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On Approximation of Entire Functions and Generalized Orders

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Abstract. A characterization of the generalized order of entire functions of several complex variables by means of polynomial approximation and interpolation is established.

Let α, β be two positive, strictly increasing to infinity differentiable functions on an interval $(d, +\infty)$. We assume that

$$\lim_{x\to+\infty} \frac{\alpha(cx)}{\alpha(x)} = 1 \text{ provided } c>0;$$

$$\lim_{x\to+\infty} \frac{\beta\big((1+\omega(x))x\big)}{\beta(x)} = 1$$

for every function ω such that $\omega(x) \to 0$ as $x \to +\infty$.

Given an entire transcendental function g in C^N, $N\geqslant 1$, we put

$$S(r, g) = \sup\{|g(z)|: |z_1|^2+\ldots+|z_N|^2 = r^2\}, \quad r>0.$$

The quantity

$$\varrho(\alpha, \beta, g) = \limsup_{r\to+\infty} \frac{\alpha(\log S(r, g))}{\beta(\log r)}$$

is called a *generalized order* of g (see [5] and [2]). If $\alpha(x) = \log x$, $\beta(x) = x$, we get the classical definition of the order of an entire function.

The aim of this paper is to characterize the generalized order $\varrho(\alpha, \beta, g)$ by means of polynomial approximation and interpolation to g on compact subsets of C^N.

Let K be a compact set in C^N and let $\| \ \|_K$ denote the supremum norm on K. The function

$$\Phi_K(z) = \sup\{|\varrho(z)|^{\frac{1}{n}}: p\text{-polynomial, } \deg p\leqslant n,\ \|p\|_K\leqslant 1,\ n\in N\},\ z\in C^N,$$

is called the Siciak extremal function of the compact K ([3], [4]).

Given a function f, defined and bounded on K, we put for $n \in N$

$$E_n^{(1)}(f, K) = \|f - t_n\|_K;$$
$$E_n^{(2)}(f, K) = \|f - l_n\|_K;$$
$$E_{n+1}^{(3)}(f, K) = \|l_{n+1} - l_n\|_K,$$

where t_n denotes the n-th Čebyšev polynomial of the best approximation to f on K and l_n denotes the n-th Lagrange interpolation polynomial for f with nodes at extremal points of K (see [3] or [4]).

THEOREM. *Let K be a compact set in C^N such that Φ_K is locally bounded in C^N. Assume that for every $c>0$ there exist positive constants a, b such that*

$$\left|\frac{d(\beta^{-1}(c\alpha(x)))}{d(\log x)}\right| \leqslant b, \quad x \geqslant a.$$

Then the function f, defined and bounded on K, is the restriction to K of an entire function g of the finite generalized order $\varrho(\alpha, \beta, g)$ if and only if

$$\varrho(\alpha, \beta, g) = \limsup_{n \to +\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\log E_n^{(s)}(f, K)\right)}, \quad s = 1, 2, 3.$$

This Theorem was obtained by S. M. Shah [2] in the case $N = 1$, $K = [-1, 1]$ and by Winiarski [6] for $\alpha(x) = \log x$, $\beta(x) = x$.

We start with the following

LEMMA. *Under the assumptions of Theorem let $(p_n)_{n \in N}$ be a sequence of polynomials in C^N such that*

(i) $\deg p_n \leqslant n$, $n \in N$;

(ii) *there exists $n_0 \in N$ such that*

$$\|p_n\|_K \leqslant \exp\left(-n\beta^{-1}\left(\frac{1}{\mu}\alpha(n)\right)\right), \quad n \geqslant n_0.$$

Then $\sum_{n=0}^{\infty} p_n$ is an entire function and $\varrho(\alpha, \beta, \sum_{n=0}^{\infty} p_n) \leqslant \mu$ provided $\sum_{n=0}^{\infty} p_n$ is not a polynomial.

Proof. By the assumptions

$$\|p_n\|_K r^n \leqslant r^n \exp\left(-n\beta^{-1}\left(\frac{1}{\mu}\alpha(n)\right)\right), \quad n \geqslant n_0, \quad r>0.$$

By the methods of the calculus we obtain that the maximum of the function

$$\left\{(0, +\infty) \ni x \to r^x \exp\left(-x\beta^{-1}\left(\frac{1}{\mu}\alpha(x)\right)\right) \in R\right\}$$

is reached for $x = x_r$, where x_r is the solution of the equation

$$x = \alpha^{-1}\left(\mu\beta\left(\log r - \frac{d\left(\beta^{-1}\left(\frac{1}{\mu}\alpha(x)\right)\right)}{d(\log x)}\right)\right).$$

Thus

$$(1) \qquad \|p_n\|_K r^n \leqslant \exp\big(b\alpha^{-1}(\mu\beta(\log r + b))\big), \quad n \geqslant n_0, \quad r > 0.$$

Write

$$K_r = \{z \in \boldsymbol{C}^N\colon\ \Phi_K(z) < r\}, \quad r > 1.$$

For every polynomial p of degree $\leqslant n$ ([3], [4])

$$(2) \qquad |p_n(z)| \leqslant \|p_n\|_K \Phi_K^n(z),\ z \in \boldsymbol{C}^N.$$

So the series $\sum_{n=0}^{\infty} p_n$ is convergent in every set K_r, $r>1$, whence $\sum_{n=0}^{\infty} p_n$ is an entire function. Put

$$M^*(r) = \sup\{\|p_n\|_K r^n\colon\ n \in N\}, \quad r > 0.$$

On account of (1) for every $r>0$ there exists a positive integer $\nu(r)$ such that

$$M^*(r) = \|p_{\nu(r)}\|_K r^{\nu(r)}$$

and

$$M^*(r) > \|p_n\|_K r^n, \quad n > \nu(r).$$

Observe that $\nu(r)$ increases with r. Suppose first that $\nu(r) \to +\infty$ as $r \to +\infty$. Then, putting $n = \nu(r)$ in (1) we get for sufficiently large r

$$(3) \qquad M^*(r) \leqslant \exp\big(b\alpha^{-1}(\mu\beta(\log r + b))\big).$$

Put

$$F_r = \{z \in \boldsymbol{C}^N\colon\ \Phi_K(z) = r\}, \quad r > 1;$$

$$M(r) = \sup\{|\sum_{n=0}^{\infty} p_n(z)|\colon\ z \in F_r\}, \quad r > 1.$$

The inequality (2) implies

$$(4) \qquad M(r) \leqslant \sum_{n=0}^{\infty} \|p_n\|_K r^n = \sum_{n=0}^{\infty} \|p_n\|_K (2r)^n 2^{-n} \leqslant 2M^*(2r)$$

Moreover, [6, (3.3)],

$$(5) \qquad S(r, \sum_{n=0}^{\infty} p_n) \leqslant M(kr)$$

for some positive constant k. Combining (3), (4) and (5) we get

$$\log S(r, \sum_{n=0}^{\infty} p_n) \leqslant \log 2 + b\alpha^{-1}(\mu\beta(\log r + \log 2ke^b)),$$

whence

$$\frac{\alpha\left(\frac{1}{b}\log S(r, \sum_{n=0}^{\infty} p_n) - \frac{1}{b}\log 2\right)}{\beta(\log r + \log 2ke^b)} \leqslant \mu .$$

Owing to the properties of the functions α and β, after passing to the upper limit we obtain

$$\varrho(\alpha, \beta, \sum_{n=0}^{\infty} p_n) \leqslant \mu .$$

It remains to consider the case where $\nu(r)$ is bounded. But then $M^*(r)$ is also bounded, whence $\sum_{n=0}^{\infty} p_n$ is a polynomial. So the proof is completed.

Proof of Theorem. Let g be an entire transcendental function. Write

$$\varrho = \varrho(\alpha, \beta, g);$$

$$\gamma_s = \limsup_{n \to +\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\log E_n^{(s)}\right)}, \qquad s = 1, 2, 3;$$

here $E_n^{(s)}$ stands for $E_n^{(s)}(g|_K, K)$, $s = 1, 2, 3$. We claim that $\varrho = \gamma_s$, $s = 1, 2, 3$. It is known (see e.g. [6]) that

$$E_n^{(1)} \leqslant E_n^{(2)} \leqslant (n_* + 2) E_n^{(1)}, \qquad n \geqslant 0; \tag{6}$$

$$E_n^{(3)} \leqslant 2(n_* + 2) E_{n-1}^{(1)}, \qquad n \geqslant 1, \tag{7}$$

where $n_* = \binom{n+N}{n}$. Thus $\gamma_3 \leqslant \gamma_2 = \gamma_1$ and it suffices to prove that $\gamma_1 \leqslant \varrho \leqslant \gamma_3$.

1° $\gamma_1 \leqslant \varrho$. Owing to the definition of the generalized order for every $\mu > \varrho$

$$\log S(r, g) \leqslant \alpha^{-1}(\mu\beta(\log r))$$

provided r is sufficiently large. We may suppose obviously that

$$K \subset B = \{z \in \boldsymbol{C}^N \colon |z_1|^2 + \ldots + |z_N|^2 \leqslant 1\} .$$

Then $E_n^{(1)} \leqslant E_n^{(1)}(g, B)$. By Lemma 3.4 of [1]

$$E_n^{(1)}(g, B) \leqslant r^{-n} S(r, g), \qquad r \geqslant 2, \qquad n \geqslant 0 .$$

Thus

$$\log E_n^{(1)} \leqslant -n\log r + \alpha^{-1}(\mu\beta(\log r))$$

for every $n \in \boldsymbol{N}$ and for every sufficiently large r. Putting

$$r = \exp\left(\beta^{-1}\left(\frac{1}{\mu}\alpha(n)\right)\right)$$

we get

$$\log E_n^{(1)} \leqslant -n\left(\beta^{-1}\left(\frac{1}{\mu}\alpha(n)\right)\right) + n\,,$$

whence

$$\frac{\alpha(n)}{\beta\left(1-\frac{1}{n}\log E_n^{(1)}\right)} \leqslant \mu\,.$$

Letting $n \to +\infty$ and $\mu \to \varrho$ we obtain the required inequality.

2° $\varrho \leqslant \gamma_3$. Suppose $\gamma_3 < \varrho$. Then for every $\lambda \in (\gamma_3, \varrho)$

$$\frac{\alpha(n)}{\beta\left(-\frac{1}{n}\log E_n^{(3)}\right)} \leqslant \lambda$$

provided n is sufficiently large. Thus

$$E_n^{(3)} \leqslant \exp\left(-n\beta^{-1}\left(\frac{1}{\lambda}\alpha(n)\right)\right)$$

and by Lemma $\varrho \leqslant \lambda$. Since λ has been chosen less than ϱ, we get a contradiction, so $\varrho \leqslant \gamma_3$.

Now let f be a function defined and bounded on K and such that for $s = 1, 2$ or 3

$$\gamma_s = \limsup_{n \to +\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\log E_n^{(s)}\right)}$$

is finite. We claim that the function

$$g = l_0 + \sum_{n=1}^{\infty} (l_n - l_{n-1})$$

is the required entire continuation of f and $\varrho(\alpha, \beta, g) = \gamma_s$. Indeed, for every $\lambda > \gamma_s$

$$\frac{\alpha(n)}{\beta\left(-\frac{1}{n}\log E_n^{(s)}\right)} \leqslant \lambda$$

provided n is sufficiently large. Hence

$$E_n^{(s)} \leqslant \exp\left(-n\beta^{-1}\left(\frac{1}{\lambda}\alpha(n)\right)\right).$$

Owing to the inequalities (6), (7) and Lemma the function g is entire and $\varrho(\alpha, \beta, g)$ is finite. So by the first part of the proof $\varrho(\alpha, \beta, g) = \gamma_s$, as claimed.

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