ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
CCCCXLI PRACE MATEMATYCZNE, ZESZYT 18 1977

Wiesław Niedoba

On the Difference Method for a Non-Linear Parabolic Functional-Differential Equation

1. In this paper we shall consider the following functional-differential equation

$$\frac{\partial u}{\partial t} = f\left(t, x_1, \ldots, x_p, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_p}, \frac{\partial^2 u}{\partial x_1^2}, \ldots, \frac{\partial^2 u}{\delta x_p^2}, u(t, \cdot)\right) \tag{1.1}$$

We construct the corresponding difference equation and we prove the convergence of the difference method.

2. We shall consider the following sets

$$E:\ 0 \leqslant t \leqslant d,\ 0 \leqslant x_j \leqslant \sigma,\ d > 0,\ \sigma > 0\ (j = 1, \ldots, p) \tag{2.1}$$

$$P = \mathop{X}_{j=1}^{p} [0, \sigma] \tag{2.2}$$

$$D = \begin{cases} 0 \leqslant t \leqslant d,\ 0 \leqslant x_j \leqslant \sigma,\ -\infty < u < +\infty \\ -\infty < q_j < +\infty,\ -\infty < \omega_j < +\infty,\ s \in B(P) \end{cases} \tag{2.3}$$

where $B(P)$ is the set of the bounded functions for $x \in P$ with norm

$$\|s\| = \max_{x \in P} |s(x)| \tag{2.4}$$

and the nodal points

$$\begin{cases} t^{\mu} = \mu \cdot k,\ x_j^{v} = v \cdot h,\ \mu = 0, 1, \ldots, N_h,\ v = 0, 1, \ldots, N_k\ (j = 1, \ldots, p) \\ 0 < k = \dfrac{d}{N_k},\ 0 < h = \dfrac{\sigma}{N_h} \end{cases} \tag{2.5}$$

N_h, N_k being two natural numbers.
The nodal points with coordinates

$$(t^{\mu}, x^{m}) \quad \text{where} \quad x^{m} = (x_1^{m_1}, \ldots, x_p^{m_p}) \tag{2.6}$$

are characterized by the sequence of indices

$$M = (\mu, m) \quad \text{where} \quad m = (m_1, \ldots, m_p) \tag{2.7}$$

We shall define the following sets
Z = the set of the sequence $M = (\mu, m)$, such that $0 \leqslant \mu \leqslant N_k$ and $0 \leqslant m_j \leqslant N_h$ $(j = 1, \ldots, p)$
and
Z^+ = the set of the sequences $M = (\mu, m)$ such that $0 \leqslant \mu \leqslant N_k$, $0 \leqslant m_j \leqslant N_h$, for all indices j except at most one value j, for which $-1 \leqslant m_j \leqslant N_h + 1$. We shall also use the nodal points characterized by the sequences

(2.8) $$M = (\mu+1, m), j(M) = (\mu, j(m)), \; -j(M) = (\mu, -j(M))$$

where

$$j(m) = (m_1, \ldots, m_{j-1}, m_j+1, m_{j+1}, \ldots, m_p)$$
$$-j(m) = (m_1, \ldots, m_{j-1}, m_j-1, m_{j+1}, \ldots, m_p)$$

Suppose that to each nodal point characterized by the sequence there corresponds a number v^M.
We shall introduce the following definitions

(2.9) $$\begin{cases} v^M = \frac{1}{k}(v^{\omega(M)} - v^M) \\ v^{M_j} = \frac{1}{2h}(v^{j(M)} - v^{-j(M)}) \\ v^{M\Delta} = (v^{M_1}, \ldots, v^{M_p}) \\ v^{M_{jj}} = \frac{1}{h^2}(v^{j(M)} - 2v^M + v^{-j(M)}) \\ v^{M\Box} = (v^{M_{11}}, \ldots, v^{M_{pp}}) \; (j = 1, \ldots, p) \\ \tilde{v}^\mu(x) = \sum_{M \in \tilde{Z}^\mu} v^M \chi_{I_M}(x) \end{cases}$$

where

(2.10) $$\tilde{Z}^\mu = \{M\colon 0 \leqslant m_j \leqslant N_h - 1, (j = 1, \ldots, p)\}$$

(2.11) $$I_M = \{X\colon m_i h \leqslant x_i < (m_i+1)h \; (i = 1, \ldots, p)\}$$

and

(2.12) $$\chi_{I_M}(x) = \begin{cases} 1 & x \in I_M \\ 0 & x \notin I_M \end{cases}$$

In the sequel we shall use the following assumptions H.

3. ASSUMPTIONS H

1°. Assume that the function $f(t, x, u, q, w, s)$, where $x = (x_1, \ldots, x_p)$, $q = (q_1, \ldots, q_p)$, $w = w_1, \ldots, w_p$, is defined in the set D (2.3) and is of the class C^1 as the function of (t, x, u, q) and of the class C^0 as the function of s.

2°. The function $f(t, x, u, q, w, s)$ satisfies the conditions

$$(3.1)\qquad \left|\frac{\partial f}{\partial q_j}\right| \leqslant \Gamma, \quad 0 < g \leqslant \frac{\partial f}{\partial w_j} \leqslant G$$

and

$$(3.2)\qquad \begin{cases} f(t, x, u, q, w, s) - f(t, x, \tilde{u}, q, w, s) \leqslant L|u - \tilde{u}| \\ f(t, x, u, q, w, s) - f(t, x, u, q, w, \tilde{s}) \leqslant K\|s - \tilde{s}\| \end{cases}$$

3°. The time interval k and the space interval h are chosen to satisfy

$$(3.3)\qquad \frac{g}{h} - \frac{\Gamma}{2} \geqslant 0, \quad \frac{2pG}{h^2} - \frac{1}{k} \leqslant 0$$

The function $u(t, x)$ is of the class C^2 in the set E, and satisfies the functional — equation (1.1) and the boundary conditions.

$$(3.4)\qquad u(0, x) = \varphi(x)$$

$$(3.5)\qquad \begin{cases} \beta_1^j \dfrac{\partial u}{\partial x_j} + \gamma_1^j u(t, x) = 0, & x_j = 0 \\ & \qquad (j = 1, \ldots, p) \\ \beta_2^j \dfrac{\partial u}{\partial x_j} + \gamma_1^j u(t, x) = 0, & x_j = \sigma \end{cases}$$

where

$$(\beta_1^j)^2 + (\gamma_1^j)^2 \neq 0 \quad \text{and} \quad \beta_1^j \cdot \gamma_1^j = 0$$

and

$$(\beta_2^j)^2 + (\gamma_2^j)^2 \neq 0, \quad \beta_2^j \cdot \gamma_2^j = 0.$$

We suppose first that $\beta_1^j \neq 0$ and $\beta_2^j \neq 0$.

Denote by u^M the value of the solution of equation (1.1) at the nodal point (2.6).

We shall define the value of the solution at the points $x_j = -h$ and $x_j = \sigma + h$.

$$(3.6)\qquad \begin{cases} u^{-j(M)} = u^{j(M)} & \text{for} \quad M = (\mu, m_1, \ldots, m_{j-1}, 0, m_{j+1}, \ldots, m_p) \in Z \\ u^{j(M)} = u^{-j(M)} & \text{for} \quad M = (\mu, m_1, \ldots, m_{j-1}, N_h, m_{j+1}, \ldots, m_p) \in Z \end{cases}$$

Hence it follows that u^M satisfies the following boundary conditions.

$$(3.7)\qquad \begin{array}{lll} \beta_1^j u^{M_j} + \gamma_1^j u^M = 0 & \text{for} & M = (\mu, m_1, \ldots, m_{j-1}, 0, m_{j+1}, \ldots, m_p) \in Z \\ \beta_2^j u^{M_j} + \gamma_2^j u^M = 0 & \text{for} & M = (\mu, m_1, \ldots, m_{j-1}, N_h, m_{j+1}, \ldots, m_p) \in Z \end{array}$$

where

$$(\beta_i^j)^2 + (\gamma_i^j)^2 \neq 0 \quad \text{and} \quad \beta_i^j \cdot \gamma_i^j = 0, \ (j = 1, \ldots, p), \ (i = 1, \ldots, p)$$

The boundary conditions for the approximate solution v^M are:

$$(3.8)\qquad \begin{cases} v^M = \varphi(x^m) & \text{for} \quad M = (0, m) \\ \beta_1^j v^{M_j} + \gamma_1^j v^M = 0 & \text{for} \quad M = (\mu, m_1, \ldots, m_{j-1}, 0, m_{j+1}, \ldots, m_p) \in Z \\ \beta_2^j v^{M_j} + \gamma_2^j v^M = 0 & \text{for} \quad M = (\mu, m_1, \ldots, m_{j-1}, N_h, m_{j+1}, \ldots, m_p) \in Z \end{cases}$$

where

$$(\beta_i^j)^2+(\gamma_i^j)^2 \neq 0 \quad \text{and} \quad \beta_i^j\cdot\gamma_i^j = 0, \ (j = 1, \dots, p), \ (i = 1, \dots, p)$$

The values v^M at the remaining nodal points we define successively starting from (3.8) with the aid of the difference equation

$$v^{M\sim} = f(t^\mu, x^m, v^M, v^{M\triangle}, v^{M\square}, \tilde{v}^\mu(\cdot)) \tag{3.9}$$

4. LEMMA 1

Let us suppose that the assumptions H are fulfilled, then the numbers u^M satisfy the difference equation

$$\begin{cases} u^{M\sim} = f(t^\mu, x^m, u^M, u^{M\triangle}, u^{M\square}, \tilde{u}^\mu(\cdot))+\eta^M \\ \text{for} \quad M\in Z \ (0<\mu\leqslant N_{k-1}), \quad \text{where} \quad \max\limits_m |\eta^M|\to 0 \\ \text{as } h\to 0, \quad \text{when} \quad \beta_i^j \text{ and } \beta_2^j \neq 0 \quad (j = 1, \dots, p) \end{cases} \tag{4.1}$$

Proof: This lemma is obvious for $M = (\mu, m)$, where $1\leqslant m_j\leqslant N_h-1$, $(j = 1, \dots, p)$, since f is of the class C^1 and the corresponding differences and sum (2.9) tend to the corresponding derivatives $u(t, x)$ and to $u(t, \cdot)$. If for some j, $m_j = 0$ or $m_j = N_h$, then u^{Mj} is equal to the derivative $\left.\frac{\partial u}{\partial x_j}\right|_{x_j=0}$ or $\left.\frac{\partial u}{\partial x_j}\right|_{x_j=N_h}$ by the boundary conditions (3.5) and (3.7). It is necessary to prove that for $m_j = 0$ or $m_j = N_h$

$$u^{Mjj} - \frac{\partial^2 u}{\partial x_j^2}(t^\mu, x^m)\to 0 \quad \text{as} \quad h\to 0 \tag{4.2}$$

We suppose that $m_j = 0$.

Then we have:

$$u(P) = u(Q)+\frac{\partial u}{\partial x_j}(Q)\cdot x_j+\tfrac{1}{2}x_j^2\frac{\partial^2 u}{\partial x_j^2}(Q)+\text{remainder} \tag{4.3}$$

where

$$P = (t, x_1, \dots, x_p) \text{ and } Q = (t, x_1, \dots, x_{j-1}\, 0, x_j, \dots, x_p).$$

It is true, by (3.7), that

$$\frac{\partial u(Q)}{\partial x_j} = 0 \tag{4.4}$$

From (4.3) we have

$$\frac{2}{x_j^2}(u(P)-u(Q))-\frac{\partial u}{\partial x_j}(Q)\cdot\frac{2}{x_j}-\frac{\partial^2 u}{\partial x_j^2}(Q)\to 0 \tag{4.5}$$

as $x_j \to 0$ and hence, by (4.4)

$$\frac{2}{x_j^2}(u(P)-u(Q))-\frac{\partial^2 u}{\partial x_j^2}(Q)\to 0 \quad \text{as} \quad x_j\to 0 \tag{4.6}$$

If P and Q are the corresponding nodal points, it is necessary to prove that

$$\frac{2}{x_j^2}(u(P)-u(Q)) = u^{Mjj} \quad \text{for} \quad m_j = 0 \tag{4.7}$$

It is equivalent to

$$\frac{2}{h^2}(u^{j(M)}-u^M) = u^{Mjj} \quad \text{for} \quad M = (\mu, m_1, \ldots, m_{j-1}, 0, m_{j+1}, \ldots, m_p) \tag{4.8}$$

We have by the definition of u^{Mjj} and the definition (3.6)

$$u^{Mjj} = \frac{u^{j(M)}+2u^M-u^{-j(M)}}{h^2} = \frac{u^{j(M)}-2u^M+u^{j(M)}}{h^2} = \frac{2}{h^2}(u^{j(M)}-u^M). \tag{4.9}$$

This ends the proof that $u^{Mjj}\to \frac{\partial^2 u}{\partial x_j^2}(Q)$ as $h\to 0$.

In the case when for certain j, $m_j = N_h$ lemma can be proved in a similar manner.

Remark 1

If in the boundary conditions (3.5) $\beta_1^j = \beta_2^j = 0$ $(j = 1, \ldots, p)$ then the numbers u^M satisfy

$$\begin{cases} u^M = 0, & M = (\mu, m_1, \ldots, m_{j-1}, 0, m_{j+1}, \ldots, m_p) \\ u^M = 0, & M = (\mu, m_1, \ldots, m_{j-1}, N_h, n_{j+1}, \ldots, m_p) \\ u^M = \varphi(x^m), & M = (0, m) \end{cases} \tag{4.10}$$

We shall accept precisely the same conditions for v^M. Then the exact solution is equal to the approximate solution at the boundary points. The thesis of lemma 1 is true for $M \in Z$ such that $0 \leqslant \mu \leqslant N_k - 1$, $1 \leqslant m_j \leqslant N_k - 1$ $(j = 1, \ldots, p)$.

Remark 2

In the case when $\beta_1^j = 0$ and $\beta_2^j \neq 0$ (or $\beta_1^j \neq 0$, $\beta_2^j = 0$) we define the value of the solution at the points. $x_j = \sigma + h$ (or $x_j = -h$) as in (3.6) and we find that u^M satisfies

$$\begin{cases} u^M = \varphi(x^m), & M = (0, m) \\ u^M = 0, & M = (\mu, m_1, \ldots, m_{j-1}, 0, m_{j+1}, \ldots, m_p) \\ u^{Mj} = 0, & M = (\mu, m_1, \ldots, m_{j-1}, N_h, m_{j+1}, \ldots, m_p) \end{cases} \tag{4.9}$$

or

$$(4.10)\qquad \begin{cases} u^M = \varphi(x^m), & M = (0, m) \\ u^{Mj} = 0, & M = (\mu, m_1, \dots, m_{j-1}, 0, m_{j+1}, \dots, m_p) \\ u^M = 0, & M = (\mu, m_1, \dots, m_{j-1}, N_h m_{j+1}, \dots, m_p) \end{cases}$$

We shall accept exactly the same conditions for the numbers v^M. The exact solution is equal to the approximate solution at the points of E such that $x_j = 0$ (or $x_j = \sigma + h$) $(j = 1, \dots, p)$.

Lemma 1 is then true for $M \in Z$ such that

$$0 \leqslant \mu \leqslant N_h - 1,\ 1 \leqslant m_j \leqslant N_h,\ (\text{or } 0 \leqslant \mu \leqslant N_{k-1},\ 0 \leqslant m_j \leqslant N_h - 1).$$

5. LEMMA 2

Suppose that the numbers $R^\mu (\mu = 0, 1, \dots)$ satisfy the difference inequality

$$(5.1)\qquad R^{\mu\sim} \leqslant L_1 R^\mu + \varepsilon,\ \mu = 0, 1, \dots$$

and the initial condition $R^0 = 0$, the difference $R^{\mu\sim}$ being defined by

$$(5.2)\qquad R^{\mu\sim} \leqslant \frac{1}{H}(R^{\mu+1} - R^\mu)\ \ \mu = 0, 1, \dots$$

for $0 < H = \text{const}$, $0 < L_1 = \text{const}$, $0 < \varepsilon = \text{const}$.

Under these assumptions

$$(5.3)\qquad R^\mu \leqslant \frac{\varepsilon}{L_1}(e^{L_1 H_\mu} - 1)\ \ \mu = 0, 1, \dots$$

This lemma is due to Z. Kowalski [1].

6. LEMMA 3

Suppose that v^M are defined by (3.8) and (3.9). Suppose also that the assumptions H are satisfied and denote

$$(6.1)\qquad r^M = v^M - u^M$$

$$(6.2)\qquad s^\mu = \max_m r^M \quad z^\mu = \min_m r^M \quad \text{for} \quad M = (\mu, m) \in Z$$

$$(6.3)\qquad R^\mu = \max_m |r^M|$$

Under these assumptions s^μ and z^μ satisfy the conditions

$$(6.4)\qquad s^0 = z^0 = R^0 = 0$$

and the inequalities

$$(6.5)\qquad \begin{cases} s^{\mu\sim} \leqslant (L+K)R^{\mu}+\varepsilon(h) \\ Z^{\mu\sim} \geqslant (L+K)R^{\mu}-\varepsilon(h), \quad \mu = 0, 1, \ldots, N_h-1 \end{cases}$$

Proof: The condition (6.4) follows from the initial conditions (3.7) and (3.8). We shall prove the first part of (6.5).

The values $s^{\mu+1}$ and s^{μ} are attained at certain nodal points

$$(6.6)\qquad \begin{cases} s^{\mu+1} = \max\limits_{m} r^{\mu+1,m} = r^{\mu+1} = r^{\omega(A)} \\ s^{\mu} = \max\limits_{m} r^{\mu,m} = r^{\mu,b} = r^{B} \end{cases}$$

where

$$A = (\mu, a) \in Z,\ B = (\mu, b) \in Z.$$

Hence

$$(6.7)\qquad s^{\mu\sim} = \frac{1}{k}(s^{\mu+1}-s^{\mu}) = \frac{1}{k}(r^{\omega(A)}-r^{B}) = \frac{1}{k}(r^{\omega(A)}-r^{A})+\frac{1}{k}(r^{A}-r^{B}) = r^{A}+\frac{1}{k}(r^{A}-r^{B}).$$

We shall first consider the case when $\beta_1^j \neq 0$ and $\beta_2^j \neq 0$ $(j = 1, \ldots, p)$.

From (7.1), (3.9) and (5.1) we have

$$(6.8)\qquad r^{A\sim} = u^{A\sim}-v^{A\sim} = \eta^{A}+f(t^{\mu}, x^{a}, u^{A}, u^{A\Delta}, u^{A\Box}, \tilde{u}^{\mu}(\cdot))-f(t^{\mu}, x^{a}, v^{A}, v^{A\Delta}, v^{A\Box}, \tilde{v}^{\mu}(\cdot)).$$

The right side of (13.8) may be written as

$$\begin{aligned}(6.9)\qquad r^{A\sim} = \eta^{A} &+f(t^{\mu}, x^{a}, u^{A}, u^{A\Delta}, u^{A\Box}, \tilde{u}^{\mu}(\cdot))-f(t^{\mu}, x^{a}, v^{A}, u^{A\Delta}, u^{A\Box}, \tilde{u}^{\mu}(\cdot)) \\ &+f(t^{\mu}, x^{a}, v^{A}, u^{A\Delta}, u^{A\Box}, \tilde{u}^{\mu}(\cdot))-f(t^{\mu}, x^{a}, v^{A}, u^{A\Delta}, u^{A\Box})\, \tilde{v}^{\mu}(\cdot)) \\ &+f(t^{\mu}, x^{a}, v^{A}, u^{A\Delta}, u^{A\Box}, \tilde{v}^{\mu}(\cdot))-f(t^{\mu}, x^{a}, v^{A}, v^{A\Delta}, v^{A\Box}, \tilde{v}^{\mu}(\cdot)).\end{aligned}$$

From the mean value theorem and by the assumptions (3.2) it follows that

$$(6.10)\qquad r^{A\sim} \leqslant \eta^{A}+L|u^{A}-v^{A}|+K\|\tilde{u}^{\mu}(x)-\tilde{v}^{\mu}(x)\|+ \\ +\sum_{j=1}^{p} f_{q_j}(\sim)(u^{Aj}-v^{Aj})+\sum_{j=1}^{p} f_{\omega_j}(\sim)(u^{Ajj}-v^{Ajj})$$

From the definitions 2, 9 it follows that

$$(6.11)\qquad \begin{cases} u^{Aj}-v^{Aj} = \dfrac{1}{2h}(r^{j(A)}-r^{-j(A)}) \\ u^{Ajj}-v^{Ajj} = \dfrac{1}{h^2}(r^{j(A)}-2r^{A}+r^{-j(A)}) \end{cases}$$

and by (2.9) and (2.4) we have

$$\|\tilde{u}^{\mu}(x)-\tilde{v}^{\mu}(x)\| = \max_{x\in p} |\sum_{M\in\tilde{Z}^{\mu}} (u^{M}\chi_{I_M}(x)-v^{M}\chi_{I_M}(x))| = \max_{M\in\tilde{Z}^{\mu}} |u^{M}-v^{M}|$$

Now we introduce (6.11) in (6.10) and (6.7) and we obtain

$$(6.12)\quad s^{\mu\sim} \leqslant \eta^A + L|r^A| + \sum_{j=1}^{p} \frac{\partial f}{\partial q_j}(\sim)\frac{1}{2h}(r^{j(A)} - r^{-j(A)})$$

$$+\sum_{j=1}^{p} \frac{\partial f}{\partial w_j}(\sim)\left[\frac{1}{h^2}(r^{j(A)} - 2r^A + r^{-j(A)})\right] + K \max_{M \in \bar{Z}^\mu} |u^M - v^M|$$

$$+\frac{1}{k}(r^A - r^B) = \eta^A + Lr^A + K \max_{M \in \bar{Z}^\mu} |r^M|$$

$$+\sum_{j=1}^{p} \frac{\partial f}{\partial q_j}(\sim)\frac{1}{2h}\cdot[r^{j(A)} - r^B - (r^{-j(A)} - r^B)]$$

$$+\sum_{j=1}^{p} \frac{\partial f}{\partial w_j}(\sim)\frac{1}{h^2}[(r^{j(A)} - r^B) - 2(r^A - r^B) + (r^{-j(A)} - r^B)]$$

$$+\frac{1}{k}(r^A - r^B) = A + L|r^A| + K \max_{M \in Z^\mu} |r^M| + (r^{j(A)} - r^B)$$

$$\cdot\sum_{j=1}^{p} \frac{1}{2h}\left(\frac{\partial f}{\partial q_j}(\sim) + \frac{1}{h^2}\frac{\partial f}{\partial w_j}(\sim)\right) + (r^{-j(A)} - r^B)$$

$$\cdot\sum_{j=1}^{p} \left(\frac{1}{h^2}\frac{\partial f}{\partial w_j}(\sim) - \frac{1}{2h}\frac{\partial f}{\partial q_j}(\sim)\right)$$

$$+(r^B - r^A)\left(\frac{2}{h^2}\sum_{j=1}^{p} \frac{\partial f}{\partial^{q_j}}(\sim)\cdot\frac{1}{k}\right)$$

By (6.6) we have

$$(6.13)\qquad r^{j(A)} - r^B \leqslant 0$$

and by (3.3)

$$(6.14)\qquad \frac{1}{2h}\frac{\partial f}{\partial q_j}(\sim) + \frac{1}{h^2}\frac{\partial f}{\partial w_j}(\sim) \geqslant \frac{g}{h^2} - \frac{\Gamma}{2h} = \frac{1}{h}\left(\frac{g}{h} - \frac{\Gamma}{2}\right) \geqslant 0$$

From (6.12) and (6.13) it follows that

$$(6.15)\qquad (r^{j(A)} - r^B)\sum_{j=1}^{p}\left(\frac{1}{2h}\frac{\partial f}{\partial q_j}(\sim) + \frac{1}{h^2}\frac{\partial f}{\partial \omega_j}(\sim)\right) \leqslant 0$$

Then by (6.6) we get

$$(6.16)\qquad r^{-j(A)} - r^B \leqslant 0,$$

and by (3.3)

(6.17) $$\frac{1}{h^2}\frac{\partial f}{\partial w_j}(\sim)-\frac{1}{2h}\frac{\partial f}{\partial q_j}(\sim)\geqslant -\frac{\Gamma}{2h}+\frac{g}{h^2}=\frac{1}{h}\left(\frac{g}{2}-\frac{\Gamma}{2}\right)\geqslant 0$$

From (6.16) and (6.17) it follows that

(6.18) $$(r^{-J(A)}-r^B)\left(\frac{1}{h^2}\frac{\partial f}{\partial w_j}(\sim)-\frac{1}{2h}\frac{\partial f}{\partial q_j}(\sim)\right)\leqslant 0$$

By (6.6) we have

(6.19) $$r^B-r^A\geqslant 0,$$

and by (3.3)

(6.20) $$\frac{2}{h^2}\sum_{j=1}^{p}\frac{\partial f}{\partial w_j}(\sim)-\frac{1}{k}\leqslant\frac{2p\sigma}{h^2}-\frac{1}{k}\leqslant 0$$

From (6.19) and (6.20) it follows that

(6.21) $$(r^B-r^A)\frac{2}{h^2}\sum_{j=1}^{p}\left(\frac{\partial f}{\partial w_j}(\sim)-\frac{1}{k}\right)\leqslant 0$$

By (6.15), (6.18) and (6.21) the last three terms in (6.12) are non-positive, and by (6.12) we obtain

$$s^{\mu\sim}\leqslant\eta^A+L|r^A|+K\max_{M\in Z}|r^M|\leqslant\varepsilon(h)+(L+K)R^\mu$$

where

$$\varepsilon(h)=\max_M|\eta^M|,\ \text{since}\ |r^A|\leqslant R^\mu\ \text{and}\ \max_{M\in Z^\mu}|r^M|\leqslant R^\mu$$

In the case when $\beta_1^j=0$ (or $\beta_2^j=0$), if for some j, $a_j=0$ (or $a_j=N_h$), then the inequalities (6.5) are evident since by the boundary conditions

$$r^{\omega(A)}-r^A=0\quad\text{and}\quad r^B\geqslant 0,$$

and by (6.7) we have $s_{\mu\sim}\leqslant 0$; on the contrary the right hand of the inequality (6.5) is non-negative.

In the remaining cases, i.e. when for every j, $a_j\neq 0$ ($a_j\neq N_h$), by remark 1 or remark 2 we may use lemma 1 and the proof of lemma 3 is analogous, as in the case when β_1^j and $\beta_2^j\neq 0$.

The second inequality in (6.5) can be proved in a similar manner.

7. LEMMA 4

Let us suppose that s^μ and z^μ are defined by (7.2). Under these assumptions

(7.1) $$[\max_m r^M]^\sim\leqslant\max(s^{\mu\sim},\,-z^{\mu\sim}),\quad\text{for}\quad M=(\mu,m).$$

This lemma can, easily be proved starting from the relation $\max_m|r^M|=\max(s^\mu,\,-z^\mu)$.

8. LEMMA 5

Let us suppose that the assumptions H are fulfilled and R^{μ} is defined by (6.3). Under these assumptions R^{μ} satisfies the difference inequality

$$R^{\mu\sim} = (L+K)R^{\mu}+\varepsilon(h)(\mu = 0, 1, \ldots, N_k-1)\,. \tag{8.1}$$

From the definition (6.3) and lemma 4 we obtain

$$R^{\mu\sim} = (\max_m |r^M|)^{\sim} \leqslant \max(s^{\mu\sim}, -z^{\mu\sim})\,. \tag{8.2}$$

The quantities s^{μ} and z^{μ} satisfy (6.5), therefore

$$\begin{aligned} R^{\mu\sim} &\leqslant \max\big((L+K)R^{\mu}+\varepsilon(h), (L+K)R^{\mu}+\varepsilon(h)\big) \\ &= (L+K)R^{\mu}+\varepsilon(h) \quad \text{for} \quad (\mu = 0, 1, \ldots, N_k-1)\,. \end{aligned} \tag{8.3}$$

This ends the proof of Lemma 5.

9. Theorem 1. Suppose that the assumptions H are fulfilled. Then we have the following conclusions:

(1) the error estimate

$$r^M \leqslant \frac{\varepsilon(h)}{L+K}(e^{(L+K)k\mu}-1)$$

(2) the difference method is convergent i.e.

$$\lim_{h\to 0} r^M = 0\,.$$

Proof (2) follows from (1), since $k\mu \leqslant d$ and $\varepsilon(h)\to 0$ as $h\to 0$.
Lemmas 5 and 1 imply that

$$R^{\mu} \leqslant \frac{\varepsilon(h)}{L+K}(e^{(L+K)k\mu}-1), (\mu = 0, 1, \ldots, N_k-1)$$

But $|r^M| \leqslant R^{\mu}$, because of the definition of R^{μ}, hence $|r^M| \leqslant \frac{\varepsilon(h)}{L+K}(e^{(L+K)k\mu}-1)$. This ends the proof of theorem 1.

REFERENCE

[1] Z. Kowalski, *A difference method for a non linear partial differential equation of the first order*, Ann. Polon. Math. 12 (1966).