ON COUPLING OF A FIRST ORDER HYPERBOLIC SYSTEM WITH AN ELLIPTIC EQUATION

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We investigate boundery conditions for coupling of two simplified models of gas dynamics: system of acoustics and potential flow equations[1]:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + div\vec{u} = 0, \\ \dfrac{\partial \vec{u}}{\partial t} + gradP = 0, \end{cases} \tag{1}$$

$$P = c^2\rho, c = const, \vec{u} = (u_1, u_2, u_3),$$

and

$$\Delta\phi = 0. \tag{2}$$

The first one is a system of hyperbolic equations and another is a scalar elliptic one which are valid in domains $(0,T)\times\Omega_2$ and Ω_1 respectively, $T>0$, $\Omega_{12} = \Omega_1\cap\Omega_2$, mes $(\Omega_{12})\neq 0$, $\gamma_{12} = \partial\Omega_{12}\cap\ \Omega_1$, $\gamma_{21} = \partial\Omega_{12}\cap\Omega_2$, $\gamma_{12}\cap\gamma_{21} = \emptyset$. Let us denote the remaining part of boundaries of Ω_1 and Ω_2 by γ_1 and γ_2 respectively.

Note that the relationship between the velocity vector $\vec{u}$ and the potential function Φ is given by the formulae

$$\vec{u} = grad\,\Phi. \tag{3}$$

In [2] formulae (3) was used as a boundury condition for the overlapping domain and the existence and uniqueness of solution was proved for coupled second order elliptic and hyperbolic problems in one space dimension. The similar result is valid for (1)-(3) in a general case. Let us specify initial and boundary conditions that will be used below for equations (1) and (2):

$$\frac{\partial\Phi}{\partial n}|_{\gamma_{21}} = \varphi_{21}|_{\gamma_{21}}, \tag{4}$$

$$\Phi|_{\gamma_1} = \varphi_1|_{\gamma_1}, \tag{5}$$

$$\rho(0,x) = \rho_0(x), x\in\ \Omega_2, \tag{6}$$

$$\vec{u}(0,x) = \vec{u}_0(x), x\in\Omega_2, \tag{7}$$

$$\rho(t,s) = \rho_2(t,s), s\in\gamma_2, \tag{8}$$

$$\frac{\partial\rho(t,s)}{\partial n} = \rho_{12}(t,s), s\in\gamma_{12}, \tag{9}$$

$$u(t,s) = grad\ \phi(t,s), s \in \gamma_{12} \cup \gamma_{21}, t \in (0,T), \tag{10}$$

where $\varphi_{21}, \varphi_1, u_0, \rho_0, \rho_2, \rho_{12}$ are given sufficiently smooth functions.

Note that the system of acoustics can be reduced to the following equation for the density ρ :

$$\frac{\partial \rho}{\partial t} = c^2 \Delta \rho. \tag{11}$$

Evidently, after ρ is resolved, one can easily restore the velocity vector $\vec{u}$ from (1). This approach, equation (11) and theory of characteristics are essentially used in the proof of the following

Proposition. Let the classical solution of problems (2),(4),(5) and (6), (8), (9), (11) exist and let it be unique. Then the coupled problem (1),(2),(5)-(8),(10) has a unique classical solution as well.

Finally note that the use of the same technique as for the proof of the above proposition yields other possibilities for correct coupling of equations (1) and (2), e.g. using (3) as an interface boundary condition on γ_{21} and introducing a non local boundary condition for the velocity vector ensuring coincidence of its values on γ_{21} and γ_{12}.

References

[1] Landau L., Lipschitz E., Theoretical Physics, VI, Hydromechanics, M., Nauka, 1986, 786, (Russian).

[2] Bochorishvili R., Jaiani D., On a Coupling of Differential Equations Via Interface Boundary Conditions, BULLETIN of TICMI, 2, 1998, 25-27, (electronic version: http://www. viam.hepi.edu.ge/others/TICMI).