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Some Remarks on Domains of Holomorphy

1. Introduction. We shall denote by $\mathcal{O}(\Omega)$ the vector space of all functions holomorphic in a domain $\Omega \subset C^n$. Given $a \in \Omega$ and $f \in \mathcal{O}(\Omega)$, let $T_a f$ denote the Taylor series of f at a. We shall denote by $\varrho(T_a f)$ the radius of the maximal ball in which the series $T_a f$ is absolutely convergent.

Let F be any non empty subset of $\mathcal{O}(\Omega)$ and K be a compact set in Ω. We say that K is F-convex if

$$K = \hat{K}_F \overset{\mathrm{df}}{=} \{z \in \Omega:\ |\varphi(z)| \leqslant \sup_{z \in K} |\varphi(z)|,\ \varphi \in F\}. \tag{1}$$

If $F = \mathcal{O}(\Omega)$ and $K = \hat{K}_{\mathcal{O}(\Omega)}$ we say that K is holomorphically convex.

We say that a function $f \in \mathcal{O}(\Omega)$ is prolongable to the outside of Ω, if there is a point $c \in \Omega$ such that $\varrho(T_c f) > \operatorname{dist}(c, \partial\Omega)$.

In this note we shall give two proofs of the following theorem due to J. P. Kahane [2].

Theorem 1. Let $\Omega \subset C^n$ be a domain, $K \subset \Omega$ be a holomorphically convex compact set, and let

$$C_*^\infty(K) \overset{\mathrm{df}}{=} \{\psi \in C^\infty(\operatorname{Int} K):\ \forall \alpha \in \mathbf{z}^n\ D^\alpha \psi \text{ is prolongable continuously to } K\}. \tag{2}$$

Then there exists a function $f \in \mathcal{O}(\operatorname{Int} K) \cap C_*^\infty(K)$ not prolongable to the outside of K.

We denote $A(K) = \mathcal{O}(\operatorname{Int} K) \cap C_*^\infty(K)$.

In the first proof we shall use the following Banach theorem [1]:

Assume that E, F are any Fréchet spaces and

$$\Phi:\ E \to F \tag{3}$$

is a linear and continuous mapping. Then either $\Phi(E) = F$ or $\Phi(E)$ is a set of the first category in F.

In the second proof we shall use the fact that $A(K)$ is a Baire space (for it is metrizable and complete).

The proof proposed by Kahane is based on some properties of the Rademacher series [2].

2. First we notice that the vector space $A(K)$ is the Fréchet space, the topology of which is given by the family of seminorms

$$q_\nu(f) \stackrel{\text{df}}{=} \sup_{z \in K} \sum_{|\alpha| \leqslant \nu} |D^\alpha f(z)|, \quad (\nu = 0, 1, \ldots), \quad f \in A(K), \tag{4}$$

where $D^\alpha f$, $\alpha \in Z^n$ denotes the continuous prolongation of the α-th derivative of f to K. If $\{f_k\}_{k \in N} \subset A(K)$ and $f \in A(K)$ we shall say that $f_k \to f$ when $k \to \infty$ in $A(K)$ if and only if for every $\nu = 0, 1, \ldots$ $\lim_{k \to \infty} q_\nu(f_k - f) = 0$. The topological vector space $A(K)$, equipped with this topology, is complete.

Lemma. Assume that K is a compact set in a domain $\Omega \subset C^n$ such that $K = \hat{K}_{\mathcal{O}(\Omega)}$. Let c be an arbitrary point of $\operatorname{Int} K$ and r be any positive number greater than $\operatorname{dist}(c, \partial K)$.

Then

$$A_r(c) = \{f \in A(K)\colon \varrho(T_c f) \geqslant r\} \tag{5}$$

is a set of the first category in $A(K)$.

Proof. Observe that

$$A_r^*(c) = \{(f, T_c f), f \in A_r(c)\} \tag{6}$$

is the Fréchet space with the following family of seminorms

$$q_\nu^*[(f, T_c f)] \stackrel{\text{df}}{=} q_\nu(f) + \sup_{z \in B_\nu} |T_c f(z)|, \quad \nu = 0, 1, \ldots \tag{7}$$

where q_ν, $\nu = 0, 1, \ldots$ are given by the formula (4) and

$$B_\nu = \left\{z \in C^n\colon \|z - c\| \leqslant \frac{r}{\sqrt[\nu]{2}}\right\}, \quad \nu = 1, 2, \ldots, B_0 \stackrel{\text{df}}{=} B_1.$$

To end this proof we remark that the mapping

$$\pi\colon A_r(c) \ni (f, T_c f) \to f \in A(K) \tag{8}$$

is linear and continuous in the sense of the topology introduced above. Now, for every $a \in (\Omega \setminus K) \cap \{z \in C^n\colon \|z - c\| < r\}$ there exists a function $\varphi \in \mathcal{O}(\Omega)$ such that $|\varphi(a)| > \sup_{z \in K} |\varphi(z)|$. Since the function $f(z) = [\varphi(z) - \varphi(a)]^{-1}$ belongs to $A(K)$ and $f \notin A_r(c)$ we have $A(K) \neq A_r(c) = \pi(A_r^*(c))$. Thus by Banach's theorem $A_r(c)$ is a set of the first category in $A(K)$.

3. The first proof of Theorem 1. Let $\{c_k\}$ be any countable and dense subset of $\operatorname{Int} K$. Let us denote

$$A_n(c_k) = \left\{f \in A(K)\colon \varrho(T_{c_k} f) \geqslant \operatorname{dist}(c_k, \partial K) + \frac{1}{n}\right\} \quad k, n = 1, 2, \ldots$$

and

(9) $$P = \{f \in A(K): f \text{ is prolongable to the outside of } K\}.$$

Then $P = \bigcup_{k,n=1}^{\infty} A_n(c_k)$. Since $A_n(c_k)$, $k, n \in \mathbf{N}$ are sets of the first category, P is a set of the first category in $A(K)$, so $A(K)\setminus P \neq \emptyset$.

4. The second proof of Theorem 1. Let $\{c_k\}$ be any countable and dense subset in $\operatorname{Int} K$. Put

$$B_{k,\nu} = \{z \in C^n\colon \|z - c_k\| < \operatorname{dist}(c_k, \partial K) + r_{k,\nu}\}, \quad k, \nu = 1, 2, \ldots$$

where for every $k \in \mathbf{N}$, $\{r_{k,\nu}\}_{\nu \in \mathbf{N}}$ is a decreasing sequence of positive numbers such that

1° $\lim_{\nu\to\infty} r_{k,\nu} = 0$, for $k = 1, 2, \ldots$

2° $B_{k,\nu} \subset \Omega$, for $k, \nu = 1, 2, \ldots$

Observe that

(10) $$P_{k,l,\nu} = \{f \in A(K)\colon \sup_{z \in B_{k,\nu}} |T_{c_k} f(z)| \leq l\}, \quad k, l, \nu = 1, 2, \ldots$$

are closed subsets of $A(K)$. Moreover for every $f, g \in P_{k,l,\nu}$ and for every $\alpha, \beta \in C$ such that $|\alpha| + |\beta| \leq 1$ we have

$$|T_{c_k}(\alpha f + \beta g)(z)| \leq |\alpha||T_{c_k} f(z)| + |\beta||T_{c_k} g(z)| \leq l, \quad z \in B_{k,\nu}.$$

Thus $P_{k,l,\nu}$ are absolutely convex. We remark that $P = \bigcup_{k,l,\nu=1}^{\infty} P_{k,l,\nu}$ where P is defined by (9). Now, we have two possibilities: either $\operatorname{Int} P_{k,l,\nu} = \emptyset$ for every $k, l, \nu \in \mathbf{N}$ and then P is the set of the first category in $A(K)$; or there exist $k, l, \nu \in \mathbf{N}$ such that $\operatorname{Int} P_{k,l,\nu} \neq \emptyset$. We shall prove that the second case is impossible. Indeed, $P_{k,l,\nu}$ is an absorbing subset in $A(K)$, because $P_{k,l,\nu}$ is absolutely convex with a non empty interior. Hence

(11) $$\forall f \in A(K)\ \exists\ M > 0\colon \lambda \in C,\ |\lambda| \geq M \Rightarrow f \in \lambda P_{k,l,\nu}.$$

Now, let us take any point $a \in (\Omega\setminus K) \cap B_{k,\nu}$ and a function $\varphi \in \mathcal{O}(\Omega)$ such that $|\varphi(a)| > \sup_{z\in K} |\varphi(z)|$. Since $f(z) = [\varphi(z) - \varphi(a)]^{-1} \in A(K)$, there exist $\lambda \in C$ and $g \in P_{k,l,\nu}$ such that $f = \lambda g$. The function g is prolongable holomorphically to the ball $B_{k,\nu}$, so also is f, but this is impossible.

Corollary. If $\operatorname{Int} K$ is not a domain of holomorphy, then there exists a ball $B = B(a, r)$, $a \in \operatorname{Int} K$, $r > \operatorname{dist}(a, \partial K)$ such that every function $f \in A(K)$ is prolongable holomorphically into B.

Proof. Indeed, $A(K) = P = \bigcup P_{k,l,\nu}$. Thus there must exist numbers $k, 1, \nu \in \mathbf{N}$ such that $\operatorname{Int} P_{k,l,\nu} \neq \emptyset$. Hence $P_{k,l,\nu}$ is an absorbing set in $A(K)$, thus

$$A(K) = \bigcup_{p \in \mathbf{N}} p P_{k,l}.$$

Using the method of second proof of Theorem 1 we may obtain the following well known result:

If $D \subset C^n$ is not a domain of holomorphy, then there exists a ball $B = B(a, r)$, $a \in D$, $r > \operatorname{dist}(a, \partial D)$ such that every function $f \in \mathcal{O}(D)$ is holomorphically prolongable into this universal ball B.

5. Let $F \subset \mathcal{O}(\Omega)$ be any non empty family. Observe that Lemma and Theorem 1 hold for any compact set $E \subset \Omega$ such that $E = \hat{E}_F$ (instead of $K = \hat{K}_{\mathcal{O}(\Omega)}$). The proofs of these facts are exactly the same as the proofs of previous versions. In particular a compact set E may be polynomially convex or an analytic polyhedron.

Now, let us denote

$$H^\infty(K) = \{f \in \mathcal{O}(\operatorname{Int} K) \colon f \text{ is bounded on } K\}$$

$$A^p(K) = \left\{ f \in \mathcal{O}(\operatorname{Int} K) \colon \begin{array}{l} \forall\, \alpha \in \mathbb{Z}_+^n,\ |\alpha| \leqslant p,\ D^\alpha f \text{ is prolongable} \\ \text{continuously to } K \end{array} \right\} \qquad 0 \leqslant p \leqslant \infty$$

Then $A^\infty(K) = A(K)$.

Since we may equip the vector spaces $\mathcal{O}(\operatorname{Int} K)$, $H^\infty(K)$ and $A^p(K)$, $(0 \leqslant p \leqslant \infty)$ with Fréchet topologies, and since

$$A(K) \subset \ldots \subset A^{p+1}(K) \subset A^p(K) \subset \ldots \subset A^0(K) \subset H^\infty(K) \subset \mathcal{O}(\operatorname{Int} K) \tag{12}$$

the following theorem holds

Theorem 2. Let F be any non empty subset of $\mathcal{O}(\Omega)$ and K be any compact set in Ω such that $K = \hat{K}_F$. If A is one of the spaces (12), then there is a function $f \in A$ not prolongable holomorphically to the outside of K.

This theorem may be also proved analogously to Theorem 1.

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