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ON THE MONOTONE CONVERGENCE OF IMPLICIT NEWTON-LIKE METHODS

Ioannis K. Argyros

Abstract. In this study, we examine the convergence of implicit Newton-like methods to a solution of a nonlinear equation on a partially ordered topological space. We generate two sequences, one increasing and one decreasing that enclose a solution of the nonlinear equation. Finally, we provide an example that shows that our method converges whereas the standard explicit Newton-like method does not.

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I. INTRODUCTION

In this study, we are concerned with the problem of approximating a solution z^* of the nonlinear operator equation

$$F(x) = 0 \tag{1}$$

where F is a continuous nonlinear operator defined on a subset D of a partially ordered topological space (POTL-space) E_1 with values in a POTL-space E_2 [2], [3], [5].

We introduce implicit Newton-like methods of the form

$$F(x_{n+1}) + A^1(x_{n-1}, x_n)(x_{n+1} - x_n) = 0 \quad (n \geq 0) \tag{2}$$

$$F(y_{n+1}) + A^2(y_n, y_{n-1})(y_{n+1} - y_n) = 0 \quad (n \geq 0) \tag{3}$$

to monotonically enclose z^*. Here operators $A^1, A^2\colon D^2 \to L(E_1, E_2)$ are satisfying certain convexity type conditions.

Cameron University, Lawton, Oklahoma
E-mail Address: ioannisa@cameron.edu

So far explicit Newton-like methods of the form

$$F(x_n) + A^1(x_{n-1}, x_n)(x_{n+1} - x_n) = 0 \quad (n \geq 0) \tag{4}$$

$$F(y_n) + A^2(y_n, y_{n-1})(y_{n+1} - y_n) = 0 \quad (n \geq 0) \tag{5}$$

as well as various explicit configurations of them have been extensively examined [1]–[5] under the natural hypothesis

$$F(y) - F(x) \leq A(w, z)(y - x) \tag{6}$$

for all $x_0 \leq x \leq y \leq w \leq z \leq y_0$ where $[x_0, y_0] \subseteq D$.

One can provide examples where iterations (2) and (3) converge to a solution z^* of equation (1) whereas iterations (4) and (5) do not. (See the example at the end of this study.) On the other hand the convergence results developed so far for (4) and (5) provide no information about the behavior of iterations (2) and (3).

Sufficient convergence conditions are provided for monotonically enclosing a solution z^* of equation (1) as well as some applications. In particular we provide an example that shows that our results apply where the earlier ones fail.

II. MONOTONE CONVERGENCE

The main result of our study is contained in the following theorem:

Theorem 1. *Consider an operator $F \colon D \subseteq E_1 \to E_2$ where E_1 is a regular POTL-space and E_2 is a POTL-space. Let x_{-1}, x_0, y_{-1}, y_0 be points of D such that*

$$x_{-1} \leq x_0 \leq y_0 \leq y_{-1}, \ [x_{-1}, y_{-1}] \subseteq D \ \text{ and } \ F(x_0) \leq 0 \leq F(y_0). \tag{7}$$

Assume there exist nonnegative operators $A^1, A^2 \colon D^2 \to L(E_1, E_2)$ with continuous nonsingular nonnegative left subinverses B^1 and B^2 respectively such that

$$B(v, w)(F(y) - F(x)) \leq (I - B(v, w)A(v, w))(y - x) \tag{8}$$

where (B, A) is (B^1, A^1) or (B^2, A^2) for all $v, w, x, y \in D$ with $x_0 \leq v \leq w \leq x \leq y \leq y_0$.

Then there exist two iterations $\{x_n\}$ $(n \geq 1)$, $\{y_n\}$ $(n \geq 1)$ and two points x^*, y^* of D such that for all $n \geq 0$ approximations (2) and (3) are solvable,

$$F(x_n) \leq 0 \leq F(y_n) \tag{9}$$

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} \leq y_{n+1} \leq y_n \leq \cdots \leq y_1 \leq y_0, \tag{10}$$

$$\lim_{n\to\infty} x_n = x^*, \ \lim_{n\to\infty} y_n = y^*, \ \text{ and } \ x^* \leq y^*. \tag{11}$$

Proof: Let B_0^1 be a continuous nonsingular nonnegative left subinverse of $A_0^1 = A^1(x_{-1},\, x_0)$ and consider the operator $H\colon [x_0, y_0] \to E_1$ defined by

$$H(x) = x - B_0^1(F(x) + A_0^1(x - x_0)).$$

Clearly H is continuous and for $x \leq y$ we deduce that H is isotone. Indeed from (8) we get

$$\begin{aligned} & B_0^1(F(y) - F(x)) \leq (I - B_0^1 A_0^1)(y - x) \Rightarrow \\ & \quad x - B_0^1(F(x)) - B_0^1 A_0^1(x - x_0) \leq y - B_0^1(F(y)) - B_0^1 A_0^1(y - x_0) \\ & \Rightarrow H(x) \leq H(y). \end{aligned}$$

We also have

$$H(x_0) = x_0 - B_0^1(F(x_0) + A_0^1(x_0 - x_0)) = x_0 - B_0^1(F(x_0)) \geq x_0$$

by (7),

$$H(y_0) = y_0 - B_0^1(F(y_0) + A_0^1(y_0 - x_0)) \leq y_0$$

by (7) and the fact that A_0^1 and B_0^1 are nonnegative operators on D^2. It follows from the Kantorovich theorem on fixed points that H has a fixed point $x_1 \in [x_0, y_0]$. That is $H(x_1) = x_1$ which shows that x_1 solves (2) for $n = 0$. It also follows from (2) for $n = 0$ that $F(x_1) \leq 0$.

Let B_0^2 be a continuous nonsingular nonnegative left subinverse of $A_0^2 = A^2(y_0,\, y_{-1})$ and define the operator $G\colon [x_1, y_0] \to E_1$ by

$$G(x) = x - B_0^2(F(x) - A_0^2(y_0 - x)).$$

Operator G is clearly continuous and exactly as above using (8) we can show that it is isotone.

Moreover we have

$$G(x_1) = x_1 - B_0^2(F(x_1) - A_0^2(y_0 - x_1)) \geq x_1$$

(since $F(x_1) \leq 0$ and A_0^2 is nonnegative),

$$G(y_0) = y_0 - B_0^2(F(y_0) - A_0^2(y_0 - x_0)) \leq y_0$$

(since $F(y_0) \geq 0$). Hence there exists $y_1 \in [x_1, y_0]$ with $G(y_1) = y_1$. That is y_1 solves approximation (3) for $n = 0$ and $F(y_1) \geq 0$.

Proceeding by induction on $n \geq 0$ we can show that there exist two iterations $\{x_n\}$ $(n \geq 1)$, $\{y_n\}$ $(n \geq 1)$ satisfying (2), (3), (9) and (10). Moreover since the space E_1 is regular it follows that there exist $x^*, y^* \in E_1$ satisfying (11).

That completes the proof of the Theorem.

In what follows we shall give some natural conditions under which the points x^* and y^* are solutions of the equation $F(x) = 0$.

Theorem 2. *Under the hypotheses of Theorem* 1 *suppose that F is continuous at x^* and y^*. If one of the following conditions is satisfied:*

(a) $x^* = y^*$;

(b) *operators* A_n^1, A_n^2 $(n \geq 0)$ *are equicontinuous*;

(c) E_2 *is normal and there exists an operator* $P\colon E_1 \to E_2$ $(P(0) = 0)$ *continuous on the origin and such that* $A_n^1 \leq S$, $A_n^2 \leq S$ *for sufficiently large* n;

(d) E_1 *is normal and there exists an operator* $Q\colon E_1 \to E_2$ $(Q(0) = 0)$ *which has an isotone inverse continuous at the origin and such that* $A_n^1 \leq Q$, $A_n^2 \leq Q$ *for sufficiently large* n.

Then we have

$$F(x^*) = F(y^*) = 0.$$

Proof. Using the continuity of F and (9) we get $F(x^*) \leq 0 \leq F(y^*)$ by letting $n \to \infty$. That is $F(x^*) = F(y^*) = 0$.

(b) By the equicontinuity of the operators A_n^1, A_n^2 it follows that $\lim_{n\to\infty} A_n^1(x_n - x_{n+1}) = \lim_{n\to\infty} A_n^2(y_n - y_{n+1}) = 0$, and by (2) and (3), by letting $n \to \infty$ and using the continuity of F at x^* and y^* we get $F(x^*) = F(y^*) = 0$.

(c) For sufficiently large n we get

$$0 \geq F(x_{n+1}) \geq P(x_n - x_{n+1}), \quad 0 \leq F(y_{n+1}) \leq P(y_n - y_{n+1}).$$

By the normality of E_2 and the continuity of F and P we get $F(x^*) = F(y^*) = 0$.

(d) By (2), (3), (9) and (10) we get

$$0 \geq F(x_{n+1}) = A_n^1(x_n - x_{n+1}) \geq Q(x_n - x_{n+1}),$$
$$0 \leq F(y_{n+1}) = A_n^2(y_n - y_{n+1}) \leq Q(y_n - y_{n+1}).$$

Hence, we get

$$\lim_{n\to\infty} Q^{-1}(F(x_{n+1})) = \lim_{n\to\infty} Q^{-1}(F(y_{n+1})) = 0,$$

since E_1 is normal and $\lim_{n\to\infty}(x_n - x_{n+1}) = \lim_{n\to\infty}(y_n - y_{n+1}) = 0$. That is by the continuity of F we finally get $F(x^*) = F(y^*) = 0$.

That completes the proof of the Theorem.

Example. Let $D = E_1 = E_2 = \mathbf{R}$ and $F(x) = \frac{1}{2}x$. Choose $A^1 = A^2 = \frac{1}{4}$, $B^1 = B^2 = 1$, $x_{-1} = -2$, $x_0 = -1$, $y_0 = 1$ and $y_{-1} = 2$. Then it can easily be seen that all hypotheses of Theorem 1 are satisfied and (2) becomes $x_{n+1} = \frac{1}{3}x_n = \left(\frac{1}{3}\right)^{n+1} x_0$ which converges to the unique solution $z^* = 0$ of equation $F(x) = 0$. However condition (6) is not satisfied and iteration (4) becomes $x_{n+1} = -x_n \not\to 0$ for any $x_0 \neq 0$.

The uniqueness of the solution z^* of the equation $F(x) = 0$ in $[x_0, y_0]$ can be proved by assuming a condition complementary to (8). In particular we can show the following:

Theorem 3. *Let E_1 and E_2 be two POL-spaces, $F\colon D \subseteq E_1 \to E_2$ be a nonlinear operator and $x_0, y_0 \in D$ with $x_0 \leq y_0$ and $[x_0, y_0] \subseteq D$. Assume:*

(a) *there exists an operator $M\colon D^2 \to L(E_1, E_2)$ such that $M(x, y)$ has a nonnegative left superinverse on D^2 and*

$$F(y) - F(x) \geq M(x, y)(y - x) \tag{12}$$

for all $(x, y) \in D^2$.

(b) *$(x^*, y^*) \in D$ and $F(x^*) = F(y^*)$.*

Then we deduce

$$x^* = y^*.$$

Proof: Let $M_1(x^*, y^*)$ be a nonnegative left superinverse of $M(x^*, y^*)$. Then by (12) we get

$$\begin{aligned} 0 \leq y^* - x^* &\leq M_1(x^*, y^*)M(x^*, y^*)(y^* - x^*) \\ &\leq M_1(x^*, y^*)(F(x^*) - F(y^*)) = 0. \end{aligned}$$

That is we deduce $x^* = y^*$.

That completes the proof of the Theorem.

Remark. Several configurations other than (2) and (3) can also be considered. See [1], [2], [4] for other configurations for explicit Newton-like methods. For example our results can apply to prove convergence for the implicit Newton-like method given by

$$\begin{aligned} F(x_{n+1}) + A(y_n, y_{n-1})(x_{n+1} - x_n) = 0 \quad (n \geq 0) \\ F(y_{n+1}) + A(y_n, y_{n-1})(y_{n+1} - y_n) = 0 \quad (n \geq 0) \end{aligned}$$

where $A\colon D^2 \to L(E_1, E_2)$.

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