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Existence and Uniqueness of Solutions of Boundary Value Problems for Functional Differential Equations

INTRODUCTION

In the present note we shall consider boundary value problems for nonlinear functional differential equations. This type of equation includes differential equations with retarded argument as a special case.

Boundary value problems for differential equations with retarded argument were formulated in 1952 by L. E. Elsgolc [3] in connection with some variational problems. Existence and uniqueness results for problems of this kind have been obtained by G. A. Kamenski ([7]–[10], see also [14], [15], [17], [19]). The two point boundary value problem has been considered by M. Miedzitow [13], R. O. Driver [2], and N. V. Sarkowa [18]. V. P. Skripnik [20] has proved the existence of solution to the Nicoletti boundary value problem for nonlinear systems. General boundary value problems for linear systems have been considered by A. Halanay [6]. K. L. Cooke [1], R. Fennell, P. Waltman [4] and L. J. Grimn, K. Schmidt [5] have investigated the boundary value problem for functional equations using fixed-point theorems.

In these papers it has usually been assumed that right hand sides of equations are continuous and bounded, or Lipschitzian. An application of the contingent technique due to A. Lasota and Z. Opial [12] permits the replacement of above assumptions by the more general Carathéodory type conditions.

In Section 1 we give the notations and introduce the notions. Next, in Section 2 we quote a generalization of the first theorems of Fredholm due to A. Lasota [11] and state a lemma which will be needed in the sequel. The main theorems are stated and proved in Section 3. The last Section contains the applications of the main theorems.

1. DEFINITIONS AND NOTATIONS

Let B denote a Banach space with the norm $\|\cdot\|$. For $u \in B$, $A \subset B$ we set

$$\delta(u, A) = \inf\{\|u-v\|: v \in A\}, \quad |A| = \sup\{\|u\|: u \in A\}$$

Denote by $c(B)$ the family of all non-empty, convex subsets of B.

Let H: $B \to c(B)$. The map H will be called compact, if for any bounded subset D of B the closure of the set $\bigcup_{u \in D} H(u)$ is compact in B. The map H will be called homogeneous if for every $u \in B$ and any real λ, $H(\lambda u) = \lambda H(u)$. The map H will be called upper semi-continuous if its graph $\{(u, v): u \in B, v \in H(u)\}$ is closed in $B \times B$. The map H will be called completely continuous if it is compact and upper semi-continuous.

It is easy to see that a homogeneous map H is compact if and only if the closure of the set $\bigcup_{\|u\|=1} H(u)$ is compact.

Let R^m be a m-dimensional real Euclidean space with the Euclidean norm $|\cdot|$ and by $cf(R^m)$ denote the set of all non-empty closed and convex subsets of R^m. In $cf(R^m)$ we introduce the Hausdorff distance setting

$$d(C, D) = \max(\sup_{q \in D} \delta(q, C), \sup_{q \in C} \delta(q, D)).$$

A map F: $B \to cf(R^m)$ will be called continuous if it is continuous in the Hausdorff topology.

We say that a map F of the compact interval $[a, b] \subset R^1$ into $cf(R^m)$ is Lebesgue-measurable if for each closed subset A of R^m the set $\{t \in A: F(t) \cap A \neq \emptyset\}$ is Lebesgue-measurable [16].

We say that a map $F(t, u)$ (resp. $f(t, u)$) of $[a, b] \times B$ into $cf(R^m)$ (resp. R^m) satisfies the Carathéodory conditions if it is measurable in t for each $u \in B$ and continuous in u for each $t \in [a, b]$.

Let $C^m_{a,b}$ denote the Banach space of all continuous functions x; $[a, b] \to R^m$ with the norm of uniform convergence $\|\cdot\|$ and let C_h, $h > 0$ be the space $C^m_{[a-h, a]}$.

For $x \in C^m_{[a-h, b]}$ and $t \in [a, b]$ define $x_t \in C_h$ by $x_t(\tau) = x(t+\tau-a)$, $\tau \in [a-h, a]$.

2. PRELIMINARIES

The following theorem due to A. Lasota [11] will be basic in the sequel.

Theorem 2.1. Let B be a Banach space, H: $B \to c(B)$ a homogeneous and completely continuous map and let the map h: $B \to B$ be completely continuous and satisfy

$$\lim_{\|u\| \to 0} \frac{\varrho(h(u), H(u))}{\|u\|} = 0. \tag{2.1}$$

If $u = 0$ is the unique vector of B satisfying the condition

$$u \in H(u) \tag{2.2}$$

then there exists at least one solution of the equation

$$u = h(u) \tag{2.3}$$

In the same paper [11] A. Lasota has proved a version of Mazur's theorem which we shall formulate as the following

Lemma 2.1. If the functions $y_n(t)$ $(n = 1, 2, \ldots)$ of the compact interval $I \subset R^1$ into R^m are measurable and satisfy the inequality $|y_n(t)| \leqslant \psi(t)$ where function $\psi(t)$ is integrable on I, then there exists a sequence of integers $\{a_n\}$ and a sequence of real numbers λ_{kn} $(n \leqslant k \leqslant a_n;\ n = 1, 2, \ldots)$ satisfying

$$a_n \geqslant n\,, \quad \lambda_{kn} \geqslant 0\,, \quad \sum_{k=n}^{a_n} \lambda_{kn} = 1 \tag{2.4}$$

such that the sequence

$$z_n(t) = \sum_{k=n}^{a_n} \lambda_{kn} y_k(t)$$

converges to a function $z_0(t)$ almost everywhere on I.

Using Lemma 2.1 we shall prove the following

Lemma 2.2. Let I be a compact interval of R^1 and let a map G of $I \times C_h$ into $cf(R^m)$ be continuous in a for each $t \in I$. If the sequence $\{v^k(t)\}$ of an absolutely continuous functions satisfies the conditions

$$\begin{aligned} &\lim v^k(t) = v(t) \qquad (t \in I) \\ &\left|(v^k(t))'\right| \leqslant p(t) \qquad (t \in I,\ k = 1, 2, \ldots) \end{aligned} \tag{2.5}$$

where $p(t)$ is integrable on I and

$$\varrho\big((v^k(t))', G(t, v_t^k)\big) \to 0\,, \tag{2.6}$$

then

$$v'(t) \in G(t, v_t)\,.$$

Proof. By Lemma 2.1, there exist the sequence a_n and numbers λ_{kn} $(n \leqslant k \leqslant a_n$, $n = 1, 2, \ldots)$ satisfying (2.4) such that the sequence

$$w_n(t) = \sum_{k=n}^{a_n} \lambda_{kn} (v^k(t))' \tag{2.7}$$

converges to a function $w(t)$ almost everywhere on I.

By integrating (2.7) over $[0, t]$ we obtain

$$\int_0^t w_n(t)\,dt = \sum_{k=n}^{a_n} \lambda_{kn} (v^k(t) - v^k(0))\,.$$

By (2.4) and (2.5), the functions $w^n(t)$ are bounded by an integrable function, hence by the Lebesgue convergence Theorem we get

$$\int_0^t w(t)\,dt = v(t) - v(0)\,.$$

Thus

$$w(t) = v'(t)\,. \tag{2.8}$$

For $\varepsilon > 0$ we write

$$G_\varepsilon(t, a) = \{q \in R^m\colon\ \delta(q, G(t, a)) \leqslant \varepsilon\}\,.$$

Obviously, the set $G_\varepsilon(t, a)$ is closed and convex. Since $G(t, a)$ is continuous in a for a fixed t, for each $t \in I$ there exists a positive integer $n_1(t, \varepsilon)$ such that $G(t, v_t^n) \subset G_\varepsilon(t, v_t)$, for $n > n_1(t, \varepsilon)$. By (2.6) $(v^n(t))' \in G_\varepsilon(t, v_t^n)$ for $n > n_2(t, \varepsilon)$. Hence $(v^n(t))' \in G_{2\varepsilon}(t, v_t)$ for $n > \max(n_1(t, \varepsilon), n_2(t, \varepsilon))$, which implies that $w^n(t) \in G_{2\varepsilon}(t, v_t)$ as $n > \max(n_1(t, \varepsilon), n_2(t, \varepsilon))$. Passing with n to infinity we get $w(t) \in G_{2\varepsilon}(t, v_t)$, and since $\varepsilon > 0$ is arbitrary, $w(t) \in G(t, v_t)$. The last condition and (2.8) complete the proof of Lemma (2.2).

Remark. The above Lemma generalizes the result of A. Pliś [16], who has considered the map $G\colon I \times R^m \to R^m$.

3. EXISTENCE AND UNIQUENESS

Consider the functional differential equation

$$x' = f(t, x_t) \quad (a \leqslant t \leqslant b) \tag{3.1}$$

with a boundary condition

$$Lx = r \quad (r \in R^m)\,, \tag{3.2}$$

where f and L are the maps defined on $[a, b] \times C_h$ and $C_{[a,b]}$ respectively with the values in R^m. We define x_t as in Par. 1, if $t + \tau - a > a$ and $x_t(\tau) = a$ for $t + \tau - a \leqslant a$.

Together with the problem (3.1), (3.2) consider the functional equation with a multi-valued right-hand side

$$x' \in F(t, x_t) \quad (a \leqslant t \leqslant b)\,, \tag{3.3}$$

and the homogeneous boundary condition

$$Lx = 0\,, \tag{3.4}$$

where $F\colon [a, b] \times C_h \to cf(R^m)$.

By a solution $x(t)$ of the boundary value problems (3.1), (3.2) (resp. (3.3), (3.4)) we mean any absolutely continuous function on $[a, b]$ satisfying the conditions $x'(t) = f(t, x_t)$, $Lx = r$ (resp. $x'(t) \in F(t, x_t)$, $Lx = 0$) almost everywhere on $[a, b]$.

F, f, L satisfy the conditions:

(i) $F(t, a)$ satisfies the Carathéodory conditions, is homogeneous with respect to a and furthermore

$$\sup_{\|a\|=1} |F(t, a)| \leqslant \varphi(t)\,,$$

where $\varphi(t)$ is an integrable function on $[a, b]$;

(ii) $f(t, a)$ satisfies the Carathéodory conditions and

$$\lim_{n\to\infty} \frac{1}{n} \int_a^b \sup \delta(f(t, \alpha), F(t, \alpha))\, dt = 0$$

(iii) the mapping L is continuous and homogeneous.

Theorem 3.1. Let (i), (ii) and (iii) hold. If $x = 0$ is the only solution of problems (3.3), (3.4), then for every $r \in R^m$ there exists at least one solution of problem (3.1), (3.2).

Proof. Let $B = C_{[a,b]} \times R^m$ be a Banach space with the norm $\|x, p\|_0 = \|x\| + |p|$. Consider the mappings h and H of B into B and $c(B)$ defined by

$$h(x, p) = \left(\int_a^t f(s, x_s)\, ds + p,\ p + r - Lx \right),$$

$$H(x, p) = \left(\int_a^t u(s)\, ds + p,\ p - Lx \right),$$

respectively, where u is any measurable function satisfying $u(s) \in F(s, x_s)$.

Since $F(t, a)$ are non-empty and convex, $H(x, p) \in c(B)$. It is easy to see that if $h(x, p) = (x, p)$, then x is a solution of (3.1), (3.2). Similarly $(x, p) \in H(x, p)$ implies that x satisfies (3.3), (3.4) and hence, by the assumption of the theorem, $(x, p) = (0, 0)$.

Thus the proof of the theorem is reduced to proving that h and H satisfy the assumptions of Theorem 2.1.

By (ii), h and H satisfy (2.1), moreover, h is completely continuous.

It is easily verified that (i) implies that the mapping H is homogeneous. To show that H is completely continuous it suffices to prove that H is compact and upper semi-continuous.

It is easy to see that $\bigcup_{\|(x,p)\|_0=1} H(x, p) \subset Z_\varphi$ where

$$Z_\varphi = \left\{ (\tilde{x}, \tilde{p}) : \begin{array}{l} \tilde{x} = \int_a^t u(s)\, ds + p,\ |u(s)| \leqslant \varphi(s),\ |p| < 1 \\ |\tilde{p}| \leqslant \sup_{\|x\|=1} |Lx| + 1 \end{array} \right\}$$

Since φ is integrable on $[a, b]$, the functions $\tilde{x}$ are bounded and equicontinuous on $[a, b]$. Furthermore L is continuous and homogeneous, $\sup_{\|x\|=1} \|Lx\|$ is bounded, and consequently the closure of Z_φ is compact. Hence the mapping H is compact.

Let $\{(x^k, p^k)\}$, $\{(y^k, q^k)\}$ be sequences of B such that $(y^k, q^k) \in H(x^k, p^k)$ for $k = 1, 2, \ldots$, and $(x^k, p^k) \to (x, p)$, $(y^k, q^k) \to (y, q)$ as $k \to \infty$.

We are going to prove that $(y, q) \in H(x, p)$. This will imply that H is upper semi-continuous.

We have

(3.5) $$y^k(t) = \int_a^t u^k(s)ds + p^k, \text{ where } u^k(s) \in F(s, x_s^k) \qquad q^k = p^k + Lx^k$$

Hence

$$(y^k(t))' \in F(t, x_t^k)$$

By Lemma 2.2

(3.6) $$y'(t) \in F(t, x_t).$$

Passing in (3.5) to the limit as $n \to \infty$ we get for $t = a$

(3.7) $$y(a) = p, \quad q = p + Lx.$$

The formulae

$$y(t) = \int_a^t y(s)ds + p,$$

(3.6) and (3.7) imply that

$$(y, q) \in H(x, p)$$

which completes the proof of Theorem 3.1.

Now we shall assume the following conditions concerning f, F and L which are sufficient not only for the existence but also the uniqueness of solutions:

(iv) f: $[a, b] \times C_h \to R^m$ is measurable in t for each $\alpha \in C_h$ and satisfies the conditions

$$f(t, \alpha) - f(t, \beta) \in F(t, \alpha - \beta), \quad \int_a^b |f(t, 0)| dt < +\infty;$$

(v) L: $C_{[a,b]} \to R^m$ is linear and continuous.

Theorem 3.2. Suppose that F, f, L satisfy (i), (iv), (v) and that $x = 0$ is the unique solution of (3.3), (3.4). Then for each $r \in R^m$ there exists exactly one solution of problems (3.1), (3.2).

Proof. Indeed, (i), (iv) and (v) imply (i), (ii), and (iii). Hence the existence of solutions of problems (3.1), (3.2) is a consequence of Theorem 3.1.

To show the uniqueness, assume that $\bar{x}(t)$ and $\bar{\bar{x}}(t)$ are solutions of (3.1), (3.2). By (iv) and (v), $x(t) = \bar{x}(t) - \bar{\bar{x}}(t)$ satisfies (3.3) and (3.4) hence by our assumptions we have $x(t) \equiv 0$ on $[a, b]$.

4. APPLICATION OF DIFFERENTIAL INEQUALITIES

Because differential inequalities may be considered as a special case of contingent equations, Theorems 3.1 and 3.2 provide sufficient conditions for the existence (uniqueness) of solutions to the boundary value problem for functional equations with right-hand sides satisfying certain kinds of inequality.

Consider the inequality

$$|x'(t)| \leqslant \omega(t, \|x_t\|) , \tag{4.1}$$

where

$$\omega\colon [a, b]\times [0, \infty)\to [0, \infty)$$

We write

$$F(t, a) = \{q \in R^m\colon |q| \leqslant \omega(t, \|a\|)\} .$$

From Theorem 3.1 follows immediately

Theorem 4.1. Let the function f: $[a, b]\times C_h \to R^m$, satisfy the Carathéodory conditions and, in addition, the inequality

$$|f(t, a)| \leqslant \varphi(t)+\omega(t, \|a\|) ,$$

where $\omega(t, u)$ is continuous in (t, u), homogeneous in u and $\varphi(t)$ is integrable on $[a, b]$. If problem (4.1), (3.4) has only a trivial solution $x = 0$, then for each $r \in R^m$ there exists at least one solution of problem (3.1), (3.2).

As a consequence of Theorem 3.2 we get

Theorem 4.2. Let ω satisfy the assumptions of Theorems 4.1. Suppose that L satisfies (v) and

$$|f(t, a)-f(t, \beta)| \leqslant \omega(t, \|a-\beta\|) .$$

If problem (4.1), (3.4) has only the trivial solution $x = 0$, then for each $r \in R^m$ problem (3.1), (3.2) has exactly one solution.

As an application of the above theorems consider the equation (3.1) with the following aperiodic boundary value problems

$$x(a)+\lambda x(b) = r \quad (\lambda > 0,\ r \in R^m) . \tag{4.2}$$

Consider also the delay-differential inequality

$$|x'(t)| \leqslant p(t)\|x_t\| , \tag{4.3}$$

with a homogeneous boundary condition

$$x(a)+\lambda x(b) = 0 \tag{4.4}$$

Lemma 4.1. If the function $p(t)$ is positive and integrable on $[a, b]$,

$$\lambda\Big(1+\int_a^b p(s)\Big(\exp\int_a^s \hat{p}(\tau)\,d\tau\Big)\,ds\Big) < 1 \tag{4.5}$$

where

$$\hat{p}(t) = \sup_{\Theta \in [t-h,\, t]} |p(\Theta)|\,,$$

then $x = 0$ is the unique absolutely continuous function satisfying (4.3), (4.4).

Proof. Integrating the inequality (4.3) over the interval $[a, t]$ we obtain

$$|x(t) - x(a)| \leqslant \int_a^t p(s)\|x_s\|\,ds\,.$$

Hence

(4.6) $$|x(t)| \leqslant |x(a)| + \int_a^t p(s)\|x_s\|\,ds\,.$$

Setting

$$u(s) = \|x_s\| = \sup_{\theta \in [s-h,\, s]} |x(\Theta)|$$

from (4.6) we obtain

$$u(t) \leqslant u(a) + \int_a^t \hat{p}(s)u(s)\,ds\,.$$

By Gronwall inequality

(4.7) $$u(t) \leqslant u(a)\exp \int_a^t \hat{p}(s)\,ds\,.$$

Hence by (4.6),

$$|x(t)| \leqslant u(a) + u(a)\int_a^t p(s)\Big(\exp \int_a^s \hat{p}(\tau)\,d\tau\Big)\,ds\,.$$

From (4.4) we have

(4.8) $$|x(a)| = \lambda |x(b)|\,.$$

From (4.8) and (4.7) we obtain for $t = b$

(4.9) $$u(a) \leqslant \lambda u(a)\Big(1 + \int_a^b p(s)\Big(\exp \int_a^s \hat{p}(\tau)\,d\tau\Big)\,ds\Big)\,.$$

By (4.5) and (4.9), it follows that $u(a) = 0$ and by (4.7) $u(t) \equiv 0$ on $[a, b]$, therefore $x = 0$. This completes the proof of Lemma.

Using Lemma 4.1 and Theorems 4.1 and 4.2 we obtain.

Theorem 4.3. If the function $f\colon [a, b] \times C_h \to R^m$ satisfies the Carathéodory conditions and the inequality

$$|f(t, \alpha)| \leqslant p(t)\|\alpha\| + \varphi(t) \quad \text{for } (t, \alpha) \in [a, b] \times C_h\,,$$

where $\varphi(t)$ is integrable and $p(t)$ is nonnegative and satisfies (4.5), then the problem (3.1), (4.2) has at least one solution.

Theorem 4.4. If the function f satisfies the Carathéodory conditions and the inequality

$$|f(t;\alpha)-f(t,\beta)| \leqslant p(t)\|\alpha-\beta\|, \quad \int_a^b |f(t,0)|\,dt < +\infty,$$

where $p(t)$ is nonnegative and integrable and if (4.5) holds, then the problem (3.1), (4.2) has exactly one solution.

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