ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
DCXXIII PRACE MATEMATYCZNE, ZESZYT 23 1982

# On some Fréchet spaces of analytic functions

Marek Jarnicki

**Abstract.** In the paper we present a characterization of domains of analyticity with respect to some Fréchet spaces of $K$-analytic functions on Riemann domains over $K^n$, $K = R$ or $K = C$.

**1. Introduction.** The aim of the paper is to generalize some results of [1] and [3]. First we shall fix some basic notations.

We denote by $K$ either the field of real numbers $R$ or complex numbers $C$. For $\zeta = (\zeta_1, \ldots, \zeta_n) \in K^n$ and $r>0$, let

$$B(\zeta, r) = \{\xi = (\xi_1, \ldots, \xi_n) \in K^n: \ |\xi_j - \zeta_j| < r, \ j = 1, \ldots, n\}.$$

Let $(X, p)$ be a connected Riemann domain over $K^n$, $p = (p_1, \ldots, p_n)\colon X \to K^n$. For $x \in X$, let $d(x)$ denote the supremum of all $r>0$ such that there exists an open neighbourhood $\hat{B}(x, r)$ of the point $x$ which is mapped homeomorphically by $p$ onto $B(p(x), r)$. Let us put $\hat{B}(x) = \hat{B}(x, d(x))$, $p_x = p|_{\hat{B}(x)}$. For $A \subset X$, let $d(A) = \inf\{d(x)\colon x \in A\}$. We denote by $\mathscr{A}(X)$ the space of all $K$-analytic functions on $X$. Let $D^\alpha$ denote the operator of $\alpha$-derivative, i.e.

$$(D^\alpha f)(x) = \frac{\partial(f \circ p_x^{-1})}{\partial\xi_1^{\alpha_1} \ldots \partial\xi_n^{\alpha_n}}(p(x)), \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in Z_+^n, \quad f \in \mathscr{A}(X), \quad x \in X.$$

For $F \subset \mathscr{A}(X)$, we put $F^* = \{p_1, \ldots, p_n\} \cup \{D^\alpha f\colon f \in F, \alpha \in Z_+^n\}$,

$$F_{xr} = \{f \in F\colon \exists \tilde{f} \in \mathscr{A}(B(p(x), r))\colon f = \tilde{f} \circ p_x \text{ in } \hat{B}(x)\},$$

$$\tilde{F}_{xr} = \{\tilde{f}\colon f \in F_{xr}\}, \quad x \in X, \quad r > d(x).$$

For a function $f\colon E \to C$ we write $\|f\|_E := \sup\{|f(x)|\colon x \in E\}$.

Definition 1. A pair $(F, t)$ is said to be *a natural Fréchet space in* $\mathscr{A}(X)$ if $F$ is a vector subspace of $\mathscr{A}(X)$, $t$ is a topology of a Fréchet space on $F$ and $t$ is stronger than the topology $t_X^c$ of uniform convergence on every compact subset of $X$.

In the sequel, we shall always denote by $\Gamma$ a family of seminorms on $F$ which generates the topology $t$ and which satisfies the following condition: for every $q_1, q_2 \in \Gamma$ there exists a norm $q \in \Gamma$ such that $q_1, q_2 \leqslant q$.

For $q \in \Gamma$ and $M, c \geq 1$, let us set:

$$H(F; q, M, c) = \{x \in X: \forall f \in F, \forall \alpha \in \boldsymbol{Z}_+^n: |D^\alpha f(x)| \leq \alpha! M c^{|\alpha|} q(f)\}.$$

Definition 2. A family $F \subset \mathscr{A}(X)$ is said to be *regular* if for every $x \in X$, $r > d(x)$ there exists a natural Fréchet space $(G^{xr}, t^{xr})$ in $\mathscr{A}(B(p(x), r))$ such that $F_{xr} \subset G^{xr}$.

The main result of the paper is the following:

THEOREM 1. *Let $(F, t)$ be a regular natural Fréchet space in $\mathscr{A}(X)$. Then the following conditions are equivalent*:

(i) *$(X, p)$ is an $F$-domain of $\boldsymbol{K}$-analyticity*;

(ii) *there exists a set $N \subset F$ of the second Baire category in $(F, t)$ such that for every $f \in N$: $(X, p)$ is an $\{f\}$-domain of $\boldsymbol{K}$-analyticity*;

(iii) *$F^*$ separates points in $X$ and*

(1) *for every $q \in \Gamma$, $M, c \geq 1$: $d(H(F; q, M, c)) > 0$*;

(iv) *$F^*$ separates points in $X$ and*

(2) *for every set $E \subset X$ with $d(E) = 0$ and for every $c \geq 1$ there exists $f \in F$ such that*
$$\sup\left\{\frac{1}{\alpha! c^{|\alpha|}} \|D^\alpha f\|_E : \alpha \in \boldsymbol{Z}_+^n\right\} = +\infty.$$

In particular cases, Theorem 1 generalizes the well-known Cartan-Thullen theorem, some results of [1] and Theorem 10.1 from [3].

**2. Domains of $\boldsymbol{K}$-analyticity.** For $f \in \mathscr{A}(X)$, let $T_x f$ denote the Taylor series of $f$ at $x$, i.e. $(T_x f)(\zeta) = \sum_{\alpha \in \boldsymbol{Z}_+^n} \frac{D_\alpha f(x)}{\alpha!}(\zeta - p(x))^\alpha$. Put $d(T_x f) = \sup\{r > 0: T_x f$ is convergent in $B(p(x), r)\}$.

For a family $F \subset \mathscr{A}(X)$, let us consider the condition:

(RC) $\quad \forall x \in X\ \exists r(x) > 0: \forall f \in F: d(T_x f) \geq r(x)$.

Obviously this condition is interesting only in the real case. Note that if $\boldsymbol{K} = \boldsymbol{R}$ then the family $F = \mathscr{A}(X)$ does not satisfy (RC).

It is easy to prove

LEMMA 1. *If $F$ satisfies* (RC) *then $(X, p)$ is an $F$-domain of $\boldsymbol{K}$-analyticity if and only if $F^*$ separates points in $X$ and for every $x \in X$, $r > d(x)$: $F_{xr} \neq F$.*

**3. Natural Fréchet spaces.** Using the Baire property, it is easy to get (cf. [1], [3]):

LEMMA 2. *If $(F, t)$ is a natural Fréchet space in $\mathscr{A}(X)$ then $F$ satisfies* (RC).

COROLLARY 1. *In the real case the whole space $\mathscr{A}(X)$ cannot be endowed with the structure of a natural Fréchet space.*

The following simple lemma will be useful in the sequel (cf. [1]):

Lemma 3 (Generalized Cauchy inequalities). *Let $(F, t)$ be a natural Fréchet space in $\mathscr{A}(X)$. Then for every compact set $K \subset X$ there exists a constant $c(K) > 0$ such that: for every $0 < \tau < c(K)$ there exist $c_\tau > 0$, $q_\tau \in \Gamma$:*

$$\|D^\alpha f\|_K \leqslant \frac{\alpha! c_\tau}{\tau^{|\alpha|}} q_\tau(f), \quad f \in F, \quad \alpha \in Z_+^n .$$

Now we shall present an important example of a natural Fréchet space for the real case (cf. [1]).

Let $\mathscr{E} = (\mathscr{E}_1, \ldots, \mathscr{E}_s)$ be a differential operator in $R^n$ such that

$$\mathscr{E}_j = \sum_{|\alpha| \leqslant N} a_{j\alpha} D^\alpha , \quad \text{where} \quad a_{j\alpha} \in \mathscr{A}(R^n), \quad j = 1, \ldots, s, \quad |\alpha| \leqslant N ,$$

(*) (ellipticity of $\mathscr{E}$) for every open set $U \subset R^n$ and for every function $f$ continuous in $U$, if $\mathscr{E}f = 0$ (in the sense of the distribution theory) then $f \in \mathscr{A}(U)$.

Set $\mathscr{E}(U) = \{f \in \mathscr{A}(U)\colon \mathscr{E}f = 0\}$. By (*), $(\mathscr{E}(U), t_U^c)$ is a Fréchet space.

For a connected Riemann domain $(X, p)$ over $R^n$, set

$$\mathscr{E}(X) = \{f \in \mathscr{A}(X)\colon \forall x \in X\colon f \circ p_x^{-1} \in \mathscr{E}(B(p(x), d(x)))\} .$$

It is easily seen that $(\mathscr{E}(X), t_X^c)$ is a natural Fréchet space in $\mathscr{A}(X)$.

**4. Regular families.** Obviously in the complex case every family $F \subset \mathscr{O}(X) = \mathscr{A}(X)$ is regular — we can put $(G^{xr}, t^{xr}) = (\mathscr{O}(B), t_B^c)$, where $B = B(p(x), r)$. In the real case this is not true even if $(F, t)$ is a natural Fréchet space in $\mathscr{A}(X)$; consider the following example (due to J. Siciak):

Let $D = \{z \in C\colon |z| < 1\}$, $X = (-1, 1) \subset R$, $p = id_X$, $F = \{f|_X\colon f \in \mathscr{O}(D)\}$. Let $t$ denote the topology on $F$ which is transported from the topology $t_D^c$ by the bijection $\mathscr{O}(D) \ni f \to f|_X \in F$. Obviously $(F, t)$ is a natural Fréchet space in $\mathscr{A}(X)$. However, $F$ is not regular; for instance, the functions $f_k(z) = (z - (1 + i/k))^{-1}$, $k \in N$, belong to $F_{02}$ but $d(T_1 f_k) = 1/k$.

Note that the space $\mathscr{E}(X)$ is regular — we can put $(G^{xr}, t^{xr}) = (\mathscr{E}(B), t_B^c)$, where $B = B(p(x), r)$.

**5. The proof of Theorem 1.**

The implication (i) $\Rightarrow$ (ii) (the method of the proof is taken from the proof of Theorem 10.1 in [3]).

Let $A = \{x_k\colon k \in N\}$ be a countable dense subset of $X$ such that $A = p^{-1}(p(A))$. For $x = x_k$, $r = d(x_k) + 1/l$, let us put $F_{kl} = F_{xr}$, $\tilde{F}_{kl} = \tilde{F}_{xr}$ and let $(G^{kl}, t^{kl})$ be a natural Fréchet space in $\mathscr{A}(B(p(x), r))$ such that $\tilde{F}_{kl} \subset G^{kl}$. Denote by $\Gamma_{kl}$ a family of seminorms generating the topology $t^{kl}$. It may easily be proved that the space $F_{kl}$ with the topology generated by the following seminorms

$$F_{kl} \ni f \to q(f), \quad q \in \Gamma ,$$

$$F_{kl} \ni f \to q(\tilde{f}), \quad q \in \Gamma_{kl}$$

is a Fréchet space. In view of Lemma 1, $F_{kl} \neq F$, so by the Banach theorem $F_{kl}$ is of the first Baire category in $F$.

Now let $H_{kl} = \{f \in F\colon \forall \alpha \in Z_+^n\colon\ D^\alpha f(x_k) = D^\alpha f(x_l)\}$, where the numbers $k, l$ run over all $k, l \in N$ for which $x_k \neq x_l$ and $p(x_k) = p(x_l)$. The set $H_{kl}$ is a closed subspace of $F$ (Lemma 3). Since $H_{kl} \neq F$ (Lemma 1), so $H_{kl}$ is nowhere dense in $F$.

It may be easily proved that it is now sufficient to put

$$N = F \setminus \big(\bigcup_{k,l} F_{kl} \cup \bigcup_{k,l} H_{kl}\big).$$

The implication (ii) $\Rightarrow$ (iii). It is seen that for any $x \in H(F; q, M, c)$ we have $d(T_x f) \geqslant 1/c$, $f \in F$. Hence, by Lemma 1, $d(x) \geqslant 1/c$.

The implication (iii) $\Rightarrow$ (iv) (the method of the proof is taken from [1]). Suppose that (2) does not hold true. Then for some $E \subset X$ with $d(E) = 0$ and for some $c \geqslant 1$ we have: $F = \bigcup_{k \in N} kS$, where

$$S := \{f \in F\colon \forall \alpha \in Z_+^n\colon \|D^\alpha f\|_E \leqslant \alpha!\, c^{|\alpha|}\}.$$

The set $S$ is closed in $F$ (Lemma 3) and absolutely convex. By the Baire property of $F$, there exist $\varepsilon > 0$ and a norm $q \in \Gamma$ such that $\{f \in F\colon q(f) < 2\varepsilon\} \subset S$. Hence

$$\|D^\alpha f\|_E \leqslant \alpha! \frac{1}{\varepsilon} c^{|\alpha|} q(f), \quad f \in F, \quad \alpha \in Z_+^n.$$

Thus $E \subset H\left(F; q, \frac{1}{\varepsilon}, c\right)$ which contradicts (1).

The implication (iv) $\Rightarrow$ (i). Suppose by absurd, that for some $x \in X$ and $r > d(x)$ we have $F_{xr} = F$. Let $(G^{xr}, t^{xr})$ be chosen accordingly to Definition 2 and let $\Gamma_{xr}$ be a family of seminorms generating $t^{xr}$. Put $K := \bar{B}(p(x), d(x))$. By Lemma 3, for $0 < \tau < c(K)$ there exist $c_\tau > 0$, $q_\tau \in \Gamma_{xr}$ such that:

$$\|D^\alpha g\| \leqslant \frac{\alpha!\, c_\tau}{\tau^{|\alpha|}} q_\tau(g), \quad g \in G^{xr}, \quad \alpha \in Z_+^n.$$

Let us set $E = \hat{B}(x)$ $(d(E) = 0,\ [2])$, $c = \frac{1}{\tau}$. We have

$$\|D^\alpha f\|_E \leqslant \|D^\alpha \tilde{f}\|_K \leqslant \alpha!\, c_\tau c^{|\alpha|} q_\tau(f), \ f \in F, \ \alpha \in Z_+^n.$$

We get a contradiction. The proof is concluded.

**6. The case when** $t = t_X^c$. Throughout this section we assume that $(F, t_X^c)$ is a natural Fréchet space in $\mathscr{A}(X)$. For a compact set $K \subset X$ and for constants $M, c \geqslant 1$, we put:

$$\hat{K}_F(M, c) := \left\{x \in X\colon \forall f \in F, \forall m \in Z_+\colon \sum_{|\alpha| \leqslant m} \frac{|D^\alpha f(x)|}{\alpha!} \leqslant M c^m \sum_{|\alpha| \leqslant m} \frac{\|D^\alpha f\|}{\alpha!} K\right\},$$

$$\hat{K}_F(M) := \{x \in X\colon \forall f \in F\colon |f(x)| \leqslant M \|f\|\ \},$$

$$\hat{K}_F := \hat{K}_F(1).$$

Observe that if $F$ is *d-stable* (i.e. $\forall f \in F, \forall \alpha \in Z_+^n: D^\alpha f \in F$) then $\hat{K}_F(M, c) = \hat{K}_F(M)$ and if $F$ is *p-stable* (i.e. $\forall f \in F, \forall m \in N: f^m \in F$) then $\hat{K}_F(M) = \hat{K}_F$ (comp. [1]).

PROPOSITION 1. *The condition* (1) *is equivalent to* $(t = t_X^c!)$
(1′) *for every compact set* $K \subset X$ *and for every* $M, c \geqslant 1$:

$$d(\hat{K}_F(M, c)) > 0 .$$

*If F is d-stable then the condition* (2) *is equivalent to*
(2′) *for every set* $E \subset X$ *with* $d(E) = 0$ *there exists a function* $f \in F$ *for which* $\|f\|_E = +\infty$.

Proof. Let us fix a constant $c_0 \geqslant 1$ such that $\sum_{|\alpha| \leqslant m} 1 \leqslant c_0^m$, $m \in Z_+$. For a compact set $K \subset X$, let $q_K$ denote the seminorm given by the formula $F \ni f \to \|f\|_K$.

It is easily seen that $H(F; q_K, M, c) \subset \hat{K}_F(M, cc_0)$, hence $(1') \Rightarrow (1)$.

Conversely, in view of Lemma 3, for every $0 < \tau < c(K)$ there exist $c_\tau \geqslant 1$ and a compact set $K_\tau \in X$ for which

$$\hat{K}_F(M, c) \subset H\left(F; q_K, Mc_\tau, \frac{cc_0}{\tau}\right),$$

hence $(1) \Rightarrow (1')$.

Obviously $(2') \Rightarrow (2)$. For the proof of the implication $(2) \Rightarrow (2')$, suppose by absurd that for some $E \subset X$ with $d(E) = 0$ we have $\|f\|_E < +\infty$, $f \in F$. Then $F = \bigcup_{k \in N} kT$, where $T := \{f \in F: \|f\|_E \leqslant 1\}$. The set $T$ is closed and absolutely convex so, as in the proof of Theorem 1, for some $\varepsilon > 0$ and $K \subset X$ we have $\|f\|_E \leqslant \frac{1}{\varepsilon}\|f\|_K$, $f \in F$. Since $F$ is $d$-stable, so in view of Lemma 3,

$$\|D^\alpha f\|_E \leqslant \frac{1}{\varepsilon}\|D^\alpha f\|_K \leqslant \frac{1}{\varepsilon} \frac{\alpha! c\tau}{\tau^{|\alpha|}} \|f\|_{K_\tau}, \; f \in F, \; \alpha \in Z_+^n .$$

We get the contradiction.

The proof is completed.

COROLLARY 2. *If* $X = D$ *is a domain in* $C^n$, $p = id_D$, $(F, t) = (\mathcal{O}(D), t_D^c)$, *then from Theorem* 1 *and Proposition* 1 *we get the well-known Cartan-Thullen theorem on the characterization of domains of holomorphy.*

COROLLARY 3. *If* $X = D$ *is a domain in* $R^n$, $p = id_D$, $(F, t) = (\mathcal{E}(D), t_D^c)$, *then from Theorem* 1 *and Proposition* 1 *we get the main results of* [1].

## References


[1] A. Andreotti, M. Nacinovich, *Théorie élémentaire de la convexité*, R. C. P. 25 (Strasbourg), vol. 24 (1977).

[2] R. Narasimhan, *Several Complex Variables*, The University of Chicago Press (1971).

[3] K. Rusek, J. Siciak, *Maximal analytic extensions of Riemann domains over topological vector spaces*, Infinite Dimensional Holomorphy and Applications, North-Holland Publishing Company (1977).



INSTITUTE OF MATHEMATICS
JAGELLONIAN UNIVERSITY
KRAKÓW (POLAND)