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Non-commutative calculus and Pilgerschritt transformation

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The Pilgerschritt (Pilgrim step) transformation is a method of solving the translation functional equation (cf. [1, 7, 8, 11, 12, 14]). In this paper we give a theory of non-commutative integration (cf. [9, 10, 13, 15, 16]) and non-commutative differentiation (cf. [2, 4, 5]) which allows us to describe the Pilgerschritt transformation in a very natural way.

Let G be a complete Hausdorff group with group operation $*$ and unit element e, and let $l(G)$ be the set of all continuous group homomorphisms φ: $R \to G$, where R denotes the additive group of reals. Let $[a, b] \subset R$ denote a compact non-empty interval. Differentiation of a function F: $V_1 \to V_2$ (V_1, V_2 real Banach spaces) in the classical sense gives a function F': $V_1 \to \mathrm{Hom}(V_1, V_2)$. Analogously we shall see that differentiation of a function F: $R \to G$ gives a function F: $R \to l(G)$.

1. Integration. Let $U^s(e)$ denote the filter base of symmetric neighbourhoods of the unit element $e \in G$. For $V \in U^s(e)$ let $V^* = \{(x, y) | x * y^{-1} \in V\}$. The set $\{V^* | V \in U^s(e)\}$ forms the basis of a filter of entourages of the so-called right uniformity on G. This uniformity is complete because G is complete. The sets V^* are symmetric.

For $V \in U^s(e)$ and $a > 0$ let be

$$V_a = \{(\varphi, \psi) | \varphi, \psi \in l(G),\ \varphi(t) * \psi(t)^{-1} \in V \text{ for } |t| < a\}$$

It is simple to show that $\{V_a | V \in U^s(e), a > 0\}$ forms a basis for the filter of entourages of a complete Hausdorff uniformity on $l(G)$.

A step function f: $[a, b] \to l(G)$ is a function such that there exists a "suitable" subdivision $a = a_0 \leqq a_1 \leqq \ldots \leqq a_n = b$ with f constant over (a_k, a_{k+1}) ($k = 0, 1, \ldots, n-1$). $S[a, b]$ denotes the set of all these step functions. A basis for the filter of entourages of a uniformity on $S[a, b]$ is formed by all sets

$$V_a^{[a,b]} = \{(f, g) | f, g \in S[a, b], (f(x), g(x)) \in V_a, x \in [a, b], V \in U^s(e), a > 0\}.$$

In general this uniformity of $S[a, b]$ is not complete.

So far we have considered three uniform spaces, namely G, $l(G)$ and $S[a, b]$. We define the group G to be admissible for integration if the function

$$\int_{[a,b]} : S[a, b] \to G;\ f \to \int_{[a,b]} f$$

with

$$\int_{[a,b]} f = \left[f\left(\frac{a_n+a_{n-1}}{2}\right)(a_n-a_{n-1})\right] * \cdots * \left[f\left(\frac{a_1+a_0}{2}\right)(a_1-a_0)\right]$$

has a continuous extension on the uniform completion $R[a, b]$ of $S[a, b]$. In this context $a = a_0 \leqq a_1 \leqq \ldots \leqq a_n = b$ denotes a suitable subdivision. Analogously to the integral introduced by Cauchy we define the (product —) integral $\widehat{\int}_{[a,b]} : R[a, b] \to G$, which is a generalization of Cauchy's integral and which is related to Volterra's product integral on the group $GL(n, R)$. Indeed, assume that G is admissible for integration. Then we define $\widehat{\int}_{[a,b]}$ as the continuous extension of $\int_{[a,b]}$ from $S[a, b]$ to $R[a, b]$.

There is a canonical way of interpreting the elements f^* of $R[a, b]$ as functions f: $[a, b] \to l(G)$. If $f^* \in R[a, b]$, then there exists a Cauchy filter $\mathscr{F}_{f^*}$ on $S[a, b]$ which represents f^*. For $F \in \mathscr{F}_f^*$ and $x_0 \in [a, b]$ let be

$$F^{x_0} = \{f(x_0) | f \in F\}.$$

Then $\{F^{x_0} | F \in \mathscr{F}\}$ is a Cauchy filter in $l(G)$ and converges to an element $\varphi \in l(G)$. We define $f(x_0) := \varphi$. A function f given in this way may be called regulated. We have the following.

THEOREM: *A function χ: $[a, b] \to l(G)$ is regulated iff for each $x_0 \in [a, b]$ there exists $\lim_{x \to x_0+0} \chi(x)$ and for each $x_0 \in (a, b]$ there exists $\lim_{x \to x_0-0} \chi(x)$.*

The proof of this theorem follows the proof of the classical analogue, e.g., in [3].

2. Differentiation and Pilgerschritt transformation. Let Φ: $[a, b] \to G$ be an arbitrary function and let be $x_0 \in [a, b)$. The right derivative $d^\varrho\Phi(\xi, x_0)$ of Φ at the argument x_0 and the increment $\xi \geqq 0$ is defined by

$$d^\varrho\Phi(\xi, x_0) = \lim_{m \to \infty} \left[\Phi\left(x_0 + \frac{\xi}{m}\right) * \Phi^{-1}(x_0)\right]^m$$

if this limit exists and the sequence converges uniformly in ξ over a compact interval $[0, \varepsilon]$ $(\varepsilon > 0)$ and is a continuous function in ξ. If $\xi < 0$ we define

$$d^\varrho\Phi(\xi, x_0) = [d^\varrho\Phi(-\xi, x_0)]^{-1}.$$

THEOREM: *The right derivative $d^\varrho\Phi(., x_0)$: $R \to G$; $\xi \to d^\varrho\Phi(\xi, x_0)$ is a continuous group homomorphism and therefore an element in $l(G)$.*

Because of the supposed continuity the proof may be reduced to proving that

$$d^\varrho\Phi(\xi + \eta, x_0) = d^\varrho\Phi(\xi, x_0) * d^\varrho\Phi(\eta, x_0)$$

for rational ξ, η, $\xi + \eta \in (0, \varepsilon]$ (cf. [8] p. 10).

There is an analogous definition and theorem for the left derivative $d^\lambda\Phi(\xi, x_0)$

3. Connection between differentiation and integration.

THEOREM: *If* $f\colon [a, b] \to l(G)$ *is regulated,* $x_0 \in [a, b)$, $F(t) = \int_{[a,t]} f$, *then*

$$d^{\varrho}F(\xi, x_0) = (\lim_{x \to x_0+0} f(x))(\xi) \quad \text{for} \quad \xi \in R\,,$$

(*if* $x_0 \in (a, b]$ *then* $d^{\lambda}F(\xi, x_0) = (\lim_{x \to x_0-0} f(x))(\xi)$ *for* $\xi \in R$).

Proof. If $m \in N$ let $g_m\colon [x_0, \xi) \to l(G)$ be such that

$$g_m(t) = f(s) \quad \text{if } t \in \left[x_0 + \frac{n\xi}{m}, x_0 + \frac{(n+1)\xi}{m}\right]$$

where $n = 0, 1, 2, \ldots, m-1$ and $t = x_0 + \dfrac{n\xi}{m} + s$. Define $\gamma_m\colon [x_0, \xi) \to l(G)$ by

$$\gamma_m(t) = \begin{cases} g_m(t) & \text{if } t \neq x_0 + \dfrac{n\xi}{m}, \\ \lim\limits_{x \to x_0+0} f(x) & \text{if } t = x_0 + \dfrac{n\xi}{m}, \quad n = (0, 1, \ldots, m-1)\,. \end{cases}$$

Then $\widehat{\int}_{[x_0,\xi]} g_m = \widehat{\int}_{[x_0,\xi]} \gamma_m$ and $\gamma_m (m \to \infty)$ tends in $R[a, b]$ to the constant function with value $\lim_{x \to x_0+0} f(x) \in l(G)$. Therefore

$$\begin{aligned} d^{\varrho}F(\xi, x_0) &= \lim_{n \to \infty} \Big[\widehat{\int}_{[a,x_0+\xi/m]} f * \Big[\widehat{\int}_{[a,x_0]} f\Big]^{-1}\Big]^m \\ &= \lim_{m \to \infty} \Big[\widehat{\int}_{[x_0,x_0+\xi/m]} f * \widehat{\int}_{[a,x_0]} f * \Big[\widehat{\int}_{[a,x_0]} f\Big]^{-1}\Big]^m \\ &= \lim_{m \to \infty} \Big[\widehat{\int}_{[x_0,x_0+\xi/m]} f\Big]^m = \lim_{m \to \infty} \widehat{\int}_{[x_0,x_0+\xi]} g_m \\ &= \lim_{m} \widehat{\int}_{[x_0,x_0+\xi]} \gamma_m = (\lim_{x \to x_0+0} f(x))(\xi)\,. \end{aligned}$$

The existence of $d^{\varrho}F(\xi, x_0)$ is obvious.

A function $f\colon [a, b] \to l(G)$ is said to be C^1 if $d^{\varrho}f(., x_0) = d^{\lambda}(., x_0)$ for $x_0 \in (a, b)$ and if the function $df(.,)\colon [a, b] \to l(G)$ defined by

$$df(., x_0) = \begin{cases} d^{\varrho}f(., x_0) & \text{for} \quad x_0 \in [a, b) \\ d^{\lambda}f(., b) & \text{for} \quad x_0 = b \end{cases}$$

is continuous.

A function $f\colon [a, b] \to l(G)$ can be multiplied by a real number. If $\tau \in R$ let $\tau f\colon [a, b] \to l(G)$ be defined as $\tau f(\xi, x_0) = f(\tau\xi, x_0)$.

Examples:

1. If $G = R$ then all these definitions and theorems prove to be classical; the set $l(R)$ has only to be identified with R by $(\varphi\colon t \to at) \leftrightarrow a$.

2. If $G = GL(n, R)$, then we shall show in the more general case of the group of units in a real Banach algebra that G is admissible for integration.

A function f: $[a, b] \to GL(n, R)$ is C^1 iff it is C^1 in the classical sense (but this classical sense is differentiability in our sense concerning the additive group of all real (n, n)-matrices). We have

$$df(\xi, x_0) = \lim_{m\to\infty}\left[f\left(x_0+\frac{\xi}{m}\right)\cdot f(x_0)^{-1}\right]^m$$
$$= \lim_{m\to\infty}\left[\left(f(x_0)+\frac{\xi}{m}f'(x_0)\right)\cdot f(x_0)^{-1}\right]^m$$
$$= \lim_{m\to\infty}\left[E+\frac{\xi}{m}f'(x_0)\cdot f(x_0)^{-1}\right]^m = \exp(\xi f'(x_0)f(x_0)^{-1}),$$

(E is the unit matrix). Let $M_n(R)$ be the set of all real (n, n)-matrices. Each element $\varphi \in l(GL(n, R))$ has the form $\varphi(t) = \exp(tA)$ with $A \in M_n(R)$. Therefore a function f: $[a, b] \to l(GL(n, R))$ may be interpreted as a function f: $[a, b] \to M_n(R)$. The elementary Euler method for numerical integration of linear differential equations shows (cf. [13] and [8]) that for a function f and $t \in [a, b]$ we have $\widehat{\int\limits_{[a,t]}} f = F(t)$, where $F(t)$ is the solution of the matrix differential equation

$$F'(t) = f(t)F(t) \text{ with initial condition } F(a) = E.$$

If we have f: $[a, b] \to M_n(R)$ and a real number τ then τf: $[a, b] \to M_n(R)$ is given by $\tau f(x_0) = \tau \cdot f(x_0)$.

Definition (Pilgerschritt transformation): Let G be a complete Hausdorff group which is admissible for integration and let f: $[0, 1] \to G$ be a C^1 function with $f(0) = e$. Then the Pilgerschritt transform $\tilde{f}$: $[0, 1] \to G$ is given by $\tilde{f}(\tau) = \widehat{\int\limits_{[0,1]}} \tau\, df$.

Examples:

1. If $G = R$ then $\tilde{f}(\tau) = \int_0^1 \tau f'(t)dt = \tau(f(1)-f(0)) = \tau f(1)$.

2. If $G = GL(n, R)$ then $\tilde{f}(\tau) = X(1)$, where X is the solution of the matrix differential equation $X'(t) = \tau f'(t)f(t)^{-1}\, X(t)$ with initial condition $X(0) = E$.

4. Examples for admissible groups.

THEOREM: *Let be $G = V$ the additive group of a locally convex linear space. Then V is admissible for integration.*

Proof. Use the fact that there exists a family $\{\|\ \|_p\}_{p\in P}$ of seminorms which generates the topology in V (cf. [6]).

THEOREM: *The group G of units of a real Banach algebra B with unit element 1 is admissible for integration.*

Proof. If $h \in l(G)$, then $h(t) = \exp(tA)$ with $A \in \boldsymbol{B}$ and conversely if $B \in \boldsymbol{B}$, then $t \to \exp(tB)$ is in $l(G)$. Therefore the notation $h(t) = \exp(th^*)$ is used and $l(g)$ is identified with $\boldsymbol{B}$. If $A, B \in \boldsymbol{B}$ and $|t|<a$ and $b = \max\{\|A\|, \|B\|\}$, then

$$\|\exp(tA)-\exp(tB)\| \leqq \|A-B\| a e^{ab} .$$

We first state two Lemmae.

LEMMA 1: *If* $s_1: [0,1] \to l(G)$ *and* $s_2: [0,1] \to l(G)$ *are step functions and* $0 = a_0 \leqq a_1 \leqq \ldots \leqq a_n = 1$ *is a subdivision suitable for* s_1 *and* s_2 *and if* $s_1(t) = \exp(tA_k)$ *and* $s_2(t) = \exp(tB_k)$ *for* $t \in (a_{k-1}, a_k)$ *then*

$$\| \int_{[0,1]} s_1 [\int_{[0,1]} s_2]^{-1} - 1 \| \leqq \exp(3\max(\|A_k\|, \|B_k\|)) \max(\|A_k - B_k\|) .$$

We say that $A, B \in \boldsymbol{B}$ and $\varepsilon>0$ fulfil condition (*) iff for every $|t|<1$

$$\|\exp(tA)\exp(-tB)-1\| < \varepsilon \quad \text{and} \quad \|\exp(tB)\exp(-tA)-1\| < \varepsilon . \tag{*}$$

If $\varepsilon<1$ and $A, B \in \boldsymbol{B}$ and ε fulfil condition (*) and if $B, C \in \boldsymbol{B}$ and ε fulfil condition (*) also, then A, C and 3ε fulfil condition (*).

LEMMA 2: *If* $M>0$ *and* $\eta>0$ *are given, then there exists* $\varepsilon>0$ *such that* $\|A-B\|<\eta$ *if* $\|A\|<M$ *and* A, B *and* ε *fulfil condition* (*).

To prove that G is admissible for integration we may suppose that $[a, b] = [0, 1]$. We have to show that each Cauchy filter on $S[0, 1]$ has a convergent image under the integral $\int_{[0,1]}$. If $\tilde{\varepsilon}>0$ and a Cauchy filter $\mathscr{F}$ on $S[0, 1]$ are given, then there exists an element $F \in \mathscr{F}$ such that $s_1, s_2 \in F$ implies

$$\| \int_{[0,1]} s_1 [\int_{[0,1]} s_2]^{-1} - 1 \| < \varepsilon$$

The Cauchy filter $\mathscr{F}$ converges pointwise to a regulated function f: $[0, 1] \to l(G)$. Each regulated function on $[0, 1]$ is bounded, which means that there exists a number $M>0$ such that $\|(f(x))\|<M$ for $x \in [0, 1]$. If η is given with $0<\eta<M$, then Lemma 2 implies the existence of a number $1>\varepsilon>0$ which ensures for all $A, B \in \boldsymbol{B}$ with $\|A\|<2M$ and A, B, ε fulfilling condition (*) that $\|A-B\|<\eta$. Choose $F \in \mathscr{F}$ such that for each $s_1, s_2 \in F$ and for each $x \in [0, 1]$ condition (*) is fulfilled for $(s_1(x))^*$, $(s_2(x))^*$ and $\varepsilon/3$. Then condition (*) is fulfilled for $(s_1(x))^*$, $(f(x))^*$ and ε. From Lemma 2 we may conclude that $\|(s_1(x))^* - (f(x))^*\| < \eta$ and therefore $\|(s_1(x))^*\| < M+\eta < 2M$. Again from Lemma 2 we conclude that $\|(s_1(x))^* - (s_2(x))^*\| < \eta$ for $x \in [0, 1]$. Therefore we have

$$\|(s_2(x))^*\| < \eta + 2M < 3M .$$

With Lemma 1 we get

$$\| \int_{[0,1]} s_1 [\int_{[0,1]} s_2]^{-1} - 1 \| < e^{9M} \eta .$$

Now we take $\eta = \min(\varepsilon e^{-9M}, M/2)$. Then F is so small that (S) holds true, q.e.d.

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