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A Criterion for Polynomial Conditions of Leja's Type in C^N

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Abstract. We give a sufficient condition for the pair (E, μ), where E is a subset of C^N and μ is a measure on E, to satisfy the Leja condition (L^*). It generalizes in particular a criterion of the L-regularity proved by Cegrell [2] and Sadullaev [9].

1. The condition (L^*). Let E be a Borel subset of the space C^N of N complex variables. Let μ be a nonnegative function defined on a family of subsets of E containing all Borel subsets, and such that $\mu(\emptyset) = 0$ (in the sequel such a function μ will be simply called a measure). The pair (E, μ) is said to satisfy *condition* (L^*) *at a point* $a \in \bar{E}$ if for every subfamily $\mathscr{F}$ of the family $\mathscr{P}(C^N)$ of all polynomials from C^N to C^1 such that

$$(*) \qquad \mu(\{z \in E: \sup_{f \in \mathscr{F}} |f(z)| = +\infty\}) = 0,$$

and for every $b > 1$, there exists a constant $C > 0$ and a neighborhood U of a such that for each $f \in \mathscr{F}$,

$$|f(z)| \leqslant C b^{\deg f} \quad \text{for } z \in U.$$

By the well-known polynomial lemma of Leja [4], if E is a subset of C^1, $a \in \bar{E}$ and for a certain $t_0 > 0$, $\mu(E) = t_0$, where $\mu(A) := m_1(\{t \in [0, t_0]: E \cap C(a, t) \neq \emptyset\})$, then (E, μ) satisfies (L^*) at a: here m_1 is the Lebesgue linear measure and $C(a, t)$ denotes the circle of radius t centered at a. Other examples of pairs (E, μ) satisfying (L^*) as well as references to papers related to this condition can be found in [5] and [8]. We notice that if ν is a measure on E that dominates μ (i.e. $\nu(A) = 0$ implies $\mu(A) = 0$) and (E, μ) satisfies (L^*) at $a \in \bar{E}$, then clearly (E, ν) also satisfies (L^*) at a. The purpose of this note is to prove a geometric criterion for a pair (E, μ) to satisfy (L^*) at a point. The criterion will generalize that given in [7] and, in particular, it will contain a criterion of the L-regularity proved recently by Cegrell [2] and Sadullaev [9].

2. The L**-regularity.** Denote by $\mathscr{L}$ the set of all plurisubharmonic functions u in C^N such that $u(z) - \log(1 + |z_1| + \ldots + |z_N|)$ is uniformly bounded from above in C^N. Given a subset E of C^N, the function

$$V_E(z) = \sup\{u(z): u \in \mathscr{L}, u \leqslant 0 \text{ on } E\}$$

is called the *extremal plurisubharmonic function* associated with E (see [11], [10]). The set E is said to be *L-regular at a point* $a \in \bar{E}$ if $V_E^*(a) = 0$, where $V_E^*(z) = \limsup_{z \to w} V_E(w)$ is the upper regularization of V_E. E is said to be *L-regular* if it is L-regular at every point $z \in \bar{E}$. By a result of Zaharjuta [11] and Siciak [10], it can be shown (see [5]) that for each measure μ, if E is a compact subset of $\mathbf{C}^N$ and (E, μ) satisfies (L^*) at $a \in E$, then E is L-regular at a.

We shall need the following important lemma (see [1])

2.1. LEMMA. *Suppose $\{K_n\}$ is an increasing sequence of compact sets in $\mathbf{C}^N$ such that the set $K = \bigcup_{n=1}^{\infty} K_n$ is compact. Then if K is L-regular at a point $a \in K$, we have* $\lim_{n \to \infty} V_{K_n}^*(a) = 0$.

2.2. Remark. As was observed by Klimek [3], Lemma 2.1 remains true if K is bounded (not necessarily closed) since then $\lim_{n \to \infty} V_{K_n}^*(z) = V_K^*(z)$, $z \in \mathbf{C}^N$. This observation can be used to show (see [3]) that for every L-regular compact set E in $\mathbf{C}^N$ the pair (E, c) satisfies (L^*), c denoting the *L-capacity* of E,

$$c(E) = \liminf_{|z| \to \infty} |z| \exp(-V_E^*(z)).$$

Observe that if $N = 1$, $c(E)$ is simply equal to the logarithmic capacity of E.

The following proposition is a slightly stronger version of Cegrell [2], prop. 3.3.

2.3. PROPOSITION (see [8], prop. 4.5). *Let E be a subset of $\mathbf{C}^N$ and h a holomorphic mapping from an open set $U \subset \mathbf{C}^1$ to $\mathbf{C}^N$. Let I be a compact subset of U, L-regular at a point $t_0 \in I$, and such that $h(I) \subset \bar{E}$. If then E is L-regular at every point $h(t)$ for $t \in I \setminus \{t_0\}$, E is also L-regular at $h(t_0)$.*

2.4. COROLLARY (analytic attainment criterion). *Let E be a subset of $\mathbf{C}^N$ and h an analytic mapping from $[0, 1]$ to $\mathbf{C}^N$. If E is L-regular at every point of the set $h((0, 1])$, then E is L-regular at $h(0)$.*

2.5. Remark. Corollary 2.4 is due to Cegrell [2], prop. 3.3 and generalizes a criterion of the L-regularity announced in [7]. The same result has been obtained by Sadullaev [9].

3. The main result.

3.1. THEOREM. *Let E be a subset of $\mathbf{C}^N$. Let μ be a measure on E and let $a \in \bar{E}$. Suppose h is a holomorphic mapping from an open set U in $\mathbf{C}^1$ to a neighborhood of a in $\mathbf{C}^N$. Let I be a compact subset of U, L-regular at a point $t_0 \in I$ such that $h(t_0) = a$. If for each $t \in I \setminus \{t_0\}$ the pair (E, μ) satisfies (L^*) at $h(t)$ then it also satisfies (L^*) at a.*

Proof. Let $\mathscr{F}$ be a subfamily of $\mathscr{P}(C^N)$ satisfying (∗). Fix $b>1$. By the assumptions, for each positive integer n there exist constants $\delta_n>0$ and $C_n>0$ such that for each $f\in\mathscr{F}$,

$$\sup_{z\in F_n}|f(z)|\leqslant C_n b^{\deg f/2},$$

where $F_n:=\{z\in C^N\colon \operatorname{dist}(z,h(I_n))\leqslant\delta_n\}$ with $I_n:=\left\{t\in I\colon |t-t_0|\geqslant\frac{1}{n}\right\}$. Since every ball in C^N is L-regular, by Proposition 2.3 the set $F=\bigcup_{n=1}^{\infty}F_n\cup\{a\}$ is L-regular at a. Hence by Lemma 2.1, setting $G_n:=\bigcup_{k=1}^{n}F_k\cup\{a\}$ gives $\lim_{n\to\infty}V^*_{G_n}(a)=0$. By [10], prop. 3.11, $V^*_{G_n}=V^*_{K_n}$, where $K_n=\bigcup_{k=1}^{n}F_k$ for $n=1,2,\ldots$ Hence, since each K_n is L-regular, there is a positive integer n_0 and a neighborhood V of a such that

$$V_{K_n}(z)<(\log b)/2\qquad\text{for } z\in V.$$

Consequently, since for every polynomial $f\in\mathscr{P}(C^N)$

$$|f(z)|\leqslant\|f\|_{K_n}\exp[\deg f\;V_{K_n}(z)],\qquad z\in C^N,$$

we get for each $f\in\mathscr{F}$,

$$\sup_{z\in V}|f(z)|\leqslant C_0 b^{\deg f}$$

with $C_0=\max\{C_1,\ldots,C_{n_0}\}$. The theorem is proved.

Let now E be a subset of $R^N:=\{z=(z_1,\ldots,z_N)\in C^N\colon \operatorname{Im} z_j=0,\ j=1,\ldots,N\}$ and μ a measure on E. Suppose (E,μ) satisfies the following requirement

(I) For each interval $I=[a_1,b_1]\times\ldots\times[a_N,b_N]\subset E$, (I,μ) satisfies (L^*).

Such a condition for the pair (E,μ) has appeared in a natural way in the theory of quasianalyticity in spaces of integrable functions (see [6]). By the polynomial lemma of Leja and Fubini's theorem, if m_N is the Lebesgue N-dimensional measure, then (E,m_N) satisfies (I). Hence in particular, since m_N is absolutely continuous with respect to the L-capacity c, (E,c) also satisfies (I) (see also Remark 2.2). From Theorem 3.1 we derive

3.2. COROLLARY (analytic attainment criterion for (L*)). *Suppose E is a subset of R^N and μ is a measure on E such that (E,μ) satisfies* (I). *If there exists an analytic mapping $h\colon[0,1]\to\bar{E}$ such that $h((0,1])\subset\operatorname{int}E$, then (E,μ) satisfies* (L*) *at $h(0)$.*

3.3. Remark. A weaker version of Corollary 3.2 (with $\mu=m_N$) was announced in [7].

3.4. Remark. By Cegrell [1], a compact subset E of C^N is L-regular at a point $a\in E$ if and only if for the counting measure μ on E, (E,μ) satisfies (L*) at a. Hence Theorem 3.1 is a generalization of Proposition 2.3.

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Added in proof

In Corollary 3.2 it suffices to assume the curve h is semianalytic (see also [12]).

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