point.*

1.3. **Proposition.** (VI. 1.2. in [4]). *Let X be a locally compact space, M a compact invariant subset of X. Then at least one of the conditions* (C. 1)—(C. 4) *holds.*

(C. 1) *M is positively asymptotically stable*

(C. 2) *M is negatively asymptotically stable*

(C. 3) *There exist $x \notin M$, $y \notin M$ such that $\emptyset \neq L^+(x) \subset M$ and $\emptyset \neq L^-(y) \subset M$*

(C. 4) *Every neighbourhood U of M contains an $x \notin M$ with $\pi(x) \subset U$.*

1.4. **Proposition.** (VII. 2.6. in [4]). *Assume that X is a 2-manifold. Then:*

(i) *for every $x \in X \setminus S$ there exists a transversal homeomorphic to I which contains x as a non-end point*

(ii) *if C is a transversal homeomorphic to I, then for sufficiently small ε the set $\{\bigcup \pi(t, x)\colon -\varepsilon < t < \varepsilon,\ x \in C\}$ is a neighbourhood of all non-end points of C.*

1.5. **Proposition.** (VII. 1.2. in [4]). *Let X be a dichotomic 2-manifold and assume that $x, y \in M$, C is a transversal containing x as a non-end point and $\{t_n\}$ is a net of real numbers. If $t_n \to \infty$ and $\pi(t_n, y) \to x$, then there exists a net $\{s_n\}$ of real numbers such that $s_n \to \infty$, $t_n - s_n \to 0$, $\pi(s_n, y) \to x$ and $\pi(s_n, y) \in C$ for every n.*

1.6. **Proposition.** (VII. 4.4.) in [4]). *Let X be a dichotomic manifold, C a transversal and $x \in X$. If C or $\pi(x)$ is a Jordan curve, then they have one intersection point at most; if both are Jordan curves, then they do not intersect.*

1.7. **Proposition.** (VII. 4.7. in [4]). *Let X be a dichotomic manifold, C a transversal and $x \in X$, $t_1, t_2 \in R$, $t_1 < t_2$. Assume that $C_1 = C \cup \{\pi(t, x)\colon t_1 < t < t_2\}$ is a Jordan curve. Then one component of $X \setminus C_1$ is positively invariant and the other negatively invariant.*

1.8. **Proposition.** (VII. 4.10. in [4]). *Let X be a dichotomic manifold, C a transversal and p a parametrization of C. Assume also that $p(\sigma_k) = \pi(t_k, x)$, $(x \in X,\ k = 1, 2, 3,\ t_1 < t_2 < t_3)$. Then C is not a Jordan curve and either $\sigma_1 = \sigma_2 = \sigma_3$, or $\sigma_1 < \sigma_2 < \sigma_3$, or $\sigma_3 < \sigma_2 < \sigma_1$.*

1.9. **Proposition.** (VIII. 4.1. and VIII. 4.5. in [4]). *Let X be a dichotomic manifold, $x \in X$ be a critical point and $\{x\}$ be positively stable. Assume also that $\{x\}$ is an isolated subset of $S(\pi)$ and that there exists $\varepsilon > 0$ such that every neighbourhood of x bounded by a transversal is not contained in $B(\{x\}, \varepsilon)$. Then x is a centre, i.e. there exists a neighbourhood U of x such that $U \subset \{x\} \cup P(\pi)$. Moreover, $\{x\}$ is negatively stable, but it is neither positively nor negatively asymptotically stable.*

Now we will give some definitions of certain classes of dynamical systems. Assume that there is given a dynamical system (M, π) on a dichotomic manifold M such that $M \neq S(\pi)$, i.e. there exists a non-critical point of this system.

1.10. **Definition.** The dynamical system is called *closed* if $\pi(x) = \overline{\pi(x)}$ for every $x \in M$.

1.11. DEFINITION. The system (M, π) satisfies the condition (TR_0) if every trajectory $\pi(x)$ and every transversal C have one intersection point at most.

1.12. DEFINITION. The system (M, π) satisfies the condition (TR_∞) if there exist $x \in M$ and a transversal C such that the set $\pi(x) \cap C$ is infinite.

1.13. DEFINITION. The system (M, π) satisfies the condition (TR_2) if it satisfies neither (TR_0) nor (TR_∞).

1.14. DEFINITION. The system (M, π) satisfies the condition (TR) if it does not satisfy the condition (TR_∞).

1.15. Remark. The above conditions suggest some further definitions. One can define the (TR_n) systems for every natural n $(n \geqslant 2)$ by induction. The system would be supposed to satisfy the condition (TR_n) if it did not satisfy conditions (TR_0), (TR_2), ..., (TR_{n-1}), (TR_∞) and there existed $x \in M$ and transversal C such that $\pi(x) \cap C$ had n points. The last section of this paper shows that definitions of (TR_n) in this or any other way are unnecessary.

1.16. Remark. It is obvious that the definitions 1.11—1.13 classify the dynamical systems on dichotomic manifolds into four disjoint classes; the fourth class is formed by the dynamical systems containing only critical points.

1.17. PROPOSITION. (Lemma 2 in [3]). *Let M be a dichotomic manifold, (M, π) a dynamical system. If $x \in M$ and C is a transversal, then every point of the set $C \cap \pi(x)$ is isolated in C with the induced topology on C.*

1.18. PROPOSITION. (Theorem 2 in [3]). *The closed planar dynamical system satisfies the condition* (TR_0).

II. We recall that throughout this paper M denotes a fixed dichotomic 2-manifold.

2.1. THEOREM. *If (M, π) does not satisfy* (TR_∞) *(in particular if (M, π) satisfies* (TR_0)*), then $L(y) \subset S(\pi)$ for every regular point $y \in M$.*

Proof. If $T(\pi) = \emptyset$ the theorem is obvious. Assume that $y \in T(\pi)$. We will show that $L^+(y) \subset S(\pi)$ (for $L^-(y)$ the proof is the same).

Suppose that there exists $x \in L^+(y) \setminus S(\pi)$. Then there exists $\{t_n\}$, $t_n \to \infty$ such that $\pi(t_n, y) \to x$ and — by 1.4 — there exists a transversal C containing x as a non-end point. Using 1.5 we can find a net $\{s_n\}$ convergent to ∞ such that $\pi(s_n, y) \in C$ for every n. But in this case the set $\pi(y) \cap C$ is infinite and (M, π) satisfies (TR_∞). This contradiction finishes the proof.

2.2. THEOREM. *If for every non-periodic point x of the dynamical system (M, π) we have $L(x) \subset S(\pi)$, then (M, π) does not satisfy* (TR_∞), *i.e. either it satisfies* (TR) *or it consists of critical points.*

Proof. (communicated to the author by Andrzej Trzepizur).

Suppose that there exist a point $x \in M$ and a transversal C such that $\pi(x) \cap C$ is infinite. By 1.6 x is regular. Take the set of all distinct points $\{\pi(t_n, x)\} \subset C$. Because of the compactness of C we can assume that $\pi(t_n, x) \to y \in C$. Suppose that there exists a subnet $\{t_{n_k}\}$ convergent to t_0. Then $y = \pi(t_0, x) \in C$ and $y \in C \cap \pi(x)$; thus y is not isolated in C, which contradicts Proposition 1.17. This shows that $y \in L(x)$ and so, by the assumptions, $y \in S(\pi)$. But this is impossible, because every transversal is disjoint with $S(\pi)$.

2.3. COROLLARY. *The dynamical system on the dichotomic manifold satisfies* (TR_∞) *if and only if there exists a non-critical point belonging to the limit set of some regular point.*

Now we will state the main theorem of this section.

2.4. THEOREM. *Let (R^2, π) be a dynamical system satisfying condition* (TR) *and let x be a periodic point of this system. Then for every $\varepsilon_0 > 0$ there exists an open, invariant neighbourhood U of $\pi(x)$, such that:*

(i) $U \subset P(\pi)$

(ii) *for every $y \in U \setminus \pi(x)$ we have either $\pi(y) \subset \operatorname{In} \pi(x)$ or $\pi(x) \subset \operatorname{In} \pi(y)$*

(iii) $U \subset B(\pi(x), \varepsilon_0)$

Proof. Take any $\varepsilon_0 > 0$. Then using Theorem 2.1 we get $L(y) \cap \pi(x) = \emptyset$ for every $y \notin \pi(x)$. For every point $y \notin T(\pi)$, $L(y) = \pi(y)$. From Schönflies Theorem it follows that there exists a homeomorphism f: $R^2 \to R^2$ such that $f(\pi(x)) = S^1$. Denote by $\tilde{\pi}$ the new system $f \circ \pi$; so we will consider $(R^2, \tilde{\pi})$. It is obvious, because of the elementary properties of homeomorphism, that $L(y) \cap S^1 = \emptyset$ for every $y \notin S^1$. The set $S(\tilde{\pi})$ is closed, so there exists $\varepsilon > 0$ such that $B(S^1, \varepsilon) \cap S(\tilde{\pi}) = \emptyset$ and $f^{-1}(B(S^1, \varepsilon)) \subset B(\pi(x), \varepsilon_0)$. Consider the restrictions of the system $(R^2, \tilde{\pi})$ to the invariant phase spaces $\overline{\operatorname{In} S^1}$ and $R^2 \setminus \operatorname{In} S^1$. Since $L(y) \cap S^1 = \emptyset$ for every $y \notin S^1$, we obtain (using Proposition 1.3) that in both cases the condition (C.4) from Proposition 1.3 holds. Applying it to $B(S^1, \varepsilon)$ we can find $y_1 \in B(S^1, \varepsilon) \cap \operatorname{In} S^1$ and $y_2 \in B(S^1, \varepsilon) \setminus \overline{\operatorname{In} S^1}$ with properties: $\pi(y)_1 \subset \operatorname{In} S^1 \cap B(S^1, \varepsilon)$ and $\pi(y_2) \subset B(S^1, \varepsilon) \setminus \overline{\operatorname{In} S^1}$. Then we have $B(0, 1-\varepsilon) \subset \operatorname{In} \tilde{\pi}(y_1)$ and $S^1 \subset \operatorname{In} \tilde{\pi}(y_2)$; it follows from Proposition 1.2 and the elementary properties of Jordan curves. The set $U_1 = \overline{\operatorname{In} \tilde{\pi}(y_2)} \setminus \operatorname{In} \tilde{\pi}(y_1)$ is compact and invariant. Suppose that there exists $z \in U_1 \cap T(\pi)$; then $\pi(z) \subset U_1$ and (by compactness) $L(z) \neq \emptyset$, so $U_1 \cap S \neq \emptyset$ which gives a contradiction. Now define $\tilde{U}$ as $U_1 \setminus [\pi(y_1) \cup \pi(y_2)]$. If any periodic trajectory (except S^1) did not fulfil the condition (ii) we would again a contradiction with Proposition 1.2. Now define U as $f^{-1}(\tilde{U})$. It is obvious that U fulfils all the required properties.

2.5. COROLLARY. *Assume that x is a periodic point of the dynamical system (R^2, π) and that this system satisfies condition* (TR). *Then $\pi(x)$ is positively and negatively stable and*

$P(\pi)$ is open. Moreover, there exists an invariant neighbourhood U of $\pi(x)$ and a closed dynamical system (R^2, π') such that the restriction π' to $R \times U$ coincides with the restriction π to the same set.

Proof. The two first properties are obvious corollaries of the previous theorem. To show the last property take points y_1 and y_2 in the same way as in the proof of Theorem 2.4. Thus $\pi(x) \subset \operatorname{In} \pi(y_2)$ and $\pi(y_1) \subset \operatorname{In} \pi(x)$. Define U as $\operatorname{In} \pi(y_2) \setminus \overline{\operatorname{In} \pi(y_1)}$; we have $U \subset P(\pi)$. By Schönflies Theorem there exists a homeomorphism $f_1 \colon R^2 \to R^2$ satisfying $f_1(\pi(y_1)) = S^1$. It is easy to show that $f(\operatorname{In} \pi(y_1)) = \operatorname{In} S^1$. Then we get a dynamical system (R^2, π_1).

Construct the new dynamical system (R^2, π_2), where

$$\pi_2(t, z) = \begin{cases} \pi_1(t, z), & z \notin \overline{\operatorname{In} S^1} \\ (r, \alpha) & (r, \varphi) = z \in \overline{\operatorname{In} S^1} \text{ (polar coordinates)} \end{cases}$$

and α is such that $\pi_1(t, (1, \varphi)) = (1, \alpha)$.

Now map R^2 onto R^2 by f_1^{-1}; all the trajectories of the resulting system in $U \cup \operatorname{In} \pi(y_1)$ are closed. In the same way transform the last system by the homeomorphism f_2 fulfilling $f_2(\pi(y_2)) = S^1$. After changing this system in the set $R^2 \setminus \operatorname{In} S^1$ by the above method and mapping by f_2^{-1} we get a closed planar dynamical system (R^2, π'). Of course the restrictions π and π' onto $R \times U$ are identical.

2.6. Remark. The above theorem and corollary characterize very well the behaviour of dynamical systems satisfying (TR) in neighbourhoods of periodic trajectory. It is interesting and surprising, because the condition (TR) concerns only regular trajectories. In fact, by Proposition 1.6 the periodic trajectory intersects any transversal in one point at most, so the unique trajectories which can make the system not fulfil (TR) are the regular trajectories. Yet the above shows that (TR) describes precisely not regular, but the periodic trajectories.

2.7. Remark. The properties shown in Theorem 2.4 and Corollary 2.5 were proved in [3] and [5] for closed dynamical systems, which, in particular, satisfy condition (TR) (compare 1.18). Thus the above is a generalization of the mentioned results.

2.8. Remark. By Corollary 2.5 every periodic trajectory in system fulfilling (TR), and in particular (TR_0), has an invariant neighbourhood which can be considered as a subset of a closed planar system. It is natural to ask whether every non-critical trajectory of such system has this property.

Of course if we investigate the phase space $R^2 \setminus S(\pi)$, then Theorem 2.1 implies that such a system is closed. Moreover, $R^2 \setminus S(\pi)$ is open. But interesting problems arise if for given regular trajectory there exists an invariant neighbourhood U, in which our system is closed, such that U is homeomorphic to R^2 or such that we can extend the system from U to R^2 obtaining a closed planar system.

The following example shows that the answers to the above questions are negative.

2.9. Example. Consider the planar dynamical system whose trajectories are shown in Figure 2.1.

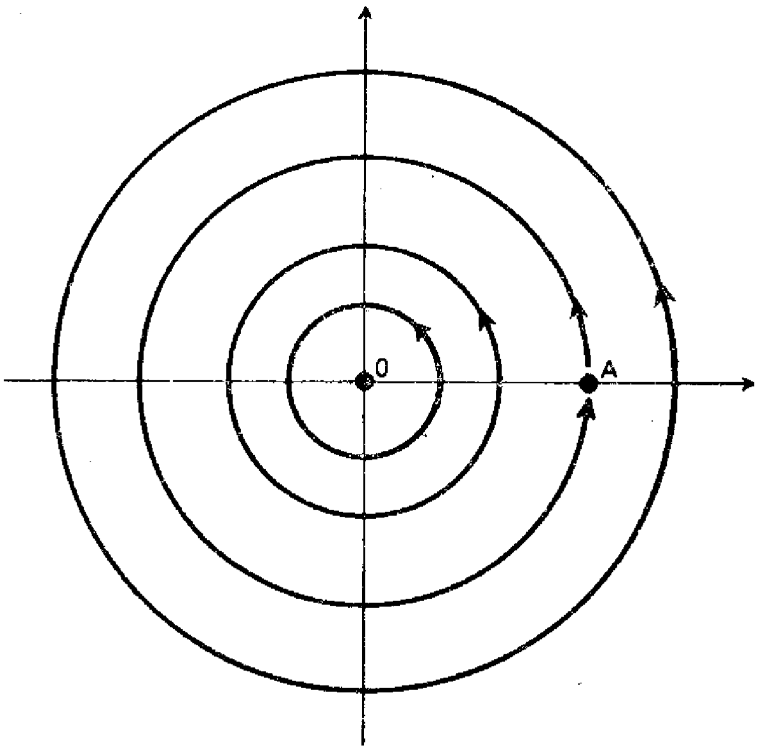

Figure 2.1

This system has two critical points 0 and A ($0 = (0, 0)$, $A = (1, 0)$ in cartesian coordinates) and one regular trajectory (contained in $S^1\setminus\{A\}$). The remaining trajectories are periodic; they are concentric circles with centres in 0. It is easy to show that such a system exists; for instance, it can be defined by the solutions of the differential system (in polar coordinates):

$$\begin{cases} \dfrac{dr}{dt} = 0 \\ \dfrac{d\varphi}{dt} = \sin^2\left(\dfrac{\varphi}{2}\right) + e^{-r}\cdot(r-1)^2\cdot r \end{cases}$$

Proposition 1.6 and Theorem 3.2 (which will be shown in the next section) yield that our system satisfies condition (TR_0). But it is easy to prove that any restriction of it to an invariant, connected and homotopically one-connected neighbourhood U of $\pi(A)$ is not closed. Then either there does not exist an invariant neighbourhood U with closed trajectories such that U is homeomorphic to R^2 or the system can be extended to a planar system without changing U, i.e. $\pi(A)$ cannot be considered as a trajectory in a closed dynamical planar system.

The example shows also that the dynamical systems satisfying (TR_0) form in reality the class larger than that formed by the closed systems.

III. In this section we will prove a theorem about the connection between condition (TR_0) and the existence of transversals homeomorphic to S^1. First we will show the following

3.1. LEMMA. *Assume that there is given a transversal C of dynamical system (M, π), a regular point $x \in M$ and $t > 0$ such that $x \in C$ and $\pi(t, x) \in C$. Let C_1 be a subarc of C having as the end points x and $\pi(t, x)$. Assume that C_1 has no more common points with $\pi(x)$. Then for every point y belonging to $C_1 \setminus \{x\}$ we have $\pi(y) \cap C_1 = \{y\}$. Moreover, every point of C_1 is regular.*

Proof. Of course $C_1 \cup \pi([0, t] \times \{x\})$ is a Jordan curve; the construction of a homeomorphism onto S^1 is obvious. Now we use Proposition 1.7 and the fact that one of the components of $M \setminus (C_1 \cup \pi([0, t] \times \{x\}))$, say A, is positively invariant. Thus $\bar{A} = A \cup C_1 \cup \pi([0, t] \times \{x\})$ is also positively invariant. Take any $y \in C_1 \setminus \pi(x)$. From definition of the transversal there exists $\theta_1 > 0$ with property $\pi((0, \theta_1) \times \{y\}) \cap C_1 = \emptyset$. Take any $\theta \in (0, \min\{\theta_1, t\})$. From the assumptions and the above properties we obtain $\pi(\theta, y) \notin C_1 \cup \pi(x)$, but $\pi(\theta, y) \in \bar{A}$, so $\pi(\theta, y) \in A$. Since A is positively invariant, we get $\pi^+(\pi(\theta, y)) \subset A$, hence $\pi^+(\pi(\theta, y)) \cap C_1 = \emptyset$. But $\pi((0, \theta] \times \{y\})$ is disjoint with C_1, which proves that $\pi^+(y) \cap C_1 = \emptyset$. We have proved that $\pi(s, y) \cap C_1 = \emptyset$ for every $s > 0$ and $y \in C_1 \setminus \pi(x)$. To finish the proof we need to show that $\pi(s, y) \cap C_1 = \emptyset$ for every $y \in C_1 \setminus \pi(x)$ and $s < 0$. Suppose, to the contrary, that there exist $s < 0$ and $z \in C_1$ such that $\pi(s, z) \in C_1$. So $-s > 0$ and by previous properties we get $z = \pi(-s, \pi(s, z)) \notin C_1$, which gives a contradiction.

Of course every point y of $C_1 \setminus \pi(x)$ is regular; otherwise we would obtain $y = \pi(s, y) \in C_1$ for $s \neq 0$, which is impossible. Now we state our theorem.

3.2. THEOREM. *The dynamical system on dichotomic manifold (M, π) satisfies condition* (TR_0) *if and only if there are no transversals of this system homeomorphic to S^1.*

Proof. 3.2.1. Suppose first that there exists a transversal C homeomorphic to S^1. Take any $\varepsilon > 0$. Let p be a parametrization of C; p: $[0, 3\pi] \to M$, such that 2π is the fundamental period of p. Thus $p(t+2\pi) = p(t)$ for every $t \in [0, \pi]$.

Define $\tilde{p}$: $[0, 3\pi] \to M$ by $\tilde{p}(t) = \pi\left(\frac{\varepsilon \cdot t}{\pi}, p(t)\right)$. Denote $\tilde{p}([0, 3\pi])$ as C_1. We will show the following four properties:

(i) $\tilde{p}$ is continuous

(ii) $\tilde{p}$ is bijective onto C_1

(iii) $\tilde{p}$ is a homeomorphism onto C_1

(iv) C_1 is transversal (with t-extend ε).

Of course $\tilde{p}$ is continuous as a composition of continuous functions. In order to show (ii) and (iv) we will prove first that $\tilde{p}$ is injective.

Assume $\tilde{p}(t_1) = \tilde{p}(t_2)$ for $t_1, t_2 \in [0, 3\pi]$. Since $\tilde{p}(t_2) \in \pi(\tilde{p}(t_1))$ and 2π is a fundamental period of p, we get $t_1 = t_2$ or $|t_1 - t_2| = 2\pi$. Thus it is sufficient to show that $\tilde{p}(t) \neq \tilde{p}(t+2\pi)$ for $t \in [0, \pi]$. By Proposition 1.6 every point of C is regular, then so is every point in C_1. We have however $\tilde{p}(t+2\pi) = \pi(2\varepsilon, \tilde{p}(t))$, which contradicts the regularity of $\tilde{p}(t)$. This proves that p is injective.

Now we show that $\pi(\theta, C_1) \cap \pi(\theta', C_1) = \emptyset$ for every $\theta \neq \theta'$, $\theta, \theta' \in [0, \varepsilon]$. In fact, suppose the contrary: that there exist $y, z \in C_1$ with $\pi(\theta, y) = \pi(\theta', z)$. Then $y \in \pi(z)$ (we may assume $y \in \pi^+(z)$) and — in the same way as above — we get $y = z = \tilde{p}(t_0)$

or $z = \tilde{p}(t_0)$ and $y = \tilde{p}(t_0+2\pi)$. In the first case $z = \pi(\theta-\theta', y)$, which contradicts the regularity of y. In the second case we obtain $y = \pi(2\varepsilon, \tilde{p}(t_0))$ and $y = \pi(\theta-\theta', \tilde{p}(t_0))$. But $\theta-\theta' \neq 2\varepsilon$ and again we get a contradiction with the regularity of y. Thus C_1 is a transversal.

The condition (iii) follows easily from (i), (ii) and the compactness of $[0, 3\pi]$.

To finish the first part of the proof we will show that the properties (i)—(iv) contradict the condition (TR_0). In fact, $p(0) \in C$, so by Proposition 1.6 $p(0)$ is regular. The points $\tilde{p}(2\pi)$ and $\tilde{p}(0)$ belong to $\pi(p(0))$, and then $\tilde{p}(2\pi) \neq \tilde{p}(0)$ but they are also in C_1. This contradicts condition (TR_0).

We proved that in every dynamical system fulfilling (TR_0) there is not any transversal homeomorphic to S^1.

3.2.2. Now we will show the inverse implication.

Assume that our system does not fulfil (TR_0), so there exists a homeomorphism $g\colon [-1, 1] \to g([-1, 1]) = C \subset M$, where C is a transversal such that for certain $a \in C$ and $t > 0$ we have $\pi(t, a) \in C$ and $a \neq \pi(t, a)$.

Define s_0 as $\inf\{s > 0\colon \pi(s, a) \in C\}$. By hypothesis and the definition of the transversal we get $s_0 > 0$; it follows immediately that $\pi(s_0, a) \in C$. Denote: $\alpha = g^{-1}(a)$ and $\beta = g^{-1}(\pi(s_0, a))$; we may assume (changing the parametrization, if necessary) that $\alpha < \beta$.

We will consider a subarc C_1 of C with the end points $g(\alpha)$ and $g(\beta)$ and show that $C_1 \cap \pi(a) = \{g(\alpha), g(\beta)\}$. Suppose that there exists $\gamma \in (\alpha, \beta)$ with $g(\gamma) = \pi(w, a)$. Then $w < 0$ or $w > s_0$. But since $\alpha < \beta$, we obtain, using Proposition 1.8, that if $w < 0$, then $\gamma < \alpha < \beta$ and if $w > s_0$, then $\alpha < \beta < \gamma$. Thus $\gamma \notin (\alpha, \beta)$, which is a contradiction.

Now we will apply Lemma 3.1. By Proposition 1.6 the point a is regular. Then $\pi(y) \cap C_1 = \{y\}$ for every $y \in C_1 \setminus \pi(a)$ and $C_1 \cap P(\pi) = \emptyset$. We can define the parametrization f of C_1 as follows: $f(t) = g(t\cdot(\beta-\alpha)+\alpha)$ for $t \in [0, 1]$. Now define a new function $h\colon [0, 1] \to M$, $h(t) = \pi(s_0 \cdot t, f(1-t))$ and C_2 as $h([0, 1])$. We have $C_2 \cap P(\pi) = \emptyset$.

We will show that, for every $y \in M$, the trajectory $\pi(y)$ intersects C_2 in one point at most. Let $t_1, t_2 \in [0, 1)$. Then $f(1-t_1) \neq f(1-t_2)$. Since $h(0) = \pi(s_0, a) = h(1)$, we obtain $f(1-t_1) \notin \pi(f(1-t_2))$ and $h(t_1) \neq h(t_2)$. Thus every trajectory has not more than one intersection point with C_2. We can identify $h(0)$ with $h(1)$ and from the compactness of the set $h([0, 1])$ is homeomorphic to S^1. So C_2 is obviously a transversal. In fact, if there are θ and θ' ($\theta \neq \theta'$) such that $\pi(\theta, C_2) \cap \pi(\theta', C_2) \neq \emptyset$, then there exist $u, v \in C_2$ such that $u = \pi(\theta-\theta', v)$. This is however impossible, because all the points of C_2 are regular. Thus we constructed a transversal homeomorphic to S^1. This finishes the proof of Theorem 3.2.

3.3. COROLLARY. *In a dynamical system on a dichotomic manifold, satisfying* (TR_0) *every critical point isolated in* $S(\pi)$ *and positively stable is a centre* (*see* 1.9).

IV. In the last part of this paper we will investigate dynamical systems on dichotomic manifolds satisfying condition (TR_2).

4.1. THEOREM. *Let* (M, π) *be a dynamical system on a dichotomic manifold. Then the following two conditions are equivalent*:

(o) *there exist a transversal* C *and a point* $x \in M$ *such that* $\pi(x) \cap C$ *has exactly two elements*

(oo) *for every natural number* n, $n \geqslant 2$ *there exist a transversal* C *and a point* $x \in M$ *such that* $\pi(x) \cap C$ *has exactly* n *elements.*

Proof. The implication (o) $\Rightarrow$ (oo) is obvious as a subarc of a transversal is a transversal. Let f be a parametrization of C and $f(\alpha), f(\beta) \in \pi(x)$. We may assume without loss of generality that $f(\alpha) = x$, $f(\beta) = \pi(t, x)$, $t > 0$. Define a function $g\colon [0, 1] \to M$ by the formula $g(t) = f[(\beta-\alpha)\cdot t+\alpha]$ and $C_2 = g([0, 1])$. Of course C_2 intersects $\pi(x)$ in two points: $g(0)$ and $g(1)$. Using Lemma 3.1 we obtain that $\pi(y)$ and C_2 have exactly one intersection point for every $y \in C_2 \setminus \pi(x)$. By Lemma 3.1 and Proposition 1.6 all the points of C_2 are regular.

Now define $C_n = \bigcup_{k=0}^{n-2} \pi(t\cdot k, C_2)$ for $n \geqslant 3$. We will show two properties:

(i) C_n and $\pi(x)$ have exactly n intersection points for every $n \geqslant 2$

(ii) C_n is a transversal for every $n \geqslant 2$.

For $n = 2$ they are obvious. Assume $n \geqslant 3$. First we will prove (i). By the definition of C_n it follows that $y \in \pi(x) \cap C_n$ if and only if $y \in \pi(x)$ and $y = \pi(k\cdot t, z)$ for certain $k = 0, 1, \ldots, n-2$ and $z \in C_2$. Thus from $C_2 \cap \pi(x) = \{x, \pi(t, x)\}$ we obtain that there exists k ($k = 0, 1, \ldots, n-1$) such that $y = \pi(k\cdot t, x)$. Of course all the points $\pi(k\cdot t, x) \in \pi(x) \cap C_n$ ($k = 0, 1, \ldots, n-1$) and then the set $\pi(x) \cap C_n$ has n elements, since x is regular.

Now we will prove (ii). For $n \geqslant 3$ it holds $C_n = p_n([0, n-1])$, where $p_n\colon [0, n-1] \to M$ is defined as $p_n(s) = \pi(k\cdot t, g(s-k))$, if $s \in [k, k+1]$, for every $k = 0, 1, \ldots, n-2$. It is obvious that p_n is well defined and continuous. We claim p_n is injective. Suppose that $p_n(s_1) = p_n(s_2)$. Then there exist k, l with $\pi(k\cdot t, g(s_1-k)) = \pi(l\cdot t, g(s_2-l))$ and $g(s_2-l) = \pi((k-l)\cdot t, g(s_1-k))$. Consequently, either $g(s_2-l) = g(s_1-k)$, or $g(s_1-k) = x$ and $g(s_2-l) = \pi(t, x)$, or $g(s_2-k) = \pi(t, x)$ and $g(s_1-l) = x$. In all the cases we obtain $s_1 = s_2$. By the definition of p_n and C_n it follows easily that $p_n([0, n-1]) = C_n$. Then by the compactness of $[0, n-1]$ we obtain that p_n is a homeomorphism onto C_n.

In order to finish the proof we will show that C_n is a transversal with t-extend $t/3$. Suppose there exist $y, z \in C_n$ and $\theta, \theta' \in -(t/3, t/3)$ with $\pi(\theta, z) = \pi(\theta', y)$. So we can find $z_1, z_2 \in C_2$ and the integers k, l such that $\pi(\theta+t\cdot k, z_1) = \pi(\theta'+t\cdot l, z_2)$. Thus $z_2 = \pi(\theta-\theta'+t\cdot(k-l), z_1)$. Since $z_1 \in C_2$, $z_2 \in C_n \cap \pi(z_1)$ there exists a natural number m with $z_2 = \pi(t\cdot m, z_1)$. From this we have $z_1 = \pi(\theta-\theta'+t\cdot(k-l-m), z_1)$. But $|\theta-\theta'| < \frac{2}{3}t$ and $\theta \neq \theta'$, so $\theta-\theta'+t\cdot(k-l-m) \neq 0$, which is impossible since z_1 is regular. Thus C_n is a transversal; this finishes the proof of Theorem 4.1.

4.2. COROLLARY. *Let* (M, π) *be a dynamical system on a dichotomic manifold. Then the following two conditions are equivalent*:

(o) *there exist a trajectory* $\pi(x)$, *a transversal* C *and a natural number* $k \geqslant 2$ *such that* $\pi(x)$ *and* C *have exactly* k *intersection points.*

(oo) *for every natural number n* ($n \geqslant 2$) *there exist a trajectory* $\pi(y)$ *and a transversal* C_n *such that the set* $\pi(y) \cap C_n$ *has n elements.*

4.3. COROLLARY. *If a dynamical system* (M, π) *fulfils condition* (TR_2), *then for every natural number n there exist a transversal C and a point* $x \in M$ *such that* $\pi(x)$ *intersects C in exactly n distinct points.*

4.4. Remark. Theorem 4.1 and Corollary 4.3 show that definitions of (TR_n) systems ($n \geqslant 3$) in any way would be useless.

References

[1] A. Beck, *Continuous Flows in the Plane*, Springer-Verlag, Berlin Heidelberg New York, 1974.

[2] N. P. Bhatia and G. P. Szegö, *Stability Theory of Dynamical Systems*, Springer-Verlag, Berlin Heidelberg New York, 1970.

[3] K. Ciesielski and A. Trzepizur, *Regular Trajectories of Closed Dynamical Systems in the Plane*, Universitatis Iagellonicae Acta Mathematica 24 (1984), 227—237.

[4] O. Hajek, *Dynamical Systems in the Plane*, Academic Press, London and New York, 1968.

[5] R. A. Knight, *Structure of Certain Closed Flows*, Annales Polon. Math., (to appear).

[6] K. Kuratowski, *Topology*, vol. I, Academic Press — PWN, New York—Warszawa 1966.

[7] K. Kuratowski, *Topology*, vol. II, Academic Press — PWN, New York—Warszawa 1968.

[8] A. Pelczar, *Ogólne układy dynamiczne*, Uniwersytet Jagielloński, skrypt uczelniany nr 293, Kraków 1978.

[9] A. Pelczar, *Wstęp do teorii układów dynamicznych*, Akademia Górniczo-Hutnicza w Krakowie, skrypt uczelniany nr 694, Kraków 1980.

UNIWERSYTET JAGIELLOŃSKI
INSTYTUT MATEMATYKI
30-059 KRAKÓW, UL. REYMONTA 4

Received February 17, 1981.