ON A CYLINDRICAL BENDING OF A PRISMATIC SHELL WITH TWO CUSPED EDGES UNDER THE ACTION OF AN INCOMPRESSIBLE VISCOUS FLUID[1]

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This paper deals with the proof of a uniqueness theorem of a problem of the interaction of the incompressible viscous fluid and of a thin elastic cusped plate of variable thickness (i.e., prismatic shell).

Let the projection of a prismatic shell be an infinite strip

$$\Omega = \{(x_1, x_2): \ -\infty < x_1 < \infty, \ 0 \le x_2 \le \ell\},$$

and let the shell thickness be given by the equation

$$2h(x_2) = h_0 x_2^{\alpha/3} (\ell - x_2)^{\beta/3}, \quad h_0, \alpha, \beta = const, \quad h_0 > 0, \quad \alpha, \beta \ge 0.$$

The edges $x_2 = 0$ and $x_2 = \ell$ are cusped if $\alpha > 0$ and $\beta > 0$, respectively.

The geometry of the elastic solid along thickness is taken into account in the coefficients of the following bending equation [1]

$$\frac{\partial^2}{\partial x_2^2}\left[x_2^2(\ell - x_2)^\beta \frac{\partial^2 w(x_2,t)}{\partial x_2^2}\right] = -\frac{2\rho^s h(x_2)}{D_0}\frac{\partial^2 w(x_2,t)}{\partial t^2} + \frac{q(x_2,t)}{D_0}, \qquad (1)$$

where $D(x_2) = D_0 x_2^\alpha(\ell - x_2)^\beta$, $D_0 = const > 0$, is a flexural rigidity of the plate, ρ^s is a density of solid, $w(x_2,t) \in C^4(\Omega) \cap C^2(t > 0) \cap C^1(t \ge 0)$ is a deflection of the plate, $q(x_2,t) \in C(\bar\Omega \times \{t \ge 0\})$ is an intensivity of a lateral load.

On $\partial\Omega$ the admissible boundary conditions of the plate theory should be given [1,2].

Let a flow of the viscous and incompressible fluid be independent of x_1 and parallel to $0x_2x_3$ (i.e., $v_1 \equiv 0$) with the following conditions at infinity

$$\begin{gathered} v_i(x_2,x_3,t)|_{x_3\to\infty} = v_{i\infty}(x_2,t), \ \ p(x_2,x_3,t)|_{x_3\to\infty} = p_\infty(x_2,t), \\ v_{i\infty}(x_2,t)|_{x_3\to-\infty} = O(1), \ \ p_\infty(x_2,t)|_{x_3\to-\infty} = O(1), \\ v_{i\infty}(x_2,t)|_{x_2^2\to\infty} = O(1), \ \ p_\infty(x_2,t)|_{x_2^2\to\infty} = O(1), \ \ i = 2,3, \end{gathered} \qquad (2)$$

here $v := (v_2, v_3) \in C^2(\Omega^f) \cap C^1(t > 0) \cap C(t \ge 0)$ is a velocity vector of the fluid, $p(x_2,x_3,t) \in C^2(\Omega^f) \cap C(t \ge 0)$ is a pressure, $v_{2\infty}(x_2,t)$, $v_{3\infty}(x_2,t)$, and $p_\infty(x_2,t) \in C(\{]-\infty,0[\cap]\ell,+\infty[\} \times \{t \ge 0\})$ are given functions.

Because of the incompressibility

$$div\ v(x_2,x_3,t) = 0, \ \ (x_2,x_3) \in \Omega^f, \ \ t \ge 0, \qquad (3)$$

[1]This research was supported by a grant of Georgian President for the university

and (see [5], p.5)

$$\sigma_{ij}(x_2, x_3, t) = -p\delta_{ij} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \tag{4}$$

where σ_{ij} is a stress tensor, μ is a coefficient of velosity, δ_{ij} is Kroneker delta.

Further, if the plate is thin, we can assume that:

– the fluid occupies the whole space R^3 but the middle plane Ω of the plate, i.e., $\Omega^f = R^3 \backslash \Omega$.

– values of a normal component of stress tensor, $\sigma^f_{n3}\left(x_2, \overset{(\pm)}{h}(x_2), t\right)$ in the fluid part Ω^f are transferred at appropriate points of Ω from corresponding sides, i.e.,

$$q(x_2, 0, t) = \sigma^f_{33}(x_2, 0_+, t) - \sigma^f_{33}(x_2, 0_-, t), \quad x_2 \in [0, \ell], \quad t \geq 0; \tag{5}$$

– transmission conditions for $v_i(x_2, x_3, t)$ $(i = 2, 3)$ have the forms as follows (see [3], [4])

$$v_2(x_2, 0, t) = 0, \quad v_3(x_2, 0, t) = \frac{\partial w(x_2, t)}{\partial t}, \quad x_2 \in [0, \ell], \quad t \geq 0. \tag{6}$$

From (3) and (4) a normal component of the stress tensor is equal to

$$\sigma^f_{33}(x_2, x_3, t) = -p(x_2, x_3, t) - 2\mu \frac{\partial v_2(x_2, x_3, t)}{\partial x_2}. \tag{7}$$

By virtue of (6), from (7) we get

$$\sigma^f_{33}(x_2, 0_\pm, t) = -p(x_2, 0_\pm, t).$$

Hence, (5) has the following form

$$-p^+(x_2, t) + p^-(x_2, t) = q(x_2, t), \quad x_2 \in [0, \ell], \quad t \geq 0. \tag{8}$$

Let the motion of the fluid flow be sufficiently slow i.e., v_i and $v_{i,j}$ $(i, j = 2, 3)$ be so small that linearization of Navier-Stokes equations (see [4,5,6,7]) be admissible. Hence,

$$\frac{\partial v_i}{\partial t} = -\frac{1}{\rho^f} grad p + \nu \Delta v_i + F_i(x_2, x_3, t), \quad \rho^f, \nu = const, \ i = 2, 3, \tag{9_i}$$

where ρ^f is a density of fluid, $\nu = \mu/\rho$, $F := (F_2, F_3)$ is a volume force.

Let the initial conditions be

$$\begin{aligned} & v_i(x_2, x_3, 0) = v^0_i(x_2, x_3), \quad i = 2, 3, \quad (x_2, x_3) \in \Omega^f, \\ & w(x_2, 0) = w^0(x_2), \quad \frac{\partial w(x_2, t)}{\partial t} = v^0_3(x_2, 0), \quad x_2 \in [0, \ell]. \end{aligned} \tag{10}$$

Theorem 1. The solution of the initial transmission boundary value problem (1), (3), (9_i) ($i = 2, 3$), (10), (6), (8), (2) is unique.

Proof. Let the difference of two admissible solutions of the problem under consideration be $v_i(x_2, x_3, t)$ ($i = 2, 3$), $p(x_2, x_3, t)$, $q(x_2, t)$, $w(x_2, t)$.

For these functions we get equations (1), (3), and

$$\frac{\partial v_i}{\partial t} = -\frac{1}{\rho^f} grad p + \nu \Delta v_i, \quad i = 2, 3, \qquad (11_i),$$

under the following conditions at infinity

$$p(x_2, x_3, t) = 0 \ when \ x_3 \to \infty; \quad p(x_2, x_3, t) = O(1) \ when \ x_2^2 + x_3^2 \to \infty, \quad (12_1)$$

$$v_i(x_2, x_3, t) = 0 \ when \ \ x_3 \to \infty;$$
$$v_i(x_2, x_3, t) = O(1) \ when \ x_2^2 + x_3^2 \to \infty, \ i = 2, 3, \qquad (12_i)$$

initial conditions

$$\begin{aligned} &v_i(x_2, x_3, 0) = 0 \ (i = 2, 3), \ \ p(x_2, x_3, 0) = 0, \ \ (x_2, x_3) \in \Omega^f, \\ &q(x_2, 0) = 0, \ w(x_2, 0) = 0 \ \ w_{,t}(x_2, 0) = 0, \ \ x_2 \in [0, \ell], \end{aligned} \qquad (13)$$

and transmission conditions (6), (8).

After differentiation of (11_i) with respect to x_i and termwise summation, in virtue of (3), we obtain that $p(x_2, x_3, t)$ is a harmonic function, i.e.,

$$\Delta p(x_2, x_3, t) = 0, \qquad (14)$$

where $\Delta = \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is a Laplace operator.

The solution of the problem (14), (8), (12_1), has the following form (see [8])

$$p(x_2, x_3, t) = -\frac{1}{2\omega\pi}\int_0^\ell \frac{q(\xi, t) x_3}{(\xi - x_2)^2 + x_3^2} d\xi;$$

$v_i(x_2, x_3, t)$ ($i = 2, 3$) can be written as follows

$$v_i(x_2, x_3, t) = -\frac{1}{\rho^f}\int_0^t p_{,i}(x_2, x_3, \tau) d\tau + \phi_i(x_2, x_3, t), \qquad (15_i)$$

where the first term is a particular solution of (11_i), and $\phi_i(x_2, x_3, t)$ ($i = 2, 3$), taking into account (11_i), (13), (3), (12), (6), are solutions of the problems

$$\phi_{i,t} - \nu \Delta \phi_i = 0,$$

$$\phi_i(x_2, x_3, 0) = 0; \ \ \phi_i|_{x_3 \to \infty} = 0, \ \ \phi_i|_{x_2^2 + x_3^2 \to \infty} = O(1), \qquad (16_i)$$

$$\phi_2(x_2,0,t)=0,\ \phi_3(x_2,0,t)=w,_t(x_2,t)+\frac{1}{\rho^f}\int_0^t p,_3(x_2,0,\tau)d\tau,\ x_2\in[0,\ell],$$

$$\phi_{2,2}(x_2,x_3,t)+\phi_{3,3}(x_2,x_3,t)=0. \tag{17}$$

After differentiating (15_i) with respect to x_j $(i,j=2,3,\ i\neq j)$ and using (14) it is easy to show that there exists the function $\psi(x_2,x_3,t)$ given by the expression

$$\psi,_i(x_2,x_3,t)=v_i(x_2,x_3,t)-\phi_i(x_2,x_3,t),\quad i=2,3. \tag{18_i}$$

By virtue of (11_i) and (17), we can find $\psi(x_2,x_3,t)$ from the the following problem

$$\Delta\psi=0,\quad \psi,_2(x_2,0,t)=0,\quad \psi,_2|_{x_2^2+x_3^2\to\infty}=O(1). \tag{19}$$

(19) means (see [8]) that $\psi,_2(x_2,x_3,t)=0$. Therefore $\psi,_{33}(x_2,x_3,t)\equiv 0$, and

$$\psi(x_2,x_3,t)=c_1(t)x_3+c_2(t).$$

Taking into account (18_3), (12_3), and (16_3), we get

$$\psi,_3=c_1(t)=v_3-\psi_3=0,\quad when\ \ x_3\to\infty.$$

Finally, we obtain $\psi,_3\equiv 0$,

$$v_i(x_2,x_3,t)-\phi_i(x_2,x_3,t)=0,\quad i=2,3. \tag{20}$$

By virtue of (15_i), (20), and $p(x_2,x_3,t)$ is a function only of t. On the other hand (see (12_i)) this function is equal to zero at infinity, i.e.,

$$p(x_2,x_3,t)=A(t)=0. \tag{21}$$

Further, the transmission condition (8) and equation (1) mean that

$$q(x_2,t)=0,\quad w(x_2,t)=0. \tag{22}$$

In view of (21), (22), from (11_i) and (6) we obtain

$$v_{i,t}(x_2,x_3,t)-\nu\Delta v_i(x_2,x_3,t)=0,\quad i=2,3. \tag{23_i}$$

$$v_i(x_2,0,t)=0,\quad x_2\in[0,\ell],\quad i=2,3. \tag{24}$$

It is easy to show that the problem (3), (23_i), (24), (13), (12) has only trivial solution, i.e., $v_i\equiv 0$, $i=2,3$.

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