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On the generic Chaos in Dynamical Systems

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This paper is devoted to some new concept of the chaos in dynamical systems. It was proposed by A. Lasota and we will call this chaos generic.

1. For the definition of "*generic chaos*", let (V, ϱ) be a metric space and let $\{S_t\}$ be a semigroup of transformations of V into V. It may be as well discrete ($t \in \boldsymbol{N}$) as continuous ($t \in \boldsymbol{R}_+$) semigroup. Let G be the set of all (v_1, v_2) in V^2 for which the following conditions hold

$$\liminf_{t \to +\infty} \varrho(S_t v_1, S_t v_2) = 0$$

$$\limsup_{t \to +\infty} \varrho(S_t v_1, S_t v_2) > 0 .$$

We call the dynamical system $\{S_t\}$ genericly chaotic iff the set G is generic in V^2, i.e. iff its complement is a set of the first category.

We shall say that a single transformation $S\colon V \to V$ is genericly chaotic iff the semigroup of iterates S^n (n nonnegative integer) is genericly chaotic.

We shall show that some two special dynamical systems are genericly chaotic.

Let us recall that chaos in dynamical systems has been defined in other forms by Li and Yorke ([4]) and by Auslander and Yorke ([1]). A. Lasota in his papers [2], [3] proves some "chaotic" properties for dynamical system introduced by a partial differential equation arising as a mathematical model of growth of blood cells' population. The mentioned equation is similar to that of n. 3 of this paper.

2. Let us recall that so-called r-adic transformation is a mapping from the unit interval to itself given by the formula

$$Tx = rx \pmod 1 , \qquad x \in [0, 1]$$

for some integer $r \geqslant 2$.

Generalizing this example we obtain so-called piece-wise monotonic transformations which, as we shall show, are genericly chaotic.

THEOREM 1. *Let us fix an integer $r \geqslant 2$ and a sequence $a_0 = 0 < a_1 < \ldots < a_r = 1$. Let T: $[0, 1] \to [0, 1] = I$ be a transformation such that*:

(i) $\varphi_i = T|_{[a_{i-1}, a_i]}$ *is continuous and is differentiable in* (a_{i-1}, a_i) *for* $i = 1, \ldots, r$,

(ii) $\varphi_i([a_{i-1}, a_i]) = I$ *for* $i = 1, \ldots, r$,

(iii) $\inf|\varphi_i'| \geqslant q > 1$ *in* (a_{i-1}, a_i) *for* $i = 1, \ldots, r$.

Then T is genericly chaotic.

Proof. Let us put $A_0 = \{a_0, a_1, \ldots, a_r\}$, $A_{n+1} = A_n \cup T^{-1}(A_n)$ for $n = 1, 2, \ldots$ and $A = \bigcup_{n=0}^{\infty} A_n$. Further, let us put

$$L_{n,\varepsilon} = \{(x, y) \in (I \setminus A)^2 |\ \inf_{k \geqslant n} |T^k x - T^k y| < \varepsilon\}, \quad \varepsilon > 0,\ n = 1, 2, \ldots$$

$$U_n = \{(x, y) \in (I \setminus A)^2 |\ \sup_{k \geqslant n} |T^k x - T^k y| > c\}, \quad n = 1, 2, \ldots$$

for some fixed $c \in (0, \frac{1}{2})$ and

$$G = \{(x, y) \in I^2 | \liminf_{n \to +\infty} |T^n x - T^n y| = 0, \quad \limsup_{n \to +\infty} |T^n x - T^n y| > 0\}.$$

We shall prove that each of the sets $L_{n,\varepsilon}$, U_n (for $n = 1, 2, \ldots, \varepsilon > 0$) is open and dense in I. And this will imply the genericity of G, since $G \supset \bigcap_{n=1}^{\infty} (L_{n,\frac{1}{n}} \cap U_n)$.

(a) To prove that $L_{n,\varepsilon}$ is open, let us fix $n \geqslant 1$, $\varepsilon > 0$ and $(x_0, y_0) \in L_{n,\varepsilon}$. There exist $\eta \in (0, \varepsilon)$ and an integer $k \geqslant n$ such that $|T^k x_0 - T^k y_0| < \varepsilon - \eta$. Also there exist a neighbourhood $\overline{M}$ of x_0 and a neighbourhood $\overline{N}$ of y_0 which are disjoined with A_k. Thus T^k is continuous on $\overline{M} \cup \overline{N}$ and there exist neighbourhoods $M \subset \overline{M}$ and $N \subset \overline{N}$ of x_0 and y_0 (respectively) such that

$$|T^k x - T^k x_0| < \eta/2 \quad \text{for } x \in M \text{ and } |T^k y - T^k y_0| < \eta/2 \quad \text{for } y \in N.$$

Then, for $(x, y) \in M \times N$

$$|T^k x - T^k y| \leqslant |T^k x - T^k x_0| + |T^k x_0 - T^k y_0| + |T^k y_0 - T^k y| < \varepsilon$$

and consequently $M \times N \subset L_{n,\varepsilon}$.

(b) In the proof of density of $L_{n,\varepsilon}$ the following lemmas will be useful.

LEMMA 1. *Let* $\delta = \max\{|a_{i-1} - a_i|: i = 1, \ldots, r\}$ *and let*

$$\delta_n = \max\{|b_{i-1} - b_i|: i = 1, \ldots, rn\}$$

where $A_n = \{b_0, \ldots, b_{rn}\}$ *and* $n \geqslant 1$. *Then* $\delta_n \leqslant q^{-n}\delta$.

To prove it, let us observe that the required inequality is valid for $n = 0$ (with $\delta_0 = \delta$) and let us suppose that it is valid for some positive integer n. For b_i, b_{i-1} in A_{n+1} we have $T(b_i), T(b_{i-1}) \in A_n$ and $[b_{i-1}, b_i] \subset [a_{j-1}, a_j]$ for some $j \in \{1, \ldots, r\}$. Then

$$|T(b_{i-1}) - T(b_i)| = |\varphi_j(b_{i-1}) - \varphi_j(b_i)| = |\varphi_j'(\theta)| \cdot |b_{i-1} - b_i| \geqslant q|b_{i-1} - b_i|$$

where $\theta \in (b_{i-1}, b_i)$ and further

$$|b_{i-1}-b_i| \leqslant q^{-1}|T(b_{i-1})-T(b_i)| \leqslant q^{-1}q^{-n}\delta \leqslant q^{-(n+1)}\delta \quad \text{for } i = 1, \ldots, rn.$$

Thus $\delta_{n+1} \leqslant q^{-(n+1)}\delta$, too.

LEMMA 2. *For* $x \in I \setminus A$ *the set* $\bigcup_{n=0}^{\infty} T^{-n}(\{x\})$ *is dense in* I.

To prove it, let us observe that by Lemma 1 the set A_n divides I into rn intervals $I(n, 1), \ldots, I(n, rn)$ which longitudes are bounded from above by $q^{-n}\delta$. Since T is invertible on each interval $[a_{i-1}, a_i]$, we conclude that the intersection $T^{-n}(\{x\}) \cap \operatorname{int} I(n, i)$ has exactly one point for $i = 1, \ldots, rn$ and $n = 1, 2, \ldots$ This conclusion finishes the proof of the lemma. Indeed, for $x_0 \in I$ and $\sigma > 0$ there is positive integer n such that $q^{-n}\delta < \frac{1}{2}\sigma$ and then, for some $i \in \{1, \ldots, rn\}$, $I(n, i) \subset (x_0-\sigma, x_0+\sigma)$.

Now the proof of the density of $L_{n,\varepsilon}$ is very easy. For (x_0, y_0) in I^2, $\tau > 0$ and $n \geqslant 1$ there exist (by Lemma 2) point $\bar{x}, \bar{y} \in I \setminus A$ such that $|x_0-\bar{x}| < \tau$, $|y_0-\bar{y}| < \tau$ and $T^i\bar{x} = T^i\bar{y}$ for $i \geqslant n$. Thus $(\bar{x}, \bar{y}) \in L_{n,\varepsilon}$ for each $\varepsilon > 0$.

(c) To prove that U_n is open, let us fix $(x_0, y_0) \in U_n$. There exist $\eta > 0$ and an integer $k \geqslant n$ such that $|T^k x_0 - T^k y_0| > c+\eta$. Choosing neighbourhoods M, N as in the point (a), we obtain:

$$c+\eta < |T^k x_0 - T^k y_0| \leqslant |T^k x_0 - T^k x| + |T^k x - T^k y| + |T^k y - T^k y_0| < \eta + |T^k x - T^k y|$$

for $(x, y) \in M \times N$. Thus $M \times N \subset U_n$.

(d) It remains to prove that U_n is dense for $n = 1, 2, \ldots$ Let us fix $(x_0, y_0) \in I^2$, $n \in N$ and $\sigma > 0$. Let us also fix some integer $p \geqslant n$ such that $q^{1-p}\delta < \sigma$. For fixed p we can choose $s, t \in \{1, \ldots, rp\}$ such that $x_0 \in I(p, s)$, $y_0 \in I(p, t)$. (Intervals $I(n,j)$ are defined as in Lemma 2.) For simplicity and without loss of generality we may assume that T^m is increasing on both intervals $I(p, s)$, $I(p, t)$ and T is increasing on $[0, a_1]$. We define two decreasing sequences of intervals $J_i(x_0)$, $J_i(y_0)$ in such a way:

$$J_1(x_0) = I(p+1, (s-1)r+1) \qquad J_1(y_0) = I(p+1, tr-1)$$
$$J_2(x_0) = I(p+2, 2(s-1)r+1) \qquad J_2(y_0) = I(p+2, 2tr-1)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$J_m(x_0) = I(p+m, (s-1)mr+1) \qquad J_m(y_0) = I(p+m, mtr-1),$$

where m is chosen so that $T^n x < \frac{1}{2}(1-c)$ for $x \in J_m(x_0)$ and $T^n y > \frac{1}{2}(1+c)$ for $y \in J_m(y_0)$ (It is possible since T^n is continuous and increasing and T is increasing on $[0, a_1]$.) Further:

$$J_{m+i}(x_0) = I\big(p+m+i, (s-1)(m+i)r+\alpha(i)\big)$$
$$J_{m+i}(y_0) = I\big(p+m+i, (t-1)(m+i)r+\beta(i)\big)$$

for $i > 0$, where $\alpha\colon N \cup \{0\} \to \{1, 2, \ldots, r\}$ and $\beta\colon N \cup \{0\} \to \{1, 2, \ldots, r\}$ are non-periodic.

Intervals $J_i(x_0)$ and $J_i(y_0)$ are closed and their diameters decrease to 0 (for $i \to +\infty$). Thus, by Cantor's Principle, there exists exactly one point $\bar{x} \in I(p, s) \cap \bigcap_{k=1}^{\infty} J_k(x_0)$ and there exists exactly one point $\bar{y} \in I(p, t) \cap \bigcap_{k=1}^{\infty} J_k(y_0)$. Then $|\bar{x}-x_0|<\sigma$, $|\bar{y}-y_0|<\sigma$, $\bar{x}, \bar{y} \notin A$ and $|T^n\bar{y}-T^n\bar{x}|>\frac{1}{2}(1+c-(1-c)) = c$. Thus $(\bar{x}, \bar{y}) \in U_n$. The proof of Theorem 1 is finished.

3. For an example of generic chaos in continuous-time dynamical system, let us consider first-order partial differential equation

$$\frac{\partial u}{\partial t}+x\frac{\partial u}{\partial x} = \lambda u\,, \qquad t\geq 0,\ 0\leq x<a\leq +\infty,\ \lambda>0$$

with initial condition

$$u(0, x) = v(x) \quad \text{where } v\geq 0 \quad \text{and} \quad v(0) = 0\,.$$

One can write the solution of the equation in explicit form:

$$u(t, x) = v(xe^{-t})e^{\lambda t}\,.$$

Let us put

$$V = \{v\colon [0, a) \to \boldsymbol{R}_+ | v(0) = 0,\ v \text{ is continuous}\}\,,$$

$$\varrho[b](u, v) = \max_{0\leq x\leq b} |u(x)-v(x)|$$

for $u, v \in V$ and $b<a$ and

$$\varrho(u, v) = \sum_{n=1}^{\infty} 2^{-n} \frac{\varrho[n](u, v)}{1+\varrho[n](u, v)}$$

for $u, v \in V$ in the case $a = +\infty$ or

$$\varrho(u, v) = \sum_{n=1}^{\infty} 2^{-n} \frac{\varrho\left[a-\frac{1}{n}\right](u, v)}{1+\varrho\left[a-\frac{1}{n}\right](u, v)}$$

for $u, v \in V$ in the case $a<+\infty$.
Then (V, ϱ) is a metric space.

Let us define a family of mappings $S_t\colon V \to V$ introduced by the above mentioned equation, i.e. mappings given by the formula

$$(S_t v)(x) = v(xe^{-t})e^{\lambda t}, \quad t\geq 0,\ v \in V.$$

It is easy to see that mappings S_t are well defined and that $\{S_t\}_{t\geq 0}$ is a semigroup, it is to say, $S_{t_1} \circ S_{t_2} = S_{t_1+t_2}$ for $t_1, t_2\geq 0$ and S_0 is the identity on V.

We shall show that for $t \geqslant 0$ the mapping S_t is Lipschitzean with respect to each of the semimetrics $\varrho[b]$ $(b<a)$ and consequently with respect to the metric ϱ. For, let us fix $b \in [0, a)$ and $v_1, v_2 \in V$. We have

$$|S_t v_1(x) - S_t v_2(x)| = e^{\lambda t}|v_1(xe^{-t}) - v_2(xe^{-t})|, \ x \in [0, b], \ t \geqslant 0.$$

Thus

$$\varrho[b](S_t v_1, S_t v_2) \leqslant e^{\lambda t}\varrho[be^{-t}](v_1, v_2) \leqslant e^{\lambda t}\varrho[b](v_1, v_2), \ t \geqslant 0$$

and

$$\varrho(S_t v_1, S_t v_2) \leqslant e^{\lambda t}\varrho(v_1, v_2), \ t \geqslant 0.$$

Now we shall prove

THEOREM 2. *The dynamical system $\{S_t\}_{t \geqslant 0}$ is genericly chaotic.*

Proof of the theorem is similar to that of Theorem 1. We define

$$L_{T,\varepsilon} = \{(v_1, v_2) | \inf_{t \geqslant T} \varrho(S_t v_1, S_t v_2) < \varepsilon\}, \quad T>0, \ \varepsilon>0,$$

$$U_T = \{(v_1, v_2) | \sup_{t \geqslant T} \varrho(S_t v_1, S_t v_2) > \alpha\}, \quad T>0$$

for some fixed $\alpha \in (0, \frac{1}{2})$. If the sets $L_{T,\varepsilon}$, U_T are open and dense in V^2, we shall be able to conclude that the set

$$G = \{(v_1, v_2) | \liminf_{t \to +\infty} \varrho(S_t v_1, S_t v_2) = 0 \wedge \limsup_{t \to +\infty} \varrho(S_t v_1, S_t v_2) > 0\}$$

is generic, since $G \supset \bigcap_{n=1}^{\infty} (L_{n,\frac{1}{n}} \cap U_n)$.

(a) In order to prove that $L_{T,\varepsilon}$ is open let us fix $T>0$, $\varepsilon>0$ and $(u_0, v_0) \in L_{T,\varepsilon}$. There exist $t \geqslant T$ and $\eta>0$ such that $\varrho(S_t u_0, S_t v_0) < \varepsilon - \eta$. Then for u, v in V with $\varrho(u_0, u) < \frac{1}{2}\eta e^{-\lambda t}$, $\varrho(v_0, v) < \frac{1}{2}\eta e^{-\lambda t}$ we have

$$\varrho(S_t u, S_t v) \leqslant \varrho(S_t u, S_t u_0) + \varrho(S_t u_0, S_t v_0) + \varrho(S_t v_0, S_t v) < \tfrac{1}{2}\eta + \varepsilon - \eta + \tfrac{1}{2}\eta = \varepsilon.$$

One can prove similarly that the sets U_T are open, too.

(b) For the proof of the density of $L_{T,\varepsilon}$ let us fix $T>0$, $\delta>0$ and $(u_0, v_0) \in V^2$. There is $\eta>0$ such that $|u_0(x) - v_0(x)| < \delta$ for $x \in [0, \eta)$. We can modify functions u_0, v_0 in the interval $[0, \eta)$ to obtain a pair $(\bar{u}, \bar{v}) \in L_{T,\varepsilon}$ close to (u_0, v_0). Indeed, we put

$$\bar{u}(x) = \begin{cases} \frac{1}{2}\left(u_0(x) + v_0(x)\right), & 0 \leqslant x \leqslant \frac{1}{2}\eta \\ \frac{1}{\eta} x u_0(x) + \left(1 - \frac{1}{\eta}x\right) v_0(x), & \frac{1}{2}\eta \leqslant x \leqslant \eta \\ u_0(x) & x \geqslant \eta \end{cases},$$

$$\bar{v}(x) = \begin{cases} \frac{1}{2}\left(u_0(x) + v_0(x)\right), & 0 \leqslant x \leqslant \frac{1}{2}\eta \\ \left(1 - \frac{1}{\eta}x\right) u_0(x) + \frac{1}{\eta} x v_0(x), & \frac{1}{2}\eta \leqslant x \leqslant \eta \\ v_0(x) & x \geqslant \eta. \end{cases}$$

Now one can see that $\varrho(u_0, \bar{u})<\delta$, $\varrho(v_0, \bar{v})<\delta$ and $\bar{u}(x) = \bar{v}(x)$ for $x \in [0, \frac{1}{2}\eta]$. Thus, for every $\varepsilon>0$, $(\bar{u}, \bar{v}) \in L_{T,\varepsilon}$, since for each $b<a$ there is $t_0 \geqslant T$ with $be^{-t_0} \leqslant \frac{1}{2}\eta$ (let us remember that S_t is Lipschitzean).

(c) It remains to prove that U_T is dense. Let us fix $T>0$, $\delta>0$ and $(u_0, v_0) \in V^2$. We need to find functions $\hat{u}, \hat{v} \in V$ such that $\varrho(u_0, \hat{u})<\delta$, $\varrho(v_0, \hat{v})<\delta$ and $|S_{t_0}\hat{u}(x_0) - S_{t_0}\hat{v}(x_0)| > \alpha$ for some $x_0 \in [0, a)$, and some $t_0 \geqslant T$. Let us observe that there is $\eta>0$ such that $|u_0(x) - v_0(x)| < \frac{1}{3}\delta$ for $x \in [0, \eta)$. We put

$$\hat{u}(x) = \begin{cases} \max(u_0, v_0)(x) + \dfrac{\delta}{\eta} x, & 0 \leqslant x \leqslant \dfrac{1}{3}\eta \\ \max(u_0, v_0)(x) + \dfrac{1}{3}\delta, & \dfrac{1}{3}\eta \leqslant x \leqslant \dfrac{2}{3}\eta \\ \left(\dfrac{3}{\eta}x - 2\right) u_0(x) + \left(3 - \dfrac{3}{\eta}x\right)\left(\max(u_0, v_0)(x) + \dfrac{1}{3}\delta\right), & \dfrac{2}{3}\eta \leqslant x \leqslant \eta \\ u_0(x), & x \geqslant \eta \end{cases}$$

and

$$\hat{v}(x) = \begin{cases} \min(u_0, v_0)(x), & 0 \leqslant x \leqslant \dfrac{2}{3}\eta \\ \left(\dfrac{3}{\eta}x - 2\right) v_0(x) + \left(3 - \dfrac{3}{\eta}x\right) \min(u_0, v_0)(x), & \dfrac{2}{3}\eta \leqslant x \leqslant \eta \\ v_0(x), & x \geqslant \eta. \end{cases}$$

It is easy to see that $\varrho(u_0, \hat{u})<\delta$ and $\varrho(v_0, \hat{v})<\delta$ and $(\hat{u}, \hat{v}) \in U_T$. And this finishes the proof of Theorem 2.

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