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Remarks on Commuting Mappings in Partially Ordered Spaces

The purpose of the present note is to give some remarks on commuting mappings of a partially ordered space into itself. Some of them are very closely related to results of J. S. W. Wong [2]. A partially ordered space X is called a *complete semilattice* if every non-empty subset of X has a supremum. A mapping $f\colon X \to X$ is called *antitone* (*decreasing*) if $x \leqslant y \Rightarrow f(y) \leqslant f(x)$, and *isotone* (*increasing*) if $x \leqslant y \Rightarrow f(x) \leqslant f(y)$. We put $P_f = \{x \in X : x \leqslant f(x)\}$, $\Phi_f = \{x \in X : x = f(x)\}$.

1. Theorem 1. *Let X be a partially ordered set, $f\colon X \to X$, $g\colon X \to X$, and let f be antitone. Suppose that either for each $x \in X$ $g(x) \leqslant f(x)$ and $(f \circ g)(x) \geqslant (g \circ f)(x)$ or for each $x \in X$ $g(x) \geqslant f(x)$ and $(f \circ g)(x) \leqslant (g \circ f)(x)$. Then $(f \circ g)(x) = (g \circ f)(x)$ for $x \in X$ and $f(y) = g(y)$ for $y \in f(X)$.*

Proof. Let $g(x) \leqslant f(x)$ *and* $(f \circ g)(x) \geqslant (g \circ f)(x)$ for $x \in X$. We have $(f \circ g)(x) \geqslant (f \circ f)(x) \geqslant (g \circ f)(x)$ and then $(f \circ g)(x) = (g \circ f)(x)$. Moreover, $(g \circ f)(x) = (f \circ f)(x)$ and hence $g(y) = f(y)$ for $y \in f(X)$. If $g(x) \geqslant f(x)$ and $(f \circ g)(x) \leqslant (g \circ f)(x)$ for $x \in X$, then $(g \circ f)(x) \geqslant (f \circ f)(x) \geqslant (f \circ g)(x)$ and the assertion holds too.

Theorem 2. *If all assumption of Theorem* 1 *are satisfied and f is onto, then $f(x) = g(x)$ for $x \in X$.*

2. Wong proved in [2] the following

Theorem W. *Let X be a partially ordered set and F be a non-empty family of commuting antitone mappings of X into itself. If* $\sup P_f \in P_f$ *and* $\inf f(P_f) \leqslant \sup P_f$ *for some $f \in F$, then there exists a unique common fixed point of F.*

One can state the following modification of this result.

Theorem 3. *Let X be a partially ordered set and f be an antitone mapping of X into itself such that* $\sup P_f \in P_f$ *and* $\inf f(P_f) \leqslant \sup P_f$ (*we suppose that* $\sup P_f$ *and* $\inf f(P_f)$ *exist*). Then: 1° $\Phi_f = \{\sup P_f\}$, 2° *for each $g\colon X \to X$ such that $f \circ g = g \circ f$, $\Phi_g \supset \Phi_f = \{\sup P_f\}$,* 3° *if $g\colon X \to X$ is antitone, $f \circ g = g \circ f$, $c \in \Phi_g$ and $c \leqslant a = \sup P_f$ or $a \leqslant c$*, then $c \in f^{-1}(a)$.

Proof. The proof of 1° and 2° is the same as that of Theorem W in [2] (for the proof of 1°, see also [1], Theorem 4). We recall this reasoning shortly. 1° Let

$a = \sup P_f$, $b = \inf f(P_f)$. We have $f(a) \leqslant f(x)$ for $x \in P_f$; hence $f(a) \leqslant b$ and since $a \leqslant f(a)$, $b \leqslant a$ and then $b = a$, $f(a) = a$. If $c \in \Phi_f$, then $c \leqslant a$ and $a \leqslant f(a) \leqslant c = f(c)$. Hence $c = a$. 2° $g(a) = g(f(a)) = f(g(a))$ and then $g(a) \in \Phi_f = \{a\}$. 3° If $c = g(c)$ and $c \leqslant a$ then $f(c) \geqslant f(a) = a$. On the other hand, $f(c) = f(g(c)) = g(f(c)) \leqslant g(a) = a$. Hence $f(c) = a$. If $c = g(c) \geqslant a$, then $f(c) \leqslant a$ and $f(c) = f(g(c)) = g(f(c)) \geqslant g(a) = a$ and finally $f(c) = a$.

Corollary 1. *If $f^{-1}(a) = \{a\}$ (for example if f is an injection) then, for each g fulfilling the assumptions of 3°, $\Phi_g = \{a\}$.*

3. In the sequel we assume the existence of each supremum and infimum which appear in the text. If X is a complete semi-lattice, then this assumption is superfluous. For $g: X \to X$, by g^n we denote its n-th iteration.

Lemma. *Let X be a partially ordered set. If $g: X \to X$ is an antitone mapping and if, for some non-negative integer k, $g^{2k+1}(x) \geqslant x$ ($g^{2k+1}(x) \leqslant x$) and $g(x) \leqslant x$ ($g(x) \geqslant x$), then $x \in \Phi_g$.*

Proof. Let $g(x) \leqslant x$. We have $g^2(x) \geqslant g(x)$, $g^3(x) \leqslant g^2(x)$, ..., $g^{2k+1}(x) \leqslant g^{2k}(x)$ and then $x \leqslant g^{2k}(x)$. Hence $g(x) \geqslant g^{2k+1}(x) \geqslant x$. Then $x = g(x)$.

In the case of the inequalities $g(x) \geqslant x$ and $g^{2k+1}(x) \leqslant x$ the proof is similar.

Theorem 4. *Let X be a partially ordered set and let $f: X \to X$. Suppose that $a = \sup P_f$ is the unique fixed-point of f. Let $g: X \to X$ be an antitone mapping such that $g^n \circ f = f \circ g^n$ for $n = 2k+1$, where k is a non-negative integer. Assume moreover that*

$$(*) \qquad (f \circ g)(x) \geqslant (g \circ f)(x) \quad \text{for} \quad x \in X.$$

Then $a \in \Phi_g$.

Proof. We have $(f \circ g^n)(a) = (g^n \circ f)(a) = g^n(a)$. Hence $g^n(a) \in \Phi_f$ and then $g^n(a) = a$. On the other hand, $(f \circ g)(a) \geqslant (g \circ f)(a) = g(a)$ and then $g(a) \in P_f$. Hence $g(a) \leqslant a$. Now, the assertion follows immediately from the Lemma.

Remark 1. It is easy to see that one can assume inequality (*) for $x = a$ only.

Theorem 5. *Let X be a partially ordered set, $f: X \to X$ be an antitone mapping such that $\sup P_f \in P_f$, $\inf f(P_f) \leqslant \sup P_f$. Let $g: X \to$ be an antitone mapping such that an iteration g^n of g for $n = 2k+1$ commutes with f. Assume moreover that $(f \circ g)(x) \geqslant (g \circ f)(x)$ for $x \in X$. Then $\sup P_f \in \Phi_f$.*

Proof. From the assumptions on f it follows directly that f has exactly one fixed-point $a = \sup P_f$ (cf. the proof of 1° in Theorem 3, cf. also [1] and [2]). Hence the assertion follows from Theorem 4.

Remark 2. Theorem 5 reduces to Theorem 3 (2°) if $k = 0$ (cf. also Theorem 4 in [1]).

4. Theorem 6. *Let f, g be two isotone mappings of X into itself, where X is a partially ordered set, such that $g(x) \leqslant f(x)$ and $(f \circ g)(x) \leqslant (g \circ f)(x)$ for $x \in X$, and $(f \circ g)(a) = a$ for $a = \sup P_f$. Then $f(a) = g(a) = a$.*

Proof. $(f \circ g \circ f)(a) \geqslant (f \circ f \circ g)(a) = f(a)$; hence $f(a) \in P_{f \circ g}$ and $f(a) \leqslant a$. From the assumptions it follows that $g(a) \leqslant f(a)$ and then $g(a) \leqslant a$. From the above and the isotonity of f we have $a=(f\circ g)(a)\leqslant f(a)$ and then $f(a)=a$. On the other hand, $a=(f\circ g)(a)\leqslant(g\circ f)(a)=g(a)$ and then finally $a=g(a)=f(a)$.

Remark 3. Theorem 6 is a simple generalization of Theorem 3 of Wong [2]. We have the following.

Corollary 2 (cf. [2]). *Let X be a complete semi-lattice and f be an isotone mapping of X into itself. If $P_{f^n}\neq 0$ for some positive integer n, then $\Phi_f\neq\emptyset$.*

5. Let X be a partially ordered set and let $g: X\to X$. We put $G_g=\{(x,k)\in X\times N: g^k(x)\leqslant x\}$, $H_g=\{(x,k)\in X\times N: x\leqslant g^k(x)\}$ (N denotes the set of positive integers).

Theorem 7. *Let X be a complete semi-lattice and g be an isotone mapping of X into itself, such that $G_g\cup H_g\neq\emptyset$. Assume that $f: X\to X$ is an antitone mapping such that for each $(x,k)\in G_g$, there exists a positive integer $n=n(k)$ such that*

$$(f\circ g^{nk})(x)\leqslant(g^{nk}\circ f)(x).$$

Then $\Phi_g\neq\emptyset$.

Proof. If $H_g\neq\emptyset$, then from Corollary 2 we obtain the assertion. Suppose that $H_g=\emptyset$. Then $G_g\neq\emptyset$ and there exists (x,k) such that $x\geqslant g^k(x)$. Since g is isotone, $x\geqslant g^{nk}(x)$ for each positive integer n, and in particular for $n=n(k)$. Then

$$f(x)\leqslant(f\circ g^{nk})(x)\leqslant(g^{nk}\circ f)(x)$$

which means that $f(x)\in H_g$ and yields a contradiction.

6. Theorem 8. *Let X be a partially ordered set and f be an isotone mapping of X into itself. Suppose that $a=\sup P_f\in\Phi_f$. If $g: X\to X$ is such that $g(x)\geqslant f(x)$ for $x\in X$ and $(g\circ f)(P_f)\subset P_f$, then $a\in\Phi_g$.*

Proof. $(g\circ f)(a)=g(a)\in P_f$ and then $g(a)\leqslant a$. On the other hand, $g(a)\geqslant f(a)=a$ and then $g(a)=a$.

Corollary 3. *Let X be a complete semi-lattice, f be an isotone mapping of X into itself such that $\sup P_f\in P_f$. If $g: X\to X$ is such that $g(x)\geqslant f(x)$ for $x\in X$ and $(g\circ f)(P_f)\subset P_f$, then $g(\sup P_f)=\sup P_f$.*

Corollary 4. *If f fulfills the assumptions of Theorem 8 and $g: X\to X$ is such that $g(x)\geqslant f(x)$ for $x\in X$ and $g(P_f)\subset P_f$, then $g(\sup P_f)=\sup P_f$.*

Indeed, from the inclusion $g(P_f)\subset P_f$ we have $(g\circ f)(P_f)\subset P_f$, because $f(P_f)\subset P_f$.

Remark 3. If $g: X\to X$ is isotone and $(g\circ f)(x)\leqslant(f\circ g)(x)$ for $x\in X$, (for example if g commutes with f), then $g(P_f)\subset P_f$.

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[2] J. S. W. Wong, *Common fixed points of commuting monotone mappings* Canad. J. Math. 19 (1967), 617—620.