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Transient Analysis in Discrete Time of Markovian Queues with Quadratic Rates

R.B. Lenin, P.R. Parthasarathy

Abstract

Transient system size probabilities in discrete time are obtained in exact closed form for queueing models with constant, linear and quadratic arrival and service rates. This is achieved by using some interesting identities of tridiagonal determinants. Equilibrium probabilities are derived.

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1 Introduction

In the recent years, there has been a resurgence of interest in the analysis of discrete time queueing models due to their applications in communication systems and other related areas. These models arise naturally in the study of time sharing computing systems, where messages from a collection of terminals are assigned by different multiplexing methods to a central computer [4]. In such systems, all events occur only at definite regularly spaced time points. Due to its importance in computer communication systems, transient analysis in discrete time is also receiving considerable attention [2, 7]. One of the main reasons for the discrete time analysis is that continuous time analysis cannot accurately give the performance measures of discrete time models, e.g., in computer communication systems the elementary units are the machine cycle time of a processor, the bit or byte duration of signals on a channel. Moreover, the continuous time results can be derived from the discrete analysis as a limiting case.

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In this paper we consider Markovian queueing models with finite capacity. Such queueing models have applications in production and inventory problems, e.g., to optimize the size of the storage space, to determine the trade-off between throughput and inventory (or waiting time) and to exhibit the propagation of blockage. The performance of the produce-to-stock manufacturing facility can be obtained from the performance of the finite queueing systems [3] . The situation of a network of queues with finite buffers occurs widely in computer and telecommunications systems [1, 5, 14]. The phenomena of blocking, starvation and server breakdowns can severely restrict throughput and response time [12]. Data sources in computer networks are generally bursty in nature and it is appropriate to consider these peak arrival rates to be quadratic [6, 11, 15]. In order to process the messages rapidly to avoid congestion, service rates must also be quadratic. Hence the queueing models considered in this paper may be useful in computer networks with quadratic arrivals and service rates.

In the solution procedure, suitable probability generating functions are used to convert the system of difference equations of probabilities into a system of difference equations of generating functions. Using Cramer's rule, the required solution is represented by a ratio of two tridiagonal determinants , that is, ratio of two polynomials defined recursively by three-term recurrence relations. The roots of the denominator polynomial is obtained by reducing the order of the associated tridiagonal determinant by one and comparing the resultant determinant to the identities of tridiagonal determinants [9, 10]. Making use of these roots, the solution is obtained by partial fraction technique. These roots are calculated analytically in closed form irrespective of the order of the system capacity N which is in contrast with the computational difficulties encountered in finding these roots numerically for large value of N. Steady state probabilities are deduced.

2 Notations and solution procedure

We consider queueing models with finite capacity N where arrivals occur according to a Bernoulli process with the probability of an arrival being λ_n and service completions occur according to geometric distribution with probability of a service completion being μ_n during any time slot when there are n customers in the system and they are defined as follows:

a) constant rates

$$\begin{cases} \lambda_n = \lambda, & n = 0, 1, 2, \ldots, N-1, \\ \mu_n = \mu, & n = 1, 2, 3, \ldots, N; \end{cases} \tag{2.1}$$

b) linear rates

$$\begin{cases} \lambda_n = (N-n)a, & n = 0, 1, \ldots, N-1, \\ \mu_n = nb, & n = 1, 2, \ldots, N, \qquad a, b > 0; \end{cases} \tag{2.2}$$

c) quadratic rates

$$\begin{cases} \lambda_n = (N-n)(b-(n-1)c), & n = 0,1,\dots,N-1, \\ \mu_n = (N+n+1)(b+(n+2)c), & n = 1,2,\dots,N, \end{cases} \tag{2.3}$$

with $b > 0$, $-b/(N+2) < c < 0$ and

d)
$$\begin{cases} \lambda_n = (N-n)(b+(n-1)d), & n = 0,1,\dots,N-1, \\ \mu_n = n(c-(n-1)d), & n = 1,2,\dots,N, \end{cases} \tag{2.4}$$

where $b, c > 0$ and

$$\begin{cases} d \le 0, & \text{if } N = 1 \text{ or } 2, \\ \frac{-b}{N-2} < d \le 0, & \text{if } N > 2. \end{cases}$$

The conditions imposed on the parameters ensure the positivity of the rates.

Let X_r be the random variable denoting the number of customers in the system at discrete time epoch r. Let

$$P_r(n,m) = P(X_r = n | X_0 = m), \quad n, m = 0, 1, \dots, N.$$

For notational convenience we make the second argument of $P_r(n,m)$ implicit. The function $P_r(n)$ satisfies the following difference equations:

$$\begin{aligned} P_{r+1}(0) &= (1-\lambda_0)P_r(0) + \mu_1 P_r(1) \\ P_{r+1}(n) &= \lambda_{n-1}P_r(n-1) + (1-\lambda_n-\mu_n)P_r(n) + \mu_{n+1}P_r(n+1), \ n = 1,2,3,\dots,N-1 \\ P_{r+1}(N) &= \lambda_{N-1}P_r(N-1) + (1-\mu_N)P_r(N). \end{aligned}$$

If $G_z(n) = \sum_{r=0}^{\infty} P_r(n) z^r$; $|z| \le 1$, then the above system can be written as

$$\begin{aligned} \left[\frac{1}{z} - (1-\lambda_0)\right] G_z(0) - \mu_1 G_z(1) &= \frac{P_0(0)}{z} \\ -\lambda_{r-1}G_z(r-1) + \left[\frac{1}{z} - (1-\lambda_r-\mu_r)\right] G_z(r) - \mu_{r+1}G_z(r+1) &= \frac{P_0(r)}{z}, \\ & \quad r = 1,2,3,\dots,N-1 \\ -\lambda_{N-1}G_z(N-1) + \left[\frac{1}{z} - (1-\mu_N)\right] G_z(N) &= \frac{P_0(N)}{z} \end{aligned}$$

where $P_0(j) = \delta_{mj}$, the Kronecker delta.

If $e_m = [\delta_{m0}, \delta_{m1}, \dots, \delta_{mN}]'$, is a column vector with 1 in the mth place and 0 elsewhere, then the above system can be written in matrix form as follows:

$$M(z)\,[G_z(0), G_z(1), \dots, G_z(N)]' = \frac{e_i}{z} \ ,$$

where

$$M(z) = \begin{bmatrix} \frac{1}{z}-1+\lambda_0 & -\mu_1 & & & \\ -\lambda_0 & \frac{1}{z}-1+\lambda_1+\mu_1 & -\mu_2 & & \\ & & \ddots & & \\ & & & -\lambda_{N-1} & \frac{1}{z}-1+\lambda_N+\mu_N \end{bmatrix}_{(N+1)\times(N+1)}.$$

For the sake of simplicity we use the transformation $s = \dfrac{1}{z} - 1$ and denote $M(z)$ by $M(s)$ and $G_z(N)$ by $G_s(N)$. Hence

$$M(s) = \begin{bmatrix} s+\lambda_0 & -\mu_1 & & & & \\ -\lambda_0 & s+\lambda_1+\mu_1 & -\mu_2 & & & \\ & -\lambda_1 & s+\lambda_2+\mu_2 & -\mu_3 & & \\ & & & \ddots & & \\ & & & & -\lambda_{N-1} & s+\mu_N \end{bmatrix}_{(N+1)\times(N+1)} \tag{2.5}$$

Using Cramer's rule we have

$$G_s(j) = \frac{\frac{1}{z}Q_{m,j}(s)}{|M(s)|}, \tag{2.6}$$

where $\frac{1}{z}Q_{m,j}(s)$ is obtained from $|M(s)|$ by replacing column j by $\frac{e_m}{z}$ which can be given explicitly by

$$Q_{m,j}(s) = c_{m,j}A_{N-max(m,j)}(s)B_{min(m,j)}(s) \tag{2.7}$$

where

$$\begin{aligned} B_0(s) &= 1 \\ B_1(s) &= s+\lambda_0 \\ B_n(s) &= (s+\lambda_{n-1}+\mu_{n-1})B_{n-1}(s) - \lambda_{n-2}\mu_{n-1}B_{n-2}(s), \quad n = 2,3,\ldots,N \\ B_{N+1}(s) &= (s+\mu_N)B_N(s) - \lambda_{N-1}\mu_N B_{N-1}(s), \end{aligned} \tag{2.8}$$

$$\begin{aligned} A_0(s) &= 1 \\ A_1(s) &= s+\mu_N \\ A_n(s) &= (s+\lambda_{N+1-n}+\mu_{N+1-n})A_{n-1}(s) - \lambda_{N+1-n}\mu_{n-2}A_{n-2}(s), \quad n = 2,3,\ldots,N+1. \end{aligned}$$

and

$$c_{m,j} = \begin{cases} \lambda_m\lambda_{m+1}\ldots\lambda_{j-1}, & m < j \\ 1, & m = j \\ \mu_{j+1}\mu_{j+2}\ldots\mu_m, & m > j \end{cases}$$

Derivation of (2.7) can be had from [13].

To analyze (2.6), the roots of $|M(s)|$ are to be computed. We observe that $|M(s)|$ is zero when $-s$ is an eigenvalue of the matrix

$$\begin{bmatrix} \lambda_0 & -\mu_1 & & & & \\ -\lambda_0 & \lambda_1+\mu_1 & -\mu_2 & & & \\ & -\lambda_1 & \lambda_2+\mu_2 & \mu_3 & & \\ & & & \ddots & & \\ & & & & -\lambda_{N-1} & \mu_N \end{bmatrix}_{(N+1)\times(N+1)}.$$

This matrix can be transformed into a real symmetric positive definite tridiagonal matrix with non-zero sub-diagonal elements and therefore the eigenvalues are real and distinct [16]. Suppose $s_0, s_1, s_2, \ldots, s_N$ are the roots of $|B(s)|$ with $s_0 < s_1 < \ldots < s_N$. From (2.6) we have,

$$G_s(j) = \frac{\frac{1}{z}Q_{m,j}(s)}{\prod_{k=0}^{N}(s+s_k)}$$

Using partial fraction technique and replacing s by $\frac{1}{z}-1$ we have,

$$G_z(j) = \sum_{k=0}^{N} \frac{\frac{1}{z}Q_{m,j}(s_k)}{\left(\frac{1}{z}-1-s_k\right)\prod_{j=0,j\neq k}^{N}(s_k-s_j)}$$

Expanding binomially in terms of z and comparing the coefficients of z^r we have,

$$P_r(j) = \sum_{k=0}^{N} \frac{Q_{m,j}(s_k)}{\prod_{j=0,j\neq k}^{N}(s_k-s_j)}(1+s_k)^r$$

At the roots of $|M(s)|$, $Q_{m,j}(s)$ given by (2.7) can be written as

$$Q_{m,j}(s_k) = c_{m,N}c_{N,j}\frac{B_m(s_k)B_j(s_k)}{B_N(s_k)}, \quad k = 0, 1, \ldots, N.$$

Therefore,

$$P_r(j) = c_{m,N}c_{N,j}\sum_{k=0}^{N} \frac{B_m(s_k)B_j(s_k)}{B_N(s_k)\prod_{j=0,j\neq k}^{N}(s_k-s_j)}(1+s_k)^r. \tag{2.9}$$

Using the following two identities of tridiagonal determinants

$$\begin{vmatrix} a_1 & b_1 & & \\ c_1 & a_2 & b_2 & \\ & c_2 & a_3 & \\ & & & \ddots \end{vmatrix}_{N\times N} = \begin{vmatrix} a_1 & 1 & & \\ b_1c_1 & a_2 & 1 & \\ & b_2c_2 & a_3 & \\ & & & \ddots \end{vmatrix}_{N\times N}.$$

and

$$\begin{vmatrix} \eta+a_0 & a_0 & & & & \\ b_1 & \eta+a_1+b_1 & a_1 & & & \\ & b_2 & \eta+a_2+b_2 & a_2 & & \\ & & & \ddots & & \\ & & & & \eta+a_{N-1}+b_{N-1} & a_{N-1} \\ & & & & b_N & \eta+b_N \end{vmatrix}_{(N+1)\times(N+1)}$$

$$= \eta \times \begin{vmatrix} \eta+a_0+b_1 & b_1 & & & \\ a_1 & \eta+a_1+b_2 & b_2 & & \\ & & \ddots & & \\ & & & \eta+a_{N-2}+b_{N-1} & b_{N-1} \\ & & & a_{N-1} & \eta+a_{N-1}+b_N \end{vmatrix}_{N\times N}.$$

the determinant of the matrix $|M(s)|$ can be written as

$$M_{N+1}(s) = s \times \begin{vmatrix} s+\lambda_0+\mu_1 & \lambda_1 & & & & \\ \mu_1 & s+\lambda_1+\mu_2 & \lambda_2 & & & \\ & \mu_2 & s+\lambda_2+\mu_3 & \lambda_3 & & \\ & & & \ddots & & \\ & & & & \mu_{N-1} & s+\lambda_{N-1}+\mu_N \end{vmatrix}_{N\times N} \tag{2.10}$$

Equation (2.10) shows that one of the roots of $|M(s)|$ is zero(let $s_0 = 0$) and hence we can calculate steady state probabilities $p(j),\ j = 0, 1, \ldots, N$. When r goes to infinity, we have from (2.9),

$$p(j) = (-1)^N c_{m,N} c_{N,j} \frac{B_m(0)B_j(0)}{B_N(0) \prod_{j=1, j\neq k}^{N} (s_j)}$$

Using

$$B_k(0) = \prod_{i=0}^{k-1} \lambda_i, \quad k = 0, 1, \ldots, N \tag{2.11}$$

we have,

$$p(j) = \frac{\left(\prod_{k=j+1}^{N} \mu_k\right)\left(\prod_{k=0}^{j-1} \lambda_k\right)}{\prod_{k=1}^{N} (-s_k)}, \quad j = 0, 1, \ldots, N. \tag{2.12}$$

Therefore (2.9) can be written as

$$P_r(j) = p(j) + c_{m,N}c_{N,j}\sum_{k=1}^{N}\frac{B_m(s_k)B_j(s_k)}{B_N(s_k)\prod\limits_{j=0,j\neq k}^{N}(s_k - s_j)}(1+s_k)^r. \tag{2.13}$$

The length of time required for the system to reach steady state is called relaxation time [8] and is given by

$$T = \frac{1}{\min\limits_{1\leq k\leq N}(-s_k)}. \tag{2.14}$$

3 Queueing model with constant rates

In this section we find the roots of $M_{N+1}(s)$ for the queueing model with rates (2.1).

Theorem 1 *For the queueing model with constant λ_n and μ_n given by (2.1) the roots of $M_{N+1}(s)$ are given by $s_0 = 0, \quad s_k = \; -\lambda - \mu + 2\sqrt{\lambda\mu}\ \cos\dfrac{k\pi}{N+1}, k = 1, 2, ..., N.$*

Proof We use the following identity to prove this theorem.
Identity

$$\begin{vmatrix} \eta & a & . & . & . & . & . & . \\ b & \eta & a & . & . & . & . & . \\ . & b & \eta & a & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & b & \eta & a \\ . & . & . & . & . & . & b & \eta \end{vmatrix}_{N\times N} = \prod_{j=1}^{N}\left(\eta - 2\sqrt{ab}\ \cos\left(\frac{j\pi}{N+1}\right)\right).$$

From (2.10) we have

$$M_{N+1}(s) = s \times \begin{vmatrix} s+\lambda+\mu & \lambda & & & \\ \mu & s+\lambda+\mu & \lambda & & \\ & \mu & s+\lambda+\mu & \lambda & \\ & & & \ddots & \\ & & & \mu & s+\lambda+\mu \end{vmatrix}_{N\times N}.$$

On comparing the above determinant with the determinant in the identity we get

$$M_{N+1}(s) = s\prod_{i=1}^{N}\left(s+\lambda+\mu - 2\sqrt{\lambda\mu}\ \cos\frac{i\pi}{N+1}\right).$$

Therefore the roots of $M_{N+1}(s)$ are given by

$$0, \quad -\lambda-\mu+2\sqrt{\lambda\mu}\ \cos\frac{\pi}{N+1}, \ -\lambda-\mu+2\sqrt{\lambda\mu}\ \cos\frac{2\pi}{N+1}, \ldots, -\lambda-\mu+2\sqrt{\lambda\mu}\ \cos\frac{N\pi}{N+1}$$

□

The relaxation time for the model under consideration is given by

$$T = \frac{1}{\lambda+\mu-2(\lambda\mu)^{1/2}\cos\frac{\pi}{N+1}}.$$

In the following theorem we find the explicit expression for $B_i(s)$ defined by (2.8) at the roots of $M_{N+1}(s)$.

Theorem 2 *For the queueing model with constant λ_n and μ_n given by (2.1)*

$$B_i(s_k) = \begin{cases} \lambda^i & \text{for } k=0 \\ & \text{for } i=0,1,\ldots,N, \\ (\lambda\mu)^{i/2}\, cosec\left(\frac{k\pi}{N+1}\right)\left\{\sin\left(\frac{(i+1)k\pi}{N+1}\right)-\sqrt{\frac{\mu}{\lambda}}\sin\left(\frac{ik\pi}{N+1}\right)\right\}, & \text{for } k=1,2,\ldots,N. \end{cases}$$

Proof The result for the case $k=0$ follows from (2.11). We assume $k>0$.

$B_i(s)$ defined recursively in (2.8) can be represented in tridiagonal determinant form as follows

$$B_i(s) = \begin{vmatrix} s+\lambda_0 & 1 & & & \\ \lambda_0\mu_1 & s+\lambda_1+\mu_1 & 1 & & \\ & \lambda_1\mu_2 & s+\lambda_2+\mu_2 & 1 & \\ & & & \ddots & \\ & & & \lambda_{i-2}\mu_{i-1} & s+\lambda_{i-1}+\mu_{i-1} \end{vmatrix}_{i\times i}. \quad (3.1)$$

By substituting $\lambda_n=\lambda$ and $\mu_n=\mu$ in the above determinant and using the identity given in theorem 1 we get

$$B_i(s) = \prod_{j=1}^{i}\left\{s+\lambda+\mu-2\sqrt{\lambda\mu}\ \cos\frac{j\pi}{i+1}\right\} - \mu\prod_{j=1}^{i-1}\left\{s+\lambda+\mu-2\sqrt{\lambda\mu}\ \cos\frac{j\pi}{i}\right\}.$$

With $s_k = -\lambda-\mu+2\sqrt{(\lambda\mu)}\cos\left(\frac{k\pi}{N+1}\right)$,

$$\begin{aligned} B_i(s_k) &= 2^i(\lambda\mu)^{i/2}\prod_{j=1}^{i}\left\{\cos\left(\frac{k\pi}{N+1}\right)-\cos\left(\frac{j\pi}{i+1}\right)\right\} \\ &\quad -2^{i-1}\mu(\lambda\mu)^{(i-1)/2}\prod_{j=1}^{i-1}\left\{\cos\left(\frac{k\pi}{N+1}\right)-\cos\left(\frac{j\pi}{i}\right)\right\} \end{aligned}$$

Using the identity

$$\prod_{r=1}^{n-1}\left\{\cos\left(\frac{k\pi}{n+1}\right)-\cos\left(\frac{r\pi}{n}\right)\right\}=2^{-(n-1)}\sin\left(\frac{nk\pi}{n+1}\right)\operatorname{cosec}\left(\frac{k\pi}{n+1}\right)$$

we have

$$B_i(s_k)=(\lambda\mu)^{i/2}\operatorname{cosec}\left(\frac{k\pi}{N+1}\right)\left\{\sin\left(\frac{(i+1)k\pi}{N+1}\right)-\sqrt{\frac{\mu}{\lambda}}\sin\left(\frac{ik\pi}{N+1}\right)\right\}.$$

□

Using the results of theorems 1, 2 and (2.13) the transient system size probabilities are obtained after considerable simplification and are given in the following theorem.

Theorem 3 *For the queueing model with constant λ_n and μ_n given by (2.1) the transient system size probabilities are given by*

$$\begin{aligned} P_r(j) &= \frac{\rho^{N-n}}{\prod_{j=1}^{N}\left\{1+\rho-2\sqrt{\rho}\cos\left(\frac{j\pi}{N+1}\right)\right\}} \\ &+ (-1)^{N+1}\rho^{m-j}\lambda\mu\sum_{k=1}^{N}\frac{Y_{mk}Y_{jk}\left[1-(\lambda+\mu-2\sqrt{(\lambda\mu)}\cos\frac{k\pi}{N+1})\right]}{(N+1)\left(1+\rho-2\sqrt{\rho}\cos\left(\frac{k\pi}{N+1}\right)\right)} \end{aligned}$$

$$j=0,1,2,\ldots,N,$$

where $Y_{ik}=\sin\left(\frac{(i+1)k\pi}{N+1}\right)-\sqrt{\frac{\mu}{\lambda}}\sin\left(\frac{ik\pi}{N+1}\right)$, $\rho=\frac{\lambda}{\mu}$ and $m(\geq 0)$ is the initial state.

4 Queueing model with linear rates

In this section we find the roots of $M_{N+1}(s)$ for the queueing model with linear rates (2.2).

Theorem 4 *For the queueing model with linear λ_n and μ_n given by (2.2) the roots of $M_{N+1}(s)$ are given by*

$$s_k=-k(a+b),\quad k=0,1,\ldots,N$$

Proof We use the following identity to prove this theorem.
Identity
If $\eta_k=\xi-(k-1)(a-b)$; $\beta_k=(N-k)a$ and $\nu_k=kb$, then

$$\begin{vmatrix} \eta_1 & \beta_1 & & & \\ \nu_1 & \eta_2 & \beta_2 & & \\ & \nu_2 & \eta_3 & \beta_3 & \\ & & & \ddots & \\ & & & & \eta_N \end{vmatrix}_{N\times N}=\prod_{i=1}^{N}\{\xi-(N-i)a+(i-1)b\}.$$

On substituting (2.2) in (2.10) and comparing the resultant determinant with the determinant in the identity we have,

$$\eta_1 = s + \lambda_0 + \mu_1.$$

That is, $\xi = s + Na + b$. For this ξ we have

$$s + \lambda_{n-1} + \mu_n = \eta_n, \quad n = 2, 3, 4, \ldots, N$$

$$\lambda_k = \beta_k, \quad k = 0, 1, 2, \ldots, N-1$$

$$\mu_k = \nu_k, \quad k = 1, 2, 3, \ldots, N.$$

Therefore,

$$M_{N+1}(s) = s \prod_{i=1}^{N} (\xi - (N-i)a + (i-1)b).$$

Substituting for ξ we get,

$$M_{N+1}(s) = \prod_{i=0}^{N} (s + i(a+b)).$$

Hence the roots are

$$s_k = -k(a+b), \quad k = 0, 1, 2, \ldots, N.$$

□

The relaxation time for the model under consideration is given by

$$T = \frac{1}{a+b}.$$

Using (3.1) and the above theorem explicit expression for $B_i(s)$ $(i = 0, 1, 2, \ldots, N)$ at the roots of $M_{N+1}(s)$ is obtained by induction as given in the following theorem.

Theorem 5 *For the queueing model with linear λ_n and μ_n given by (2.2)*

$$B_i(s_k) = \begin{cases} \prod_{j=0}^{i-1} \lambda_j, & k = 0 \\ \\ i!\gamma_i^k, & k = 1, 2, \ldots, N. \end{cases} \qquad \textit{for } i = 0, 1, \ldots, N,$$

where

$$\gamma_i^k = \sum_{j=0}^{i} (-1)^j \binom{k}{j} \binom{N-k}{i-j} b^j a^{i-j}.$$

After considerable simplification we obtain the following result using theorems 4 and 5.

Theorem 6 *For the queueing model with linear λ_n and μ_n given by (2.2) we have for $r = 0, 1, \ldots, N$,*

$$P_r(j) = \left(\begin{array}{c} N \\ j \end{array} \right) \frac{\rho^j}{(1+\rho)^N} + \frac{a^{N-m-j}\rho^j (N-m)!m!}{(1+\rho)^N} \sum_{k=1}^{N} \frac{\gamma_m^k \gamma_j^k \left(\gamma_N^k\right)^{-1}}{\prod\limits_{j=0, j\neq k}^{N} (j-k)} (1-k(a+b))^r.$$

where $\rho = \frac{a}{b}$ and $m \geq 0$ is the initial condition.

As $r \to \infty$, we have from the result of theorem 4.3 the steady state probabilities,

$$p(j) = \left(\begin{smallmatrix} N \\ j \end{smallmatrix} \right) \frac{\rho^j}{(1+\rho)^N}, \quad j = 0, 1, 2, \ldots, N.$$

In the following section we find the roots of $M_{N+1}(s)$ for the queueing model with quadratic rates given by (2.3) and (2.4) and the corresponding steady state probabilities and relaxation times.

5 Queueing models with quadratic rates

In the following theorem the roots of $M_{N+1}(s)$ are found for the queueing model with quadratic rates (2.3).

Theorem 7 *For the queueing model with quadratic λ_n and μ_n given by (2.3) the roots of $M_{N+1}(s)$ are given by*

$$s_j = -2j\left[2b - c(2j-5) + 2Nc\right], \quad j = 0, 1, 2, \ldots, N.$$

Proof We use the following identity to prove this theorem.
Identity
If $\eta_k = \xi + 2(k^2 - 1);\ \beta_k = (N-k)(b-(k-1)c);\ \nu_k = (N+k+1)(b+(k+2)c)$; then

$$\left| \begin{array}{cccccc} \eta_1 & \beta_1 & & & & \\ \nu_1 & \eta_2 & \beta_2 & & & \\ & \nu_2 & \eta_3 & \beta_3 & & \\ & & & \ddots & & \\ & & & & & \eta_N \end{array} \right|_{N\times N} = \prod_{i=0}^{N-1} \left\{ X + 2iY + i[N-(i-3)]Z \right\},$$

where $X = \xi - 2(N-1)b;\ Y = 2b - 5c$ and $Z = 4c$.

On substituting (2.3) in (2.10) and comparing the resultant determinant with the determinant in the identity we have,

$$\eta_1 = s + \lambda_0 + \mu_1.$$

That is,

$$\xi = s + 2b(N+1) + 4Nc + 6c.$$

For this value of ξ we have

$$s + \lambda_{n-1} + \mu_n = \eta_n, \quad n = 2, 3, 4, \ldots, N$$

$$\lambda_k = \beta_k, \quad k = 0, 1, 2, \ldots, N-1$$

$$\mu_k = \nu_k, \quad k = 1, 2, 3, \ldots, N.$$

Therefore,

$$M_{N+1}(s) = s \prod_{i=0}^{N-1} (X + 2iY + i(N+3-i)Z).$$

On substituting the values of X, Y and Z we get,

$$M_{N+1}(s) = s \prod_{i=0}^{N-1} (s + 2(i+1)\left[2b - c(2i-3) + 2Nc\right]).$$

Hence the roots of $M_{N+1}(s)$ are given by

$$s_j = -2j\left[2b - c(2j-5) + 2Nc\right], \quad j = 0, 1, 2, \ldots, N.$$

□

The relaxation time for the model under consideration is given by

$$T = \frac{1}{4Nc + 4b - 6c}.$$

The transient system size probabilities can be calculated using the above roots from (2.13).

The steady state probabilities calculated using (2.12) are given by

$$p(j) = \frac{\prod_{i=j+1}^{N} (N+i+1)[b + (i+2)c] \prod_{i=0}^{j-1} (N-i)[b - (j-i)c]}{2^N \prod_{i=1}^{N} (i+1)[2b - c(2i-3) + 2Nc]}, \quad j = 0, 1, 2, \ldots, N.$$

In the following theorem the roots of $M_{N+1}(s)$ for the queueing model with quadratic rates (2.4) are found.

Theorem 8 *For the queueing model with quadratic λ_n and μ_n given by (2.4) the roots of $M_{N+1}(s)$ are given by*

$$s_j = -j\left[(b+c)-(N-j+1)d\right], \quad j = 0,1,2,\ldots,N.$$

Proof We use the following identity to prove this theorem.

Identity

If $\eta_k = \xi - (k-1)(b-c) + (k-1)\{N-2(k-1)\}d;\ \beta_k = (N-k)\{b+(k-1)d\}$ and $\nu_k = k\{c-(k-1)\}d$ then

$$\begin{vmatrix} \eta_1 & \beta_1 & & & \\ \nu_1 & \eta_2 & \beta_2 & & \\ & \nu_2 & \eta_3 & \beta_3 & \\ & & & \ddots & \\ & & & & \eta_N \end{vmatrix}_{N\times N} = \prod_{i=1}^{N}\{\xi-(N-i)b+(i-1)c-(i-1)(N-i)d\}\ .$$

On substituting (2.4) in (2.10) and comparing the resultant determinant with the determinant in the identity we have,

$$\eta_1 = s + \lambda_0 + \mu_1.$$

That is,

$$\xi = s + N(b-d) + c.$$

With this value of ξ we could prove

$$s + \lambda_{n-1} + \mu_n = \eta_n, \quad n = 2,3,4,\ldots,N$$

$$\lambda_k = \beta_k, \quad k = 0,1,2,\ldots,N-1$$

$$\mu_k = \nu_k, \quad k = 1,2,3,\ldots,N.$$

Therefore,

$$M_{N+1}(s) = \prod_{i=0}^{N}\left(s+i(b+c)-i(N-i+1)d\right).$$

Hence the roots of $M_{N+1}(s)$ are given by

$$s_j = -j\left[(b+c)-(N-j+1)d\right], \quad j = 0,1,2,\ldots,N.$$

□

The relaxation time for the model under consideration is given by

$$T = \frac{1}{-dN+b+c}.$$

The transient system size probabilities can be calculated using the above roots from (2.13).

The steady state probabilities calculated using (2.12) are given by

$$p(j) = \frac{\prod_{i=j+1}^{N} i[c-(i-1)d] \prod_{i=0}^{j-1}(N-i)[b+(j-i)d]}{\prod_{i=1}^{N} i[b+c+(N-i+1)d]}, \quad j=0,1,2,\ldots,N.$$

If $d=0$, this model reduces to the linear model discussed in section 4.

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References

[1] Bertsekas, D. and Gallager, R. *Data Networks*, Second edition, Prentice Hall - International Edition, 1994.

[2] Böhm, W.M. *Markovian Queueing Systems in Discrete Time: Mathematical Systems in Economics*, 137,(1993) Springer-Verlag.

[3] Buzacott, J.A. and Shantikumar, J.G. *Stochastic Models of Manufacturing Systems*, Prentice Hall, New Jersey, 1993.

[4] Hunter, J.J. *Mathematical Techniques of Applied Probability - Discrete time models: Techniques and applications*, 2, Academic Press, New York, 1993.

[5] Klieenrock, L. *Queueing Systems, Vol - II: Computer Applications*, John Wiley, New York, 1976.

[6] Lamond, B.F. Optimal admission policies for a finite queue with bursty arrivals. *Ann. Oper. Res.*, 28, (1991), 243-260.

[7] Mohanty, S.G., Parthasarathy, P.R. and Sharafali, M. Transient solution of a discrete queue, *Stat. Res. Rep.*, McMaster University, (1990), Canada.

[8] Morse, P.M. *Queues, Inventories and Maintenance*, John Wiley, New York, 1958.

[9] Muir, T. *History of the Theory of Determinants*, Dover Publications, New York, 1960.

[10] Muir, T. *A Treatise on the Theory of Determinants*, Dover Publications, New York,1960.

[11] Park, D., Perros, H.G. and Yamashita, H. Approximate analysis of discrete-time tandem queueing networks with bursty and correlated input traffic and customer loss. *Oper.Res. Lett.*, 15, (1994), 95-104.

[12] Perros, H.G. and Altiok, T. (Eds.) *Queueing Networks With Blocking*, Elsevier Science Publishers, B.V., North Holland, Amsterdam, 1989.

[13] Rosenlund, S.I. Transition probabilities for a truncated birth-death process. *Scand. J. Statist.*, 5, (1978), 119-122.

[14] Takagi, H. *Stochastic Analysis of Computer and Communication Systems, Vol - II: Finite Systems*, North-Holland, Amsterdam, 1993.

[15] Toomey, F. Bursty traffic and finite capacity queues. *Queueing networks with blocking. Ann. Oper. Res.*, 79, (1998), 45-62.

[16] Wilkinson, J.H. *The Algebraic Eigenvalue Problem*, Prentice Hall, New York, 1965.

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Fixed Point Theorem by Altering Distance between the Points

M. El Amrani and A. B. Mbarki

Abstract

In this note we establish a fixed point theorem in complete metric spaces which generalizes some results in [2].

Subject Classification Codes: 47H10
Key Words: Fixed Points, Complete Metric Spaces, Generalized Contractive Conditions

1 Introduction

Delbosco [1] and Skof [3] have established fixed point theorems for self-maps of complete metric spaces by altering the distances between the points with the use of a function $\phi : [0, \infty) \to [0, \infty)$ satisfying the following properties:
(i) $\phi : [0, \infty) \to [0, \infty)$ is continuous and stricly increasing in $\Re^+$;
(ii)$\phi(t) = 0$ iff $t = 0$;
(iii)$\phi(t) \geq Mt^u$ for every $t > 0, M, u > 0$ are constants.

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In 1977, F. skof [3] gave the following theorem

Theorem 1.1 *Let T be a self-map of a complete metric space (X,d) and ϕ satisfying the conditions (i), (ii) and (iii). Suppose that T satisfies the following condition:*

$$(A.1)\ \ \phi(d(Tx,Ty)) \leq a\phi(d(x,y)) + b\phi(d(x,Tx)) + c\phi(d(y,Ty)),$$

where $x,y \in X$, a,b and c are three nonnegative constants satisfying $a+b+c<1$. Then T has a unique fixed point.

In [1], the author has considered functions ϕ satisfying the conditions (i),(ii) and (iii) such that $\phi(t)=t^{n+1},\ n \in N$ for every $t \geq 0$. Clearly, $\phi \circ d$ is not in general, a metric.
In 1984, M. S. Khan et al. [2] generalized Theorem 1.1 by using a much extensive condition than (A. 1) and removed the hypothesis (iii) which seems superfluous. They proved the following theorem:

Theorem 1.2 *Let T be a self-map of a complete metric space (X,d) and ϕ satisfying (i) and (ii). Furthermore, let a,b,c be three decreasing functions from $(0,\infty)$ into $[0,1)$ such that*

$$(B.1)\quad a(t)+2b(t)+c(t)<1$$

for every $t>0$.
Suppose that T satisfies the following condition:

$$\begin{aligned}(B.2)\ \phi(d(Tx,Ty)) &\leq a(d(x,y))\phi(d(x,y)) + b(d(x,y))[\phi(d(x,Tx)) + \phi(d(y,Ty))] \\ &+ c(d(x,y))min\{\phi(d(y,Tx)), \phi(d(x,Ty))\}.\end{aligned}$$

Where $x,y \in X$ and $x \neq y$. Then T has a unique fixed point.

In this note we shall extend this last theorem by using a much extensive hypothesis than (B. 1) and a,b and c are three decreasing functions.

2 Main result

We shall make frequent use of the following definition:

Definition 2.1 *A function ψ will be said to have property (P) if $\psi:(0,\infty)\to[0,1)$ and if for every $t>0$ there exists $\eta(t)>0$ such that $0 \leq s-t<\eta(t)$ implies $\psi(s)\leq\gamma(t)<1$. (that is, $\limsup_{s\to t^+}\psi(s)<1$).*

We establish the following:

Theorem 2.1 *Let T be a self-map of a complete metric space (X, d) and ϕ satisfying (i) and (ii). Furthermore, let a, b, c be three functions from $(0, \infty)$ into $[0, 1)$ such that $a + 2b$ and $a + c$ have both property (P). Suppose that T satisfies the condition (B. 2). Then T has a unique fixed point.*

Proof. Let x be some point in X, define

$$x_n = T^n x, \ n = 0, 1, 2, ...$$

Let $\tau_n = d(x_n, x_{n+1})$. We have for $n \geq 1$ (We may assume that $\tau_n > 0$)

$$\phi(\tau_n) = \phi(d(x_n, x_{n+1})) = \phi(d(T^n x, T^{n+1} x)) \leq A_1 + A_2 + A_3$$

Where

$$\begin{aligned} A_1 &= a(d(x_{n-1}, x_n))\phi(d(x_{n-1}, x_n)), \\ A_2 &= b(d(x_{n-1}, x_n))[\phi(d(x_{n-1}, x_n)) + \phi(d(x_n, x_{n+1}))], \\ A_3 &= c(d(x_{n-1}, x_n))min\{\phi(d(x_{n-1}, x_{n+1})), \phi(d(x_n, x_n))\}. \end{aligned}$$

So

$$\phi(\tau_n) \leq a(\tau_{n-1})\phi(\tau_{n-1}) + b(\tau_{n-1})[\phi(\tau_{n-1}) + \phi(\tau_n)]$$

Hence we have

$$\phi(\tau_n) \leq \frac{a(\tau_{n-1}) + b(\tau_{n-1})}{1 - b(\tau_{n-1})}\phi(\tau_{n-1}) < \phi(\tau_{n-1}) \ (1)$$

Since ϕ is increasing, $\{\tau_n\}$ is a decreasing sequence. Let $lim_n \tau_n = \tau$, assume that $\tau > 0$. Since $a + 2b$ has property (P), then for $0 \leq s - \tau < \eta(\tau)$ we have $a(s) + 2b(s) \leq \gamma(\tau) < 1$, which implies

$$\frac{a(s) + b(s)}{1 - b(s)} \leq \frac{\gamma(\tau) - b(s)}{1 - b(s)}.$$

It follows that for these s

$$\frac{a(s) + b(s)}{1 - b(s)} \leq \gamma(\tau). \ (2)$$

Since

$$\tau_n \geq \tau, \quad \lim_n \tau_n = \tau \quad \text{and} \quad \lim_n \frac{\phi(\tau_n)}{\phi(\tau_{n-1})} = 1,$$

there exists an integer $p \geq 1$ such that

$$0 \leq \tau_{p-1} - \tau < \eta(\tau) \quad (3) \quad \text{while} \quad \gamma(\tau) < \frac{\phi(\tau_p)}{\phi(\tau_{p-1})},$$

the last inequality with (1) imply that

$$\gamma(\tau) < \frac{a(\tau_{p-1}) + b(\tau_{p-1})}{1 - b(\tau_{p-1})}$$

and (2) with (3) imply that

$$\frac{a(\tau_{p-1}) + b(\tau_{p-1})}{1 - b(\tau_{p-1})} \leq \gamma(\tau),$$

which is a cantradiction. Consequently $\tau = 0$.
If $\{x_n\}$ is not cauchy sequence, there exists $\varepsilon > 0$, and two sequences of natural numbers $\{m(i)\}, \{n(i)\}, n(i) > m(i), m(i) \to \infty$ as $i \to \infty$ such that

$$d(x_{n(i)}, x_{m(i)}) > \varepsilon \text{ while } d(x_{n(i)-1}, x_{m(i)}) \leq \varepsilon.$$

Then we have

$$\begin{aligned} \varepsilon < d_i = d(x_{n(i)}, x_{m(i)}) &\leq d(x_{n(i)}, x_{n(i)-1}) + d(x_{n(i)-1}, x_{m(i)}) \\ &\leq \tau_{n(i)-1} + \varepsilon, \end{aligned}$$

since $\{\tau_n\}$ converges to 0, $d_i \to \varepsilon$. Futhermore, by triangular inequality, it follows that

$$d_i - \tau_{n(i)} - \tau_{m(i)} \leq d(x_{n(i)+1}, x_{m(i)+1}) \leq d_i + \tau_{n(i)} + \tau_{m(i)}$$

and therefore the sequence $\{d(x_{n(i)+1}, x_{m(i)+1})\}$ converges to ε
From (B. 2), we also deduce:

$$\begin{aligned} &\phi(d(x_{n(i)+1}, x_{m(i)+1})) = \phi(d(T^{n(i)+1}x, T^{m(i)+1}x)) \\ &\quad \leq C_1 + C_2 + C_3 \\ &\quad \leq C_4 + C_5 + C_6 \quad (4) \end{aligned}$$

Where

$$\begin{aligned}
C_1 &= a(d_i)\phi(d_i),\\
C_2 &= b(d_i)[\phi(\tau_{n(i)}) + \phi(\tau_{m(i)}],\\
C_3 &= c(d_i)min\{\phi(d(x_{n(i)}, x_{m(i)+1}), \phi(d(x_{n(i)+1}, x_{m(i)})\},\\
C_4 &= a(d_i)\phi(d_i + \tau_{n(i)}),\\
C_5 &= [\phi(\tau_{n(i)}) + \phi(\tau_{m(i)})],\\
C_6 &= c(d_i)\phi(d_i + \tau_{n(i)}).
\end{aligned}$$

Since $d_i > \varepsilon$, $d_i \to \varepsilon$ and $a+c$ has property (P), there exists an integer $q \geq 1$ such that

$$0 \leq d_i - \varepsilon < \eta(\varepsilon), \text{ implies } a(d_i) + c(d_i) \leq \gamma(\varepsilon) < 1$$

for all $i \geq q$

From the last inequality and (4) we have

$$\phi(d(x_{n(i)+1}, x_{m(i)+1}) \leq \gamma(\varepsilon)\phi(d_i + \tau_{n(i)}) + \phi(\tau_{n(i)}) + \phi(\tau_{m(i)}),$$

for all $i \geq q$

As $i \to \infty$, we obtain:

$$\phi(\varepsilon) \leq \gamma(\varepsilon)\phi(\varepsilon) < \phi(\varepsilon).$$

This is a contradiction. Hence $\{x_n\}$ is a cauchy sequence in a complete metric space X, then there exists a point $\tilde{x} \in X$ such that $x_n \to \tilde{x}$ as $n \to \infty$. Now we shall show that $T\tilde{x} = \tilde{x}$.

¿From (B.2), we have :

$$\begin{aligned}
&\phi(d(x_n, T\tilde{x})) = \phi(d(T^n x, T\tilde{x}))\\
&\quad \leq a(d(x_{n-1}, \tilde{x}))\phi((d(x_{n-1}, \tilde{x}))\\
&\quad +b((d(x_{n-1}, \tilde{x})))[\phi(d(x_{n-1}, x_n) + \phi(d(T\tilde{x}, \tilde{x}))]\\
&\quad +c((d(x_{n-1}, \tilde{x})))min\{\phi((d(x_{n-1}, T\tilde{x})), \phi(d(x_n, \tilde{x}))\} \quad (5)
\end{aligned}$$

Using $b < \frac{1}{2}$ and letting $n \to \infty$ in (5), we have

$$\phi(d(\tilde{x}, T\tilde{x})) \leq \frac{1}{2}\phi(d(\tilde{x}, T\tilde{x})).$$

That is $\phi(d(\tilde{x}, T\tilde{x})) = 0$ and therefore $d(\tilde{x}, T\tilde{x}) = 0$, .i.e $\tilde{x} = T\tilde{x}$. We claim that $\tilde{x}$ is unique fixed point of T. For this, we suppose that y ($\tilde{x} \neq y$) is another fixed point of T . Then

$$\begin{aligned}
\phi(d(y, \tilde{x})) &= \phi(d(Ty, T\tilde{x})) \\
&\leq a(d(\tilde{x}, y))\phi(d(\tilde{x}, y)) + b(d(\tilde{x}, y))[\phi(d(\tilde{x}, T\tilde{x})) + \phi(d(y, Ty))] \\
&+c(d(\tilde{x}, y))min\{\phi(d(\tilde{x}, Ty)), \phi(d(y, T\tilde{x}))\} \\
&\leq [a(d(\tilde{x}, y)) + c(d(\tilde{x}, y))]\phi(d(\tilde{x}, y)) < \phi(d(\tilde{x}, y)),
\end{aligned}$$

a cantradiction. This completes the proof of the Theorem.

We now present some consequences of Theorem 2.1

Corollary 2.1 *Let T be a self-map of a complete metric space (X, d) and ϕ satisfying (i) and(ii). Furthermore,let a, b, c be three functions from (0 ,∞) into [0,1) satisfying at least one of the following conditions:*
(1) $a + 2b + c$ is decreasing function.
(2) $a + 2b + c$ has property (P).
(3)(4) $a + 2b$ and $a + c : (0, \infty) \to [0, 1)$ are decreasing functions.
(4) $a+2b$ has property (P) and $a+c : (0, \infty) \to [0, 1)$ is decreasing function.
(5) $a+c$ has property (P) and $a+2b : (0, \infty) \to [0, 1)$ is decreasing function.
Suppose that T satisfies the condition (B. 2). Then T has a unique fixed point.

Remark. We first note that Theorem 1.2 is a special case of each result of the above Corollary. Also, it is obvious that in cases (3), (4) and (5) of the Corollary, the function $a + 2b + c$ can take values larger than 1.

REFERENCES

1. D. Delbosco, Un estensione di un teorema sul punto fisso di S. Reich, Rend. Sem. Mat. Univers. Politecn. Torino **35** (1976-77), 233-239.
2. M. S. Khan, Fixed point theorems by altering distance between the points, Bull. Austral. Math. Soc. **30** (1984), 1-9.
3. F. Skof, Teorema di punti fisso per applicazioni negli spazi metrici, Atti. Accad. Aci. Torino **111** (1977), 323-329.

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AN EXAMPLE FOR MAPPINGS RELATED TO SEMI-CONFLUENCE

Pavel Pyrih

Abstract. A semi-confluence of a mapping between topological spaces can be defined in several ways. J.J. Charatonik asked if two definitions of the semi-confluence, using components and quasi-components respectively, are equivalent for surjective mappings with compact point inverses. We give a negative answer to this question in Example 2.1.

A.M.S. Subject Classification Codes: 54C10,54F15.

Key Words: Semi-Confluent, Quasi-Component.

1. Introduction.

We recall from [1] the notation. All *mappings* considered in this paper are continuous. A mapping $f : X \to Y$ between metric continua X and Y is called *semi-confluent* provided for each subcontinuum Q of Y and for any two components C_1 and C_2 of the inverse image $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

A general topological space admits several definitions of semi-confluent mappings. We recall from [1] three definitions using connectedness.

A topological space X is said to be *connected between two its subsets A and B* provided there is no closed and open subset in X that contains A and is disjoint from B. Clearly, connectedness of a space X between points is an equivalence relation on X. The equivalence classes of this relation are called *quasi-components.* In other words, a *quasi-component* of a space X containing a point $p \in X$ is the intersection of all closed and open subsets of X containing p.

A semi-confluence of a mapping $f : X \to Y$ between topological spaces X and Y can be defined (see [1]) by the following conditions :

(1) (SC1) For each connected closed nonempty subset Q of Y and for any two *components* C_1 and C_2 of the inverse image $f^{-1}(Q)$ we have either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

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(2) (SC2) For each connected closed nonempty subset Q of Y and for any two *quasi-components* C_1 and C_2 of the inverse image $f^{-1}(Q)$ we have either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

(3) (SC3) For each connected closed nonempty subset Q of Y and points x_1 and x_2 in $f^{-1}(Q)$ the set $f^{-1}(Q)$ is either *connected between* $\{x_1\}$ *and* $f^{-1}(f(x_2))$ or *connected between* $\{x_2\}$ and $f^{-1}(f(x_1))$.

J.J. Charatonik proved in [1], p.90 the following results.

Theorem 1.1. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y. Then:*

(a) (SC1) $\Rightarrow$ (SC2) $\Rightarrow$ (SC3).

(b) *conditions* (SC1) , (SC2) *and* (SC3) *are equivalent for compact Hausdorff spaces X and Y.*

(b) *each condition* (SC1) , (SC2) *and* (SC3) *is equivalent to semi-confluence for continua X and Y.*

J.J. Charatonik asked in [1], Question 3.4, the following question.

Question 1.3. *Assume a mapping $f : X \to Y$ between topological spaces X and Y has compact point inverses. Does* (SC2) *imply* (SC1) *?*

We give a negative answer in Example 2.1.

2. Counterexample.

The following example is a simple modification of Example 2.1 in [2].

Example 2.1. *There exist topological spaces X and Y and a surjective $f : X \to Y$ with compact point inverses such that* (SC2) *is satisfied and* (SC1) *is not satisfied.*

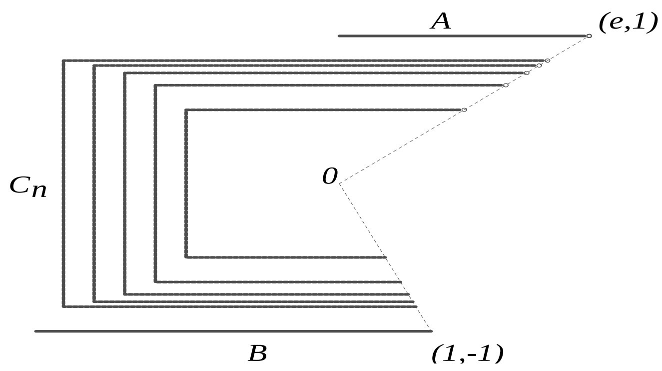

FIGURE 1 (EXAMPLE 2.1)

Proof. Let

$$X = A \cup B \cup \bigcup_{n=1}^{\infty} C_n \quad , \quad Y = A \cup \{(e, 1)\} \quad ,$$

where $A = \{(x,1) \in \mathbb{R}^2 : 0 \le x < e\}$, $B = \{(x,-1) \in \mathbb{R}^2 : x \le 1\}$ and

$$\begin{aligned} C_n = \{(x,y) \in \mathbb{R}^2 &: y = 1 - 1/n, -n \le x < e(1-1/n)\} \\ \cup\{(x,y) \in \mathbb{R}^2 &: y = -1 + 1/n, -n \le x \le 1 - 1/n\} \\ \cup\{(x,y) \in \mathbb{R}^2 &: |y| \le 1 - 1/n, x = -n\} \end{aligned}$$

as in Figure 1. (X is a simple modification of the *nested rectangles* topological space in [3], p.137.) Let both X and Y inherit the topology from the Euclidean plane.

For the definition of the mapping $f : X \to Y$ we need some notation:

For any $(x,1) \in Y$ denote by a_x the plane segment joining $(x,1)$ with the origin and by b_x the plane segment joining $(\log x, -1) \in B$ with the origin.

If $0 \le x < e$ denote $x_n^A = a_x \cap C_n$ and $x_0^A = a_x \cap A$ (we have $x_0^A = (x,1)$). For convenience let $A_n = a_0 \cap C_n$.

If $0 < x \le e$ denote $x_0^B = b_x \cap B$. If $1/n \le x \le e$ denote $x_n^B = b_x \cap C_n$. For convenience let $B_n = b_{1/n} \cap C_n$.

For any two points $S, T \in C_n$ denote by ST the arc in C_n joining S with T and by $d(S,T)$ the linear measure of the arc ST.

If $0 < x < 1/n$ denote by x_n^B the point of A_nB_n such that $d(A_n, x_n^B)/d(A_n, B_n) = nx$.

Now we define $f : X \to Y$ by the formula $f(x_j^A) = f(x_j^B) = (x,1) \in Y$ for $j \ge 0$. See Figure 2.

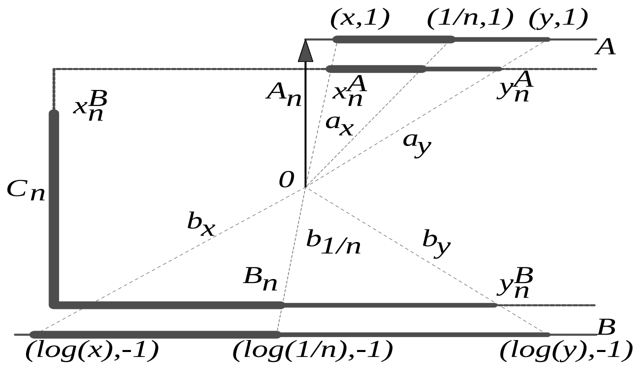

FIGURE 2 (PROOF OF EXAMPLE 2.1)

We claim the following properties of f :

(i) *f is continuous on X.*

(Proof. X is a subset of the space X in Example 2.1 in [2] and f is defined by the same way. ♡)

(ii) *f has compact point inverses.*

(Proof. Clearly the point inverse of $(e,1) \in Y$ is the sequence $\{e_j^B\}_0^\infty$ and all points e_j^B belong to the segment b_e. The sequence $\{e_j^B\}_0^\infty$ is converging to $e_0^B = (1,-1) \in B$. Hence the point inverse set of $(e,1) \in Y$ is compact in X. For all remaining points see [2]. ♡)

(iii) *For* $f : X \to Y$ *the condition* (SC2) *holds.*

(Proof. Let Q be a closed nonempty connected subset of Y. We have 4 cases :

Case (1) : When $(0,1) \notin Q$ and $(e,1) \notin Q$.

Then $f^{-1}(Q) \cap C_n$ is the set consisting of just two disjoint arcwise connected sets $Q_n^A = \{x_n^A : (x,1) \in Q\}$ and $Q_n^B = \{x_n^B : (x,1) \in Q\}$, both of them being closed and open in $f^{-1}(Q)$. Hence both Q_n^A and Q_n^B are quasi-components of $f^{-1}(Q)$ both being mapped onto Q under f.

Similarly, $f^{-1}(Q) \cap A$ is the set $Q^A = \{x_0^A : (x,1) \in Q\} = Q$. Clearly, Q^A is a quasi-component of $f^{-1}(Q)$, since it is arcwise connected and all sufficiently small ε-neighborhoods of Q^A in $f^{-1}(Q)$ are both open and closed in $f^{-1}(Q)$. Moreover, Q^A is mapped onto Q under f.

Similarly, $f^{-1}(Q) \cap B$ is the set $Q^B = \{x_0^B : (x,1) \in Q\}$. Clearly, Q^B is a quasi-component of $f^{-1}(Q)$, since it is arcwise connected and all sufficiently small ε-neighborhoods of Q^B in $f^{-1}(Q)$ are both open and closed in $f^{-1}(Q)$. Moreover, Q^B is mapped onto Q under f. See Figure 2.

All quasi-components of $f^{-1}(Q)$ are mapped onto Q, hence their images are equal and (SC2) holds.

Case (2) : When $(0,1) \in Q$ and $(e,1) \notin Q$.

The set $f^{-1}(Q) \cap C_n$ contains all arcs joining x_n^A with x_n^B in C_n for any $(x,1) \in Q$. It is again a quasi-component of $f^{-1}(Q)$ and is mapped onto Q under f.

The set $Q^A \cup Q^B$ is a quasi-component of $f^{-1}(Q)$ because no closed and open set of $f^{-1}(Q)$ containing the arcwise connected set Q^A can be disjoint from the arcwise connected set Q^B due to the fact that all arcs joining x_n^A with x_n^B in C_n for any $(x,1) \in Q$ are in $f^{-1}(Q)$, see Figure 1 and Figure 2. Moreover, the set $Q^A \cup Q^B$ is mapped onto Q under f.

All quasi-components of $f^{-1}(Q)$ are mapped onto Q, hence their images are equal and (SC2) holds.

Case (3) : When $(0,1) \notin Q$ and $(e,1) \in Q$.

Again $f^{-1}(Q) \cap C_n$ is the set consisting of just two disjoint arcwise connected sets $Q_n^A = \{x_n^A : (x,1) \in Q\}$ and $Q_n^B = \{x_n^B : (x,1) \in Q\}$, both of them being closed and open in $f^{-1}(Q)$. Hence both Q_n^A and Q_n^B are quasi-components of $f^{-1}(Q)$. We see that Q_n^A is mapped onto $Q \setminus \{(e,1)\}$ and Q_n^B is mapped onto Q.

Clearly, Q^A is a quasi-component of $f^{-1}(Q)$, since it is arcwise connected and all sufficiently small ε-neighborhoods of Q^A in $f^{-1}(Q)$ are both open and closed in $f^{-1}(Q)$. Moreover, Q^A is mapped onto $Q \setminus \{(e,1)\}$ under f.

Clearly, Q^B is a quasi-component of $f^{-1}(Q)$, since it is arcwise connected and all sufficiently small ε-neighborhoods of Q^B in $f^{-1}(Q)$ are both open and closed in $f^{-1}(Q)$. Moreover, Q^B is mapped onto Q under f.

All quasi-components of $f^{-1}(Q)$ are mapped either onto Q or onto $Q \setminus \{(e,1)\}$, hence their images differ at most at one point and (SC2) holds.

Case (4) : When $(0,1) \in Q$ and $(e,1) \in Q$.

We have $Q = Y$, $C_n = f^{-1}(Q) \cap C_n$, $f^{-1}(Q) \cap A = A$ and $f^{-1}(Q) \cap B = B$ in this case.

The set C_n is again a quasi-component of $f^{-1}(Q)$ and is mapped onto Q under f.

The set $A \cup B$ is a quasi-component of $f^{-1}(Q)$ because no closed and open set of $f^{-1}(Q)$ containing the arcwise connected set A can be disjoint from the arcwise connected set B due to the fact that all arcs joining x_n^A with x_n^B in C_n for any

$(x,1) \in Q$ are in $f^{-1}(Q)$, see Figure 2 and Figure 1. Moreover, the set $A \cup B$ is mapped onto Q under f.

All quasi-components of $f^{-1}(Q)$ are mapped onto Q, hence their images are equal and (SC2) holds. ♡)

(iv) *For* $f : X \to Y$ *the condition* (SC1) does not hold.

(Proof. If $Q = Y$, then the set $Q^B = B$ is a component of $f^{-1}(Q)$ because it is arcwise connected and no larger subset of $f^{-1}(Q)$ containing Q^B is connected. We see that $(0,1) \notin f(Q^B)$ and $f(Q^B) = Y \setminus \{(0,1)\}$. Similarly $Q^A = A$ is a component of $f^{-1}(Q)$ because it is arcwise connected and no larger subset of $f^{-1}(Q)$ containing Q^A is connected. We see that $(e,1) \notin f(Q^A)$ and $f(Q^A) = Y \setminus \{(e,1)\}$. Now we have two components Q^A and Q^B of $f^{-1}(Q)$ such that neither $f(Q^A) \subset f(Q^B)$ nor $f(Q^B) \subset f(Q^A)$. The condition (SC1) does not hold. See Figure 1. ♡)

The proof is finished. □

References

1. Charatonik J.J., *Mappings related to confluence*, Arch. Math. (Brno) **32** (1996), 85 – 103.

2. Pyrih P., *An example for mappings related to confluence*, Arch. Math. (Brno) **35,2** (1999) (to appear).

3. Steen, L.A., Seebach, J.A. Jr., *Counterexamples in topology*, Springer-Verlag, New-York, 1978.

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On Cantor's singular moments

H. Prodinger

Abstract

We evaluate a constant explicitly, thereby answering a question raised in [1].

Subject Classification Codes: *11,32,43*
Key Words: *Cantor Distribution,Dirichlet Series,Mellin-Perron Summation Formula*

In problem **10621** *of the American Mathematical Monthly, Cantor's singular moments* J_n *were to be computed. In the published answers* [1] *they come out as*

$$J_n = \frac{2}{3(n+1)} \sum_{j=0}^{n} \binom{n+1}{j} \frac{B_j}{3 \cdot 2^{j-1} - 1} \quad \textit{for} \quad n \geq 1$$

and $J_0 = 1$*, with Bernoulli numbers* B_n.

The editor asked, whether it is possible to compute

$$J_{-1} = \sum_{n \geq 0} J_n$$

exactly.

The purpose of this note is to do that. In [5] *we considered a similar problem, and the gentle reader is invited to consult this paper for more background about the technique, as well as* [6, 4] *for more information about the Cantor distribution.*

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Following the method described in [3], we can write J_n as a contour integral viz.

$$
\begin{aligned}
J_n &= \frac{2}{3(n+1)} \cdot \frac{1}{2\pi i} \int_{-\frac12 - i\infty}^{-\frac12 + i\infty} \frac{\Gamma(n+2)\Gamma(1-s)}{\Gamma(n+2-s)} \frac{\zeta(1-s)}{3\cdot 2^{s-1}-1}\, ds \\
&= \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac12 - i\infty}^{-\frac12 + i\infty} \frac{\Gamma(n+1)\Gamma(1-s)}{\Gamma(n+2-s)} \frac{\zeta(1-s)}{3\cdot 2^{s-1}-1}\, ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sum_{n=0}^{N} J_n = J_0 &+ \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac12 - i\infty}^{-\frac12 + i\infty} \frac{\Gamma(N+2)\Gamma(1-s)}{\Gamma(N+2-s)s} \frac{\zeta(1-s)}{3\cdot 2^{s-1}-1}\, ds \\
&- \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac12 - i\infty}^{-\frac12 + i\infty} \frac{\Gamma(2)\Gamma(1-s)}{\Gamma(2-s)s} \frac{\zeta(1-s)}{3\cdot 2^{s-1}-1}\, ds.
\end{aligned}
$$

From this form, one could even compute the asympotics as $N \to \infty$. However, here, we only have to note that the first integral is of order $N^{1-\log_2 3}$, which means that it goes to zero. Consequently

$$
\begin{aligned}
\sum_{n\ge 0} J_n &= 1 - \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac12 - i\infty}^{-\frac12 + i\infty} \frac{1}{(1-s)s} \frac{\zeta(1-s)}{3\cdot 2^{s-1}-1}\, ds \\
&= \frac{4}{3} + \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{\frac32 - i\infty}^{\frac32 + i\infty} \frac{1}{s(s-1)} \frac{\zeta(s)}{3\cdot 2^{-s}-1}\, ds \\
&= \frac{4}{3} + \frac{2}{3} \sum_{k,m\ge 1} 3^{-k} \cdot \frac{1}{2\pi i} \int_{\frac32 - i\infty}^{\frac32 + i\infty} \frac{1}{s(s-1)} \Big(\frac{2^k}{m}\Big)^s\, ds.
\end{aligned}
$$

The last step was by using the Dirichlet series for $\zeta(s)$ and the geometric series, both valid for $\Re s = \frac{3}{2}$. A simple application of residue calculus, as it is often used in the context of the Mellin–Perron summation formula (see [7, 2]) evaluates the integrals inside the summation:

$$
\frac{1}{2\pi i} \int_{\frac32 - i\infty}^{\frac32 + i\infty} \frac{1}{s(s-1)} t^s ds = t - 1 for\ t \ge 1 0 for\ t < 1\ .
$$

Therefore

$$
\begin{aligned}
\sum_{n\ge 0} J_n &= 1 + \frac{2}{3} \sum_{k\ge 1} \sum_{1\le m\le 2^k} 3^{-k} \Big(\frac{2^k}{m} - 1\Big) \\
&= 1 + \frac{2}{3} \sum_{k\ge 1} \Big(\frac{2}{3}\Big)^k H_{2^k} - \frac{2}{3} \sum_{k\ge 1} \Big(\frac{2}{3}\Big)^k \\
&= -\frac{1}{3} + \frac{2}{3} \sum_{k\ge 1} \Big(\frac{2}{3}\Big)^k H_{2^k} = 3.36465\ 07281\ 00925\ 16083\ 89349\ 6289\ldots,
\end{aligned}
$$

with harmonic numbers $H_n = \sum_{1\le k\le n} \frac{1}{k}$.

References

[1] H. G. Diamond and B. Reznick. Problems and solutions. *American Mathematical Monthly*, 106:175–176, 1999.

[2] P. Flajolet, P. Grabner, P. Kirschenhofer, H. Prodinger, and R.F. Tichy. Mellin transforms and asymptotics: Digital sums. *Theoretical Computer Science*, 123:291–314, 1994.

[3] P. Flajolet and R. Sedgewick. Mellin transforms and asymptotics: Finite differences and Rice's integrals. *Theoretical Computer Science*, 144:101–124, 1995.

[4] P. Grabner and H. Prodinger. Asymptotic analysis of the moments of the Cantor distribution. *Statistics and Probability Letters*, 26:243, 1996.

[5] P. Grabner and H. Prodinger. An asymptotic study of a recursion occurring in the analysis of an algorithm on broadcast communication. *Information Processing Letters*, 65:89–93, 1998.

[6] A. Knopfmacher and H. Prodinger. Exact and asymptotic formulæ for average values of order statistics of the Cantor distribution. *Statistics and Probability Letters*, 27:189, 1996.

[7] G. Tenenbaum. *Introduction to analytic and probabilistic number theory.* Cambridge University Press, 1995.

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Computational Complexity of Integrated Models of Network Design and Facility Location

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Abstract

Transportation Network Design and Location Theory are two related areas that have been extensively studied in the past, primarily in isolation of each other. This is unfortunate because when the network is designed improperly, customers get extremely poor service even when service facilities are located optimally. Thus there is a need to investigate models where transportation networks are designed with future facility locations in consideration. In a partial attempt to address this gap, we develop a framework for an integrated model of transportation network design and facility location. In this model, transportation network is to be designed to allow the residents of the network to avail of the service provided by a single facility whose location is predetermined. Our principal contributions are to show that (i) our model reduces to a constrained minimal spanning tree problem and that (ii) it is NP Complete. We demonstrate that this complexity result holds even in the simple case where the customer sites are assumed to be given by the vertices of a convex polygon. This result thus shows that integrated models of facility location and network design can be computationally hard even in the simplest case, where only one facility needs to be located.

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Keywords: Computational complexity; network design; facility location.

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1 Introduction

Transportation Network Design and *Facility Location Theory* are two (seemingly) different areas of study. The former refers to the body of methods that exist for designing a transportation network that is supposed to be used by a set of customers. When conceived in its broadest perspective, the study of network design captures virtually all important aspects of transportation planning: strategic, tactical and operational. As a result, it is an area that has been heavily studied by researchers in the past few decades - see Magnanti and Wong (1984)[9] for a comprehensive review. As mentioned in the above paper, most network design problems are computationally hard (NP Hard); however special instances, such as the Minimal Spanning Tree (MST) problem can be solved efficiently; see Prim (1957)[13], Kruskal (1956)[8]. On the other hand, *Facility Location Theory* seeks to develop methods that are designed to find optimal locations of facilities, both private and public, in order to serve a set of customers. These customers are assumed to avail of the facilities by traveling on a transportation network *whose design is already given.* Location theory is also another well established area with over 1000 published papers - see Mirchandani and Francis (1990)[12] and Drezner (1995)[3]. As mentioned in these texts, most Location Models are also NP Hard.

However, despite their popularity and umbilical connection, these two fields have been studied almost entirely in isolation. This is unfortunate because the very definition of optimal locations is constrained by the structure of the designed transportation network to be used by the customers. Furthermore, as shown in Bandelt and Labbe (1988)[1], when the network is designed improperly, customers get extremely poor service even when service facilities are located optimally. Thus there is a need to investigate models where transportation networks are designed with future facility locations in consideration.

In a partial attempt to address this gap, in this paper we begin by proposing a model that relates these two areas by incorporating facility location in the decision-making process involved in the design of a transportation network. Since this is essentially a first step in the confluence of these two areas, we begin by developing a simple model that seeks to design a transportation network that will be used by the customers in order to avail of the service provided by a *single* facility whose location has been pre-determined. Our principal contributions are to show that **(i)** our proposed network design problem reduces to a constrained MST problem and **(ii)** this problem is NP Complete. In fact, we demonstrate the computational complexity of this problem by showing that it remains NP Complete even in the simple case where all the users are assumed to be located on the vertices of a convex polygon. This result is surprising for two reasons: first and foremost, this result firmly establishes, for the first time, the computational complexity of integrated models of network design and facility location even in the simplest case where only one facility needs to be located. Second, in contrast to our problem, other NP Complete constrained MST problems are sometimes polynomially solvable in the Euclidean case (see for example, Ho et. al. (1989)[7]). By starting out with a different problem, we have also managed to demonstrate a Euclidean Constrained MST problem that remains NP Complete even in a simple instance where all the given points are vertices of a convex polygon.

The remaining paper is divided into three sections. The next section formally introduces our problem and discusses the preliminaries. Section three then demonstrates our primary result, namely, the NP Completeness of our model. Finally, the fourth section summarizes our findings.

2 Preliminaries

In this paper we study a problem of transportation network design around a given fixed location of a user-friendly facility. We begin by assuming that a Central Planning Agency (CPA), such a government, is responsible for constructing a transportation network to facilitate the usage of a given facility by a group of customers. The customers themselves are assumed to be dispersed on the N vertices of a given complete graph $G = (V, A)$ where V and A represent the set of vertices and arcs of G respectively. The arc connecting vertices v_i and v_j is denoted as (i, j). Associated with each vertex $v_i \in V$, is a non-negative weight w_i, which is referred to as the "weight" of the vertex v_i and represents the number of customers located at v_i. The total number of customers in G will be denoted by W. We assume that initially, there is no transportation network to allow travel by these customers. Thus, every arc $(i,j) \in A$, represents a *potential* link that can be constructed by the CPA to connect vertices v_i and v_j directly. The length of this link will be represented by l_{ij} and the cost of constructing it by c_{ij}.

The facility itself is assumed to be located at a vertex $v_m \in V$ and the CPA has to design the transportation network with this given location in mind - in practice such choices for facility location would probably have been made on the basis of factors such as location costs of the facility, available labor and material resources at this site or other externalities such as the case when governments target specific under-developed regions for development. This assumption however is not restrictive; it is possible to restate our model in the framework of an integrated model of facility location and network design where the facility location is also a decision variable. However, that problem can be tackled by repeatedly solving our model, once for each potential site of the facility. Hence, as it is a simple extension and does not alter the results of this paper, we do not discuss it here.

Suppose now that the CPA constructs a set of links $A' \subseteq A$; the resulting transportation network will be denoted by the subgraph $G' = (V, A')$ of G. We will say that an arc (i,j) is *present* in G' if and only if it represents a constructed link i.e., if and only if $(i,j) \in A'$ and the total cost of constructing G' will refer to the cost of constructing the links in A', and is denoted by $c(G')$. Thus,

$$c(G') = \sum_{(i,j) \in A'} c_{ij}$$

Given this, $d^{G'}(i,j)$ will be used to represent the length of any shortest path between vertices v_i and v_j in V, if travel is only allowed on the arcs that are present in G'. Thus, if G' is not a connected subgraph of G, then it is possible that such a path between v_i and v_j may not exist at all; when this happens, $d^{G'}(i,j)$ will be assumed to be infinity. Thus, with the given transportation network G' that has been constructed by the CPA, the total weighted distance of the facility at v_m to all the customers will be given by

$$F(m, G') = \sum_{i=1}^{N} w_i d^{G'}(i,m)$$

It is well known from location theory (Mirchandani and Francis (1990)[12]) that v_m is referred to as the *1-median* of G' if

$$F(m, G') \leq F(i, G') \ \forall \ v_i \in V$$

With these notations, we now turn to a description of two desirable properties of the transportation network that will be constructed by the CPA.

Connectedness Property: Any transportation network const-ructed by the CPA must be a connected subgraph of G. This property is obviously desirable to allow for communication with all the customer sites that this facility should service.

Median Constrained Property: Any transportation network constr-ucted by the CPA must have the property that the facility site v_m is the 1-median of this transportation network. In other words, the design of the transportation network is constrained by the location of this facility; hence the name of this problem. This property is desirable for two reasons: first, as evidenced by its definition, transportation networks with this property ensure that the average travel distance of all the customers to the service facility is minimized. The second reason is based on the voting paradigm ; see Hansen, Thisse and Wendell (1990)[6]. As we will shortly prove, the optimal transportation network is always given by a spanning tree of G'. Given that, the median constrained property ensures that the location of this facility at v_m is a "stable" location (see Hansen, Thisse and Wendell (1990)[6] for a proof). This means that it has the desirable property that once the transportation network has been constructed by the CPA, no alternate location of this facility will be strictly closer to more than half the total number of customers and thus be able to "defeat" the location at v_m in a pairwise election.

Given this, the objective of our model is to find the cheapest transportation network that has the connectedness and median constrained property. In other words, we assume that the CPA wishes to find the optimal set of arcs $A^* \subseteq A$ which, when constructed to yield the transportation network $G^* = (V, A^*)$ will satisfy

$$c(G^*) \leq c(G')$$

for all subgraphs G' of G that satisfy the connectedness and the median constrained property. We begin with the following interesting characterization of this optimal transportation network G^*.

Lemma 1 *It can be assumed without loss of generality that G^* is a spanning tree of G.*

Proof: It is readily seen that if only the connectedness property needs to be satisfied, then G^* can be assumed to be a spanning tree. However, what is non-trivial is to demonstrate that the assumption is still is true even if the median-constrained condition is imposed; that will be the focus of the proof.

Assume therefore, to the contrary, that G^* is not a tree. Clearly, v_m is a 1-median of G^*. Consider T', the shortest path tree from v_m to every other node in G^*. Since $d^{T'}(m,i) = d^{G^*}(m,i)$ for all vertices v_i, we have:

$F(m,T') = \sum_{i=1}^{n} w_i d^{T'}(m,j) = \sum_{i=1}^{n} w_i d^{G^*}(m,i) = F(m,G^*)$.

For any alternative vertex $v_p \in V$,

$F(p,T') = \sum_{i=1}^{n} w_i d^{T'}(p,i) \geq \sum_{i=1}^{n} w_i d^{G^*}(p,i) = F(p,G^*)$.

(since $d^{T'}(p,i) \geq d^{G^*}(p,i) \ \ \forall v_i \in V$)

Since $F(p,G^*) \geq F(m,G^*) = F(m,T')$ this implies that

$F(p,T') \geq F(m,G^*) = F(m,T') \ \ \forall p \in V$

which implies in turn that v_m is also a 1-median of T'. Thus T' satisfies both the connected and median-constrained property and further, $c(T') \leq c(G^*)$. ∎

It is interesting to note that a similar result also exists for the UFLNDP - see for example, Property 2 in Melkote and Daskin (1997a)[10]. Given the above result however, henceforth we will only consider spanning trees of G. To that extent, let T be a given spanning tree of G. Then, the subtrees of T that are obtained by removing the vertex v_m will be denoted by $T_{m,1}, T_{m,2}, ..T_{m,f}$ where f is the degree of v_m in T. Without loss of generality, we will assume that the numbering of the subtrees obeys:

$$w(T_{m,1}) \geq w(T_{m,2}) \geq ...w(T_{m.f})$$

where $w(T_{m,k}), k = 1, ..f$ represents the "weight" of the k^{th} subtree and is given by the sum of the weights of all vertices in $T_{m,k}$. With this definition, Goldman (1971)[5] has given an elegant property of the 1-median of a tree in term of the total weight W of all vertices of G.

Goldman's Property: The vertex v_m is a 1-median of T if and only if $w(T_{m,1}) \leq W/2$.

Since every transportation network in consideration must satisfy the median constrained property and hence, the Goldman's property, define $T(m)$ as the set of all spanning trees of G that have the vertex v_m as their 1-median and let $T^*(m)$ be the cheapest one among them, i.e.,

$$c(T^*(m)) \leq c(T)$$

for all spanning trees $T \in T(m)$. Then the optimization problem of the CPA can be stated as the following decision problem:

Median Constrained Minimum Spanning Tree Problem (MCMST)

Instance: A graph $G = (V, A)$ and a specified vertex $v_m \in V$.
Question: Find $T^*(m)$

When nodes are specified points in the plane, and the weight of an arc (i, j) is taken as the Euclidean distance between its nodes we get the Euclidean median constrained minimum spanning tree problem (**EMCMST**).

3 Complexity of MCMST

It is well known that while the MST problem is easily solvable, (Kruskal (1956)[8], Prim (1957)[13]), constrained MST problems are usually computationally hard; refer to section A2.1 of Garey and Johnson (1979)[4] for a detailed list of NP Complete constrained MST problems. To establish the complexity of MCMST we reduce two partitioning problem (2-P) to EMCMST.

2-Partitioning Problem (2-P)
Instance: Set D of n numbers $a_1, a_2, \ldots, a_n$

Question: Does there exist a subset B of D such that

$$\sum_{a_i \in B} a_i = \sum_{a_i \in D \setminus B} a_i$$

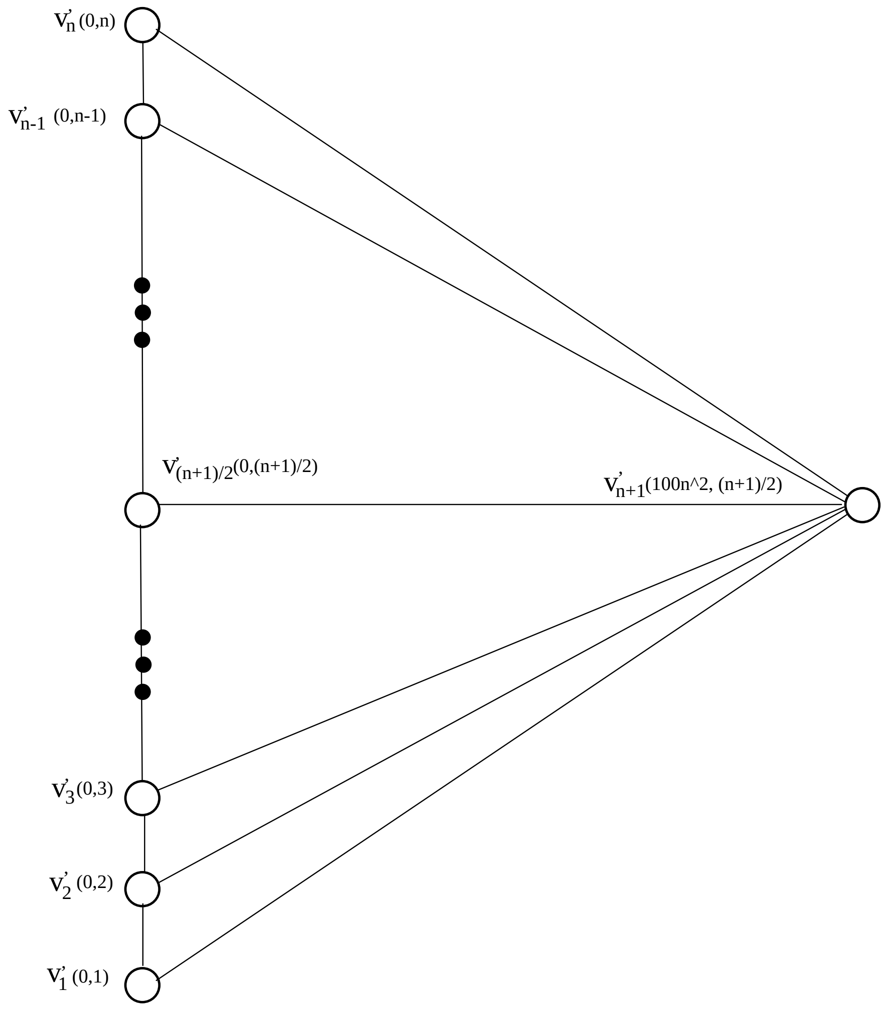

Figure 1: Illustrating the Construction of Graph G'

From an instance of 2-P we construct a complete Euclidean graph $G'(V', A')$ of $n+1$ vertices, where $V' = \{v'_i | a_i \in D\} \cup \{v'_{n+1}\}$. The first n nodes are placed equally spaced apart along a vertical line segment h with end points at coordinates $(0,1)$ and $(0,n)$. Specifically, the coordinates of $v'_1, v'_2, \ldots, v'_n$ are $(0,1), (0,2), \ldots, (0,n)$, respectively. The vertex v'_{n+1} is placed sufficiently far away to the the right of the vertical line segment. It is enough to place v'_{n+1} at $(100n^2, \frac{n+1}{2})$. This is illustrated in Figure 1 (the figure is not to the scale.) The weight of arc (i,j) is taken as $d(v'_i, v'_j)$ - the Euclidean distance between v'_i and v'_j. It can be easily verified that the graph $G'(V', A')$ as constructed above satisfies the following properties.

Property 1: $d(v'_{n+1}, v'_1) = d(v'_{n+1}, v'_n) = \sqrt{(\frac{n-1}{2})^2 + 10000n^4}$.

Property 2: $d(v'_{n+1}, v'_{\frac{n+2}{2}}) = 100n^2$

Property 3: $\max\limits_{1 \le i \le n} d(v'_{n+1}, v'_i) = d(v'_{n+1}, v'_1) = d(v'_{n+1}, v'_n)$

Property 4: $\min\limits_{1 \le i \le n} d(v'_{n+1}, v'_i) = d(v'_{n+1}, v'_{\frac{n+1}{2}}) = 100n^2$

Property 5: Let G^c denote the complete graph with nodes $v'_1, v'_2, \ldots, v'_n$. The maximum spanning tree of G^c is given by arcs $(1,n), (n,2)$, $(2, n-1), \ldots, (\frac{n+1}{2}, \frac{n+1}{2}+1)$. The weight of the maximum spanning tree is no more than $(n-1) + (n-2) + (n-3) + \ldots + 1 = \frac{n(n+1)}{2}$. This is shown in Figure 2.

Property 6: Any spanning tree with v'_{n+1} as median must have at least two arcs incident at v'_{n+1}.

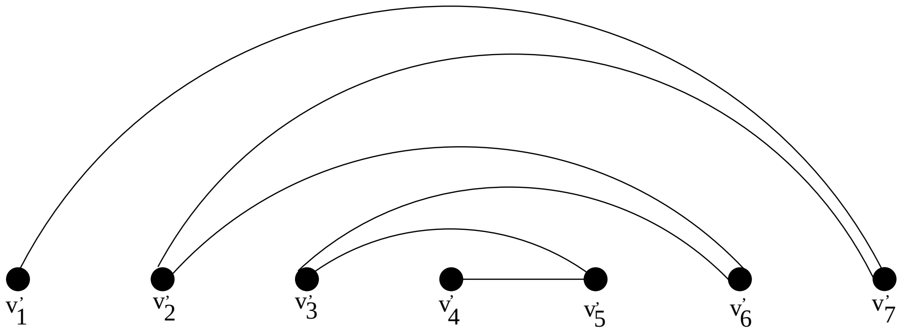

Figure 2: Illustrating the Maximum Spanning Tree

We use $T(m, v'_{n+1})$ to denote a spanning tree of $G'(V', A')$ with v'_{n+1} as median and with exactly m arcs incident at v'_{n+1}. Property 6 implies that $T(1, v'_{n+1})$ does not exist. The following lemma establishes that it is enough to consider only $T(2, v'_{n+1})$'s for finding a minimum weight spanning tree constrined to have median at v'_{n+1}.

Lemma 2 *Length of any $T(2, v'_{n+1})$ is less than the length of any $T(m, v'_{n+1})$, $m > 2$.*

Proof: Let U_2 be an upper bound of the weight of $T(2, v'_{n+1})$. Similarly, let L_3 be a lower bound of the weight of $T(m, v'_{n+1})$, $m > 2$. We observe that the weight of each arc of any $T(2, v'_{n+1})$ incident at v'_{n+1} must have length no more than the length of arcs $(n+1, 1)$ or $(n+1, n)$. Also the total length of the remaining $n-3$ arcs of any $T(2, v'_{n+1})$ is no more than the length of the maximum spanning tree of G^c (Property 5). Hence we can write:

$$U_2 = 2\sqrt{\frac{(n-1)^2}{4} + 10,000n^4} + \frac{n(n+1)}{2} \tag{1}$$

The total length of m arcs of $T(m, v'_{n+1})$, $m > 2$, incident at v'_{n+1} must be at least three times the length of the arc $(n+1, \frac{n+1}{2})$. Hence we can write:

$$L_3 = 300n^2 \tag{2}$$

Combining the above two relations we get:

$$\begin{aligned} L_3 - U_2 &\geq 300n^2 - 2\sqrt{\frac{(n-1)^2}{4} + 10,000n^4} \qquad (3) \\ &\geq 98n^2 \text{ (a positive quantity)} \end{aligned}$$

■

Let A be the set of numbers representing the capacities of n nodes $v'_1, v'_2, \ldots, v'_n$of $G'(V', E')$. The following lemma reduces the partitioning problem (2-P) to EMCMST.

Lemma 3 *There exist a 2-partition of D if and only if there exist a minimum spanning tree of $G'(V', A')$ with v'_{n+1} as median and total length $\leq U_2$.*

Proof: (2-P $\Rightarrow$ EMCMST): Suppose that we are given a 2-partitioning of D as $D_1 = \{ a'_1, a'_2, \ldots, a'_k \}$ and $D_2 = \{ a"_1, a"_2, \ldots, a"_r \}$, $k+r=n$. We refer to the nodes corresponding to D_1 as "first subset" and those corresponding to D_2 as "second subset". Given D_1 and D_2, we construct a spanning tree T' as follows. We construct two "special arcs" incident at v'_{n+1}. One of the special arcs is connected to a node in the first subset and the other to a node in the second subset. T' is obtained by adding special arcs to the spanning trees of the first subset and the second subset. Since the total weights of the first subset and the second subset are $\frac{W}{2}$ each, v'_{n+1} must be the median of T' (Goldman's condition); and the length of T' is no more than U_2.

(EMCMST $\Rightarrow$ 2-P): Suppose we are given a spanning tree of $G'(V', A')$ with v'_{n+1} as its median and total length $\leq U_2$. Since the total length is $\leq U_2$, Lemma 2 implies that there must be exactly two arcs incident at v'_{n+1}. Also by Goldman's condition each of the two subtrees of v'_{n+1} must have weight $\leq \frac{W}{2}$. Since the weight of both subtrees of v'_{n+1} can not be less than $\frac{W}{2}$, we must have that each subtree is of weight exactly $\frac{W}{2}$. Thus the capacities of the nodes of the subtrees of v'_{n+1} in T' give a two partitioning of D. ■

All the construction described in Lemma 3 take polynomial time. Furthermore, since 2-P is known to be NP-complete (Garey and Johnson, 1979[4]), Lemma 3 implies that EMCMST is NP-hard. Since a proposed solution for EMCMST can be verified easily in polynomial time, EMCMST belongs to the class NP. Hence we have the following theorem.

Theorem 1 *EMCMST is NP-Complete.*

ECMST remains NP-complete even when we require all the nodes to lie on the vertices of a convex polygon. This can be verified by the following simple observation. The nodes in Figure 2 lie on the boundary of a convex polygon such that all nodes except one (v'_{n+1}) are degenerate i.e., $v'_1, v'_2, ..., v'_n$ are collinear. The collinear degeneracy is not necessary, and can be removed easily, without violating Lemma 2 and Lemma3. We can slightly perturb the collinear vertices appropriately so that all vertices are in a general position (no three vertices are collinear) and also on the boundary of a convex polygon. Equation 3 is still valid in the modified construction and Lemma 3 follows. Hence we have the following corollary.

Corollary 1: *EMCMST remains NP-complete even when all nodes are required to lie on the vertices of a convex polygon.*

Remark: Since EMCMST is a restricted version of MCMST, it follows that MCMST is also NP-complete.

4 Conclusion

In this paper we have studied the problem of designing a transportation network to facilitate the use of a given service facility by the residents of the network. Although we have assumed throughout that the location of this facility is fixed, our model can also be used to solve a problem where both the facility location and the network design are decision variables. Our first major result is to show that under certain assumptions, our model reduces to a new variant of the well known family of constrained MST problems. Then we demonstrate that this problem is computationally hard, i.e. NP Complete, and remains so even when we make many simplifying assumptions.

Future work on this topic could examine the development of exact algorithms and efficient heuristics for this problem. Another strand of research could be to look for polynomially solvable instances of this problem and develop efficient solution procedures for them. This is the focus of the ongoing work in Bhadury et. al. (1998)[2].

References

1. Bendelt, H.J. and Labbe, M. *How Bad Can A Voting Location Be ?*. Journal: Social Choice and Welfare, Vol. 3, (1986) 125-145.

2. Bhadury, J., Chandrasekaran, R., Gewali L.P., Badr, M.H. *An Integrated Model of Facility Location and Network Design* (work in progress). Also presented at the *CORS/INFORMS Conference*, Montreal, (Spring 1998).

3. Drezner, Z. (editor), *Facility Location: A Survey of Applications and Methods*, Springer-Verlag New York, Inc, (1995).

4. Garey M.R. and Johnson D.S., *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman and Company, New York (1979).

5. Goldman A.J. (1971). "Optimum Center Location In Simple Networks". *Transportation Science*, 5, (1979) 212-221.

6. Hansen, P., Thisse, J.-F. and Wendell, R.E., *Equilibrium Analysis for Voting and Competitive Location Problems*, in *Discrete Location Theory* (P.B. Mirchandani and R.L. Francis (eds.)), John Wiley and Sons, New York, (1990) 479-501.

7. Ho, J., Lee, D.T., Chang, C.H., Wong, C.K. *Bounded-Diameter Minimum Spanning Trees and Related Problems*, *Proceedings of ACM Symposium on Computational Geometry*, (1989) 276-282.

8. Kruskal, J.B., *On The Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem*, Proceedings of the American Mathematical Society, 7, (1956) 48-50.

9. Magnanti, T.L. and Wong, R.T., *Network Design and Transportation Planning: Models and Algorithms*, Transportation Science, Vol. 18, No. 1, (February 1984) 1-55.

10. Melkote, S., and Daskin M.S., An Integrated Model of Facility Location and Network Design, Working Paper, School of Industrial Engineering and Management, Oklahoma State University, Stillwater, OK, (1997a).

11. Melkote, S., and Daskin, M.S., *Polynomially Solvable Cases of Combined Facility Location-Network Design Problems*, Working Paper, School of Industrial Engineering and Management, Oklahoma State University, Stillwater, OK, (1997).

12. Mirchandani P.B. and Francis, R.L. (1990) eds. *Discrete Location Theory*, John Wiley and Sons, New York (1990).

13. Prim, R.C., *Shortest Connection Network and some generalizations*, Bell System Technical Journal, 36, (1957) 1389-1401.

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A Method for Finding Common Set of Weights by Multiple Objective programming in Data Envelopment Analysis

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Abstract

This paper presents a multiple objective programming procedure for finding a common set of weights in Data Envelopment Analysis (DEA). DEA is a mathematical programming approach for assessing relative efficiencies within a group of Decision Making Units (DMUs). An important outcome of such an analysis is a set of virtual multipliers or weights accorded to each factor taken into account. These sets of weights are, typically, different for each of the participating DMUs. In this paper, by means of solving only one problem, we can determine common set of weights for all DMUs and their efficiencies. Finally, we employ this method for a group of real data to express its adventages comparing to other methods.

AMS: Operations Research. No. 90
Keywords: Multiple objective Programming, Mathematical Programming, Data Envelopment Analysis, Efficiency, Goal, Common Set of Weights

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1 Introduction

Data envelopment analysis, as developed by Charnes et al. (1978), does not require any a priori weights for inputs and outputs. In DEA the relative efficiency of a DMU, is determined by assigning weights to the inputs and outputs of DMU that the ratio of the weighted sum of outputs to the weighted sum of inputs is maximized. To this end, factor weights are allowed to vary freely (within the general constraints) in each run of the model. As the models of DEA are run for each DMU separately, the set of weights will typically be different for the various DMUs, and in some cases, it may be considered unacceptable that the same factor is accorded with different weights. A possible answer to this difficulty lies in the specification of a common set of weights. Recently, a possible approach has been put forward in Rall et al. (1991). The purpose of this paper is to introduce a new approach for determining the common set of weights. The second section of the paper, represents a brief disscusion about DEA models. In section 3, the goal mathematical programming model with multiple objectives is briefly discussed. The method for finding a common set of weights is presented in section 4. In section 5 an application of the method for real data is put forward and last section presents the conclusion.

2 Models of DEA

Consider a DMU that consume $X = (X_1, X_2, \ldots, X_m)$ inputs to produce $Y = (Y_1, Y_2, \ldots, Y_s)$ outputs and also, suppose inputs and outputs weights are w_i $(i = 1, \ldots, m)$, μ_r $(r = 1, \ldots, s)$, respectively. We also assume that the input and output components are nonnegative. The efficiency of DMU_p is obtained by solving the following ratio programming problem:

$$
\begin{aligned}
Max \qquad & e = \frac{\sum_{r=1}^{s} \mu_r y_{rp}}{\sum_{i=1}^{m} w_i x_{ip}} \\
s.t. \qquad & \frac{\sum_{r=1}^{s} \mu_r y_{rj}}{\sum_{i=1}^{m} w_i x_{ij}} \leq 1 \quad , j = 1, \ldots, n, \\
& w_i, \mu_r \geq \epsilon \qquad , r = 1, \ldots, s, \ i = 1, \ldots, m.
\end{aligned}
\tag{1}
$$

where ϵ is positive non-Archimedean infinitesimal. The first set of constraints are for limiting the evaluation scale. The above mentioned problem is solved separatly for each DMU, and by solving these problems different set of weights are obtained. Instead of solving problem (1) the following linear programming priblem is solved:

$$
\begin{aligned}
Max \qquad & e = \sum_{r=1}^{s} u_r y_{rp} \\
s.t. \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad , j = 1, \ldots, n, \\
& \sum_{i=1}^{m} v_i x_{ip} = 1, \\
& u_r, \ v_i \geq \epsilon \qquad ,r = 1, \ldots, s, \ i = 1, \ldots, m.
\end{aligned}
\tag{2}
$$

This model is called CCR input orientation multiplier side problem.

3 Goal Multiple Mathematical Programming

Consider the following problem:

$$\begin{array}{ll} Max & (f_1(x), f_2(x), \ \ldots, f_k(x)) \\ s.t. & \qquad x \in X. \end{array} \tag{3}$$

where $f_1, f_2, \ldots, f_k$ are objective functions, and X is the feasible region. Let $b_1, b_2, \ldots, b_k$ be the goals of $f_1, f_2, \ldots, f_k$ we are supposed to meet; therefore problem (3) is called goal multiple objective mathematical programming. If in this problem the weights $w_1, w_2, \ldots, w_k$ are assigned to $f_1, f_2, \ldots, f_k$, then the problem is called weighted problem. By introducing the deviational variables the problem can be written in the following form:

$$\begin{array}{llll} Min & \sum_{j=1}^{k} w_j z_j & & \\ s.t. & f_j(x) + z_j \ = \ b_j & , j = 1, \ \ldots, \ n, \\ & \quad x \in X. & & \end{array} \tag{4}$$

Instead of solving(3), the model (4) is solved. In this problem, $\sum_{i=1}^{k} w_i = 1, w_i \ \geq 0, (i = 1, 2, ..., k)$. If $z_j = 0$, then f_j is reached to the goal b_j. If $z_j \leq 0$, then f_j is higer than the goal.

4 A Method for Finding Common Set of Weights

In DEA for calculating the efficiency of different DMU's, different set of weights are obtained, which seems to be unacceptable in reality. So the following model is used to find common set of weights which has some advantages that will be discussed later on. Consider the following problem:

$$\begin{array}{lll} Max & e = \{\frac{\sum_{r=1}^{s} u_r y_{r1}}{\sum_{i=1}^{m} v_i x_{i1}}, \ldots, \frac{\sum_{r=1}^{s} u_r y_{rn}}{\sum_{i=1}^{m} v_i x_{in}}\} & \\ s.t. & \qquad \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1 & , \ j = 1, \ \ldots, \ n, \\ & \qquad u_r, \ v_i \ \geq \epsilon & , \ r = 1, \ldots, s, \ i = 1, \ldots, m. \end{array} \tag{5}$$

where U and V are the weights of outputs and inputs, respectively, and ϵ is a positive non-Archimedean infinitesimal.

Problem (5) is a fractional prgramming problem with multiple objectives, in which each objective function has goal equal to 1. As no priority is considered for the objective functions, we take $w_i = 1\,(i = 1, 2, ..., k)$. By introducing n deviational vairables z_j , $(j = 1, 2, \ldots, k)$ we have:

$$
\begin{array}{lll}
Min & \sum_{j=1}^{n} z_j & \\
s.t. & \dfrac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} + z_j = 1 & , \ j = 1, \ \ldots, \ n, \\
& \dfrac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1 & , \ j = 1, \ \ldots, \ n, \\
& u_r, \ v_i \ \geq \epsilon & , \ r = 1, \ldots, s, \ i = 1, \ldots, m.
\end{array}
\tag{6}
$$

If DMU$_p$ is efficient, then $z_p = 0$, othervise $z_p \geq 0$, so z_j is restricted to be non-negative. Therefore problem (6) is changed to the following problem:

$$
\begin{array}{lll}
Min & \sum_{j=1}^{n} z_j & \\
s.t. & \sum_{r=1}^{s} u_r y_{rj} + z_j \sum_{i=1}^{m} v_i x_{ij} - \sum_{i=1}^{m} v_i x_{ij} = 0 & , \ j = 1, \ \ldots, \ n, \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 & , \ j = 1, \ \ldots, \ n, \\
& z_j \geq 0 & , \ j = 1, \ \ldots, \ n, \\
& u_r, \ v_i \ \geq \epsilon & , \ r = 1, \ldots, s, \ i = 1, \ldots, m.
\end{array}
\tag{7}
$$

Note that instead of solving n linear programming DEA models, only one problem is solved and the efficeincy for all DMUs are obtained.

Theorem 1:

If DMU$_p$ is efficient in problem (7), then it is efficient in CCR model.

Proof: Suppose that (U^*, V^*, Z^*) is an optimal solution for problem (7), $\frac{{U^*}^t Y_p}{{V^*}^t X_p} = 1$.

It is evident that (U^*, V^*) is a feasible solution of problem (1) and since $\frac{{U^*}^t Y_p}{{V^*}^t X_p} = 1$, it is optimal.

5 Generalizing The Method for DEA Models

The above method can be generalized to all input and output oriented DEA models. Consider an BCC (Banker, Charnes and Cooper) output oriented model, which is in the following form for DMU$_p$.

$$
\begin{array}{lll}
Min & e=\dfrac{\sum_{i=1}^{m} v_i x_{ip}+v_0}{\sum_{r=1}^{s} u_r y_{rp}} & \\
s.t. & \dfrac{\sum_{i=1}^{m} v_i x_{ij}+v_0}{\sum_{r=1}^{s} u_r y_{rj}} \geq 1 & ,\ j=1,\ \ldots,\ n, \\
 & u_r,\ v_i\ \geq \epsilon & ,\ r=1,\ldots,s,\ i=1,\ldots,m.
\end{array} \tag{8}
$$

By introducing common weights for all DMUs, the output oriented BCC model with will be in the following form:

$$
\begin{array}{lll}
Min & e=\{\dfrac{\sum_{i=1}^{m} v_i x_{i1}+v_0}{\sum_{r=1}^{s} u_r y_{r1}},\ldots,\dfrac{\sum_{i=1}^{m} v_i x_{in}+v_0}{\sum_{r=1}^{s} u_r y_{rn}}\} & \\
s.t. & \dfrac{\sum_{i=1}^{m} v_i x_{ij}+v_0}{\sum_{r=1}^{s} u_r y_{rj}} \geq 1 & ,\ j=1,\ \ldots,\ n, \\
 & u_r,\ v_i\ \geq \epsilon & ,\ r=1,\ldots,s,\ i=1,\ldots,m.
\end{array} \tag{9}
$$

We introduce n deviational varibles to have.

$$
\begin{array}{lll}
Min & \sum_{j=1}^{n} z_j & \\
s.t. & \dfrac{\sum_{i=1}^{m} v_i x_{ij}+v_0}{\sum_{r=1}^{s} u_r y_{rj}} - z_j\ =1 & ,\ j=1,\ \ldots,\ n, \\
 & \dfrac{\sum_{i=1}^{m} v_i x_{ij}+v_0}{\sum_{r=1}^{s} u_r y_{rj}} \geq 1 & ,\ j=1,\ \ldots,\ n, \\
 & u_r,\ v_i\ \geq \epsilon & ,\ r=1,\ldots,s,\ i=1,\ldots,m.
\end{array} \tag{10}
$$

which is equivalent to the following:

$$
\begin{array}{lll}
Min & \sum_{j=1}^{n} z_j & \\
S.t & \sum_{i=1}^{m} v_i x_{ij}+v_0 - z_j \sum_{r=1}^{s} u_r y_{rj} - \sum_{r=1}^{s} u_r y_{rj}\ =0 & ,\ j=1,\ \ldots,\ n, \\
 & \sum_{i=1}^{m} v_i x_{ij}+v_0 - \sum_{r=1}^{s} u_r y_{rj} \geq 0 & ,\ j=1,\ \ldots,\ n, \\
 & u_r,\ v_i\ \geq \epsilon & ,\ r=1,\ldots,s,\ i=1,\ldots,m.
\end{array} \tag{11}
$$

We note that theorem 1 remains valid in the more general case.

6 Application

Data have been collected for 74 high schools in four cities and surrounding villages. More than 50 items for each school have been chosen. The inputs and outputs are given in table 1.

INPUT :	*OUTPUT* :
1. The educational space.	1. The number of students.
2. The official staff.	2. The educational performance of the period of study.
3. The quality of teachers.	3. The number of student could be accepted to the governmental universities.
4. The educational performance of the last period of the study.	

Table 1. Inputs and Outputs for Empirical Study.

The input and output data of this study are presented in table 2.

Running the CCR model for each school resulted in efficiencies and weights shown in table 3.

The common set of weights and the corresponding efficiencies when model (7) applied, are listed in table 4.

7 Conclusion

The model discussed in this paper has the following adventages:

1) By solving only one problem, the common set of weights of DMUs are obtained.

2) All efficiencies can be calculated without solving corresponding separate linear programming problems.

3) The new model is stable for weight variation, which is not true in other DEA models.

4) Some DMUs are efficient in CCR models which are not efficient in this model, which seems to be more realistic.

Sch	I1	I2	I3	I4	O1	O2	O3
S01	6.63	7.25	1.32	2.87	321.00	2.11	79.00
S02	6.63	7.75	1.35	2.50	294.00	2.04	197.00
S03	9.06	10.75	1.27	1.37	338.00	1.35	110.00
S04	7.56	9.50	1.18	2.24	503.00	2.08	81.00
S05	6.71	5.50	1.22	2.92	215.00	2.60	306.00
S06	7.29	12.50	1.34	3.80	337.00	2.84	69.00
S07	5.89	7.25	1.28	1.38	173.00	1.77	38.00
S08	5.89	5.50	1.22	1.73	134.00	2.40	11.00
S09	10.18	7.50	1.38	1.84	322.00	1.72	72.00
S10	8.07	8.25	1.26	2.28	281.00	2.19	30.00
S11	6.80	7.75	1.21	2.35	331.00	2.55	57.00
S12	6.98	7.50	1.53	2.65	267.00	2.59	19.00
S13	10.63	7.25	1.98	4.96	405.00	1.60	8.00
S14	6.00	6.25	1.76	2.43	166.00	1.38	4.00
S15	6.71	5.25	1.08	1.78	94.00	1.39	4.00
S16	6.97	7.50	1.14	1.78	294.00	2.52	91.00
S17	7.02	7.75	1.20	2.49	280.00	2.78	16.00
S18	6.97	7.75	1.09	1.96	254.00	2.07	67.00
S19	5.09	5.25	1.74	2.17	103.00	1.78	0.00
S20	10.04	7.75	1.06	3.49	363.00	2.94	88.00
S21	5.64	6.50	1.79	1.81	155.00	1.46	15.00
S22	6.15	6.50	1.12	3.49	315.00	2.81	59.00
S23	8.65	7.50	1.71	1.75	298.00	2.68	178.00
S24	6.32	7.25	1.78	1.82	194.00	2.67	173.00
S25	7.16	5.25	0.96	1.84	232.00	2.08	42.00
S26	10.38	8.00	1.04	2.36	224.00	2.25	339.00
S27	9.57	9.00	1.65	2.50	485.00	2.35	428.00
S28	9.03	8.00	1.20	2.69	618.00	2.54	155.00
S29	7.37	8.50	1.04	2.63	234.00	2.25	126.00
S30	5.40	4.00	0.91	3.02	205.00	3.31	26.00
S31	10.93	7.25	0.86	2.43	163.00	1.16	9.00
S32	9.19	5.25	0.96	2.57	119.00	3.07	18.00
S33	12.58	8.67	1.52	1.93	669.00	1.73	78.00
S34	7.51	5.58	1.10	2.43	420.00	2.74	97.00
S35	8.34	3.83	1.27	2.00	103.00	1.98	0.00
S36	6.76	3.83	1.30	2.76	196.00	1.96	8.00
S37	6.99	5.58	1.17	1.52	443.00	2.17	32.00
S38	5.74	3.83	1.07	2.33	239.00	2.64	0.00
S39	6.44	4.50	1.22	1.64	342.00	2.22	15.00
S40	10.23	3.58	1.10	2.12	275.00	2.25	8.00
S41	5.49	3.58	1.32	1.54	202.00	1.92	45.00
S42	4.88	3.83	1.45	1.61	167.00	1.65	7.00
S43	8.73	5.17	1.27	1.66	367.00	0.94	60.00
S44	6.12	4.58	1.45	3.04	141.00	3.37	165.00
S45	7.67	4.83	0.92	1.83	307.00	2.50	40.00
S46	13.38	8.25	1.37	2.31	688.00	2.01	342.00
S47	9.35	5.92	1.39	2.29	425.00	1.87	250.00
S48	7.78	6.17	1.77	1.56	359.00	1.44	27.00
S49	8.76	3.83	0.95	2.52	367.00	2.12	0.00
S50	8.86	3.83	1.47	2.16	134.00	2.28	40.00
S51	5.78	2.58	0.99	2.54	97.00	2.55	0.00
S52	7.81	6.42	1.13	1.78	426.00	1.86	27.00
S53	6.84	2.58	1.18	2.41	194.00	1.88	8.00
S54	9.79	6.17	1.15	2.38	394.00	2.08	13.00
S55	10.66	7.42	1.33	1.85	428.00	1.44	20.00
S56	7.69	3.58	1.19	3.36	101.00	3.36	80.00
S57	6.58	5.58	1.12	2.50	458.00	2.46	60.00
S58	9.22	5.92	1.30	2.02	360.00	2.34	32.00
S59	6.33	3.83	1.16	2.37	128.00	2.32	48.00
S60	5.80	2.33	1.16	1.39	116.00	2.10	8.00
S61	7.63	4.25	1.08	2.37	177.00	2.33	8.00
S62	10.07	8.00	1.13	2.68	714.00	2.44	360.00
S63	6.02	5.42	1.20	2.81	334.00	1.96	100.00
S64	8.92	6.92	1.09	2.76	478.00	2.49	38.00
S65	8.01	6.42	1.20	2.58	471.00	2.74	94.00
S66	11.76	8.00	1.15	1.82	598.00	1.79	50.00
S67	6.77	4.17	1.15	2.12	352.00	2.56	110.00
S68	10.56	6.17	1.35	1.76	382.00	1.82	68.00
S69	4.69	3.83	1.46	1.38	68.00	1.23	0.00
S70	4.77	3.58	1.55	2.28	120.00	2.28	45.00
S71	5.72	3.83	1.52	3.25	166.00	3.76	131.00
S72	8.43	7.17	1.39	2.27	140.00	2.29	45.00
S73	7.91	3.58	1.41	3.00	95.00	3.00	42.00
S74	5.57	3.58	1.42	2.71	66.00	3.25	9.00

Table 2. Inputs and outputs of high schools.

Sch	Eff	V_1	V_2	V_3	V_4	U_1	U_2	U_3
S01	0.78	0.001274	0.149605	0.000676	0.150657	0.000100	0.000100	0.000100
S02	0.84	0.000895	0.137220	0.001512	0.150655	0.000100	0.000100	0.000100
S03	0.86	0.001033	0.282509	0.001189	0.000100	0.000100	0.000100	0.728388
S04	0.94	0.001353	0.122213	0.000100	0.101812	0.000100	0.000100	0.102335
S05	1.00	0.000100	0.000100	0.003197	0.148888	0.000100	0.000100	0.000100
S06	0.83	0.001073	0.151195	0.000540	0.123931	0.000100	0.070832	0.000100
S07	0.85	0.000183	0.461621	0.000100	0.028631	0.000100	0.089069	0.519295
S08	0.99	0.000100	0.404785	0.000100	0.041465	0.002722	0.113474	0.348186
S09	0.68	0.000377	0.295474	0.000754	0.000100	0.000100	0.220268	0.377317
S10	0.73	0.000355	0.284557	0.000100	0.032120	0.000100	0.180202	0.224960
S11	0.90	0.000914	0.225218	0.000350	0.088743	0.000100	0.071851	0.131417
S12	0.79	0.000930	0.207597	0.000100	0.097789	0.000100	0.000100	0.119444
S13	0.60	0.001311	0.043840	0.000100	0.014554	0.116497	0.000100	0.000100
S14	0.50	0.001059	0.236961	0.000100	0.111514	0.000100	0.000100	0.135850
S15	0.56	0.000100	0.393804	0.000100	0.032168	0.005745	0.159922	0.326558
S16	1.00	0.000370	0.350033	0.000100	0.055925	0.000100	0.075478	0.294051
S17	0.89	0.000339	0.286270	0.000100	0.044852	0.000100	0.097297	0.227955
S18	0.80	0.000545	0.306505	0.000429	0.037238	0.000100	0.264729	0.230164
S19	0.64	0.000608	0.326452	0.000100	0.081822	0.000100	0.000100	0.268584
S20	0.90	0.000709	0.214461	0.000100	0.000100	0.000100	0.941389	0.000100
S21	0.60	0.000638	0.344731	0.000100	0.086104	0.000100	0.000100	0.283726
S22	0.94	0.001273	0.179464	0.000641	0.147127	0.000100	0.084080	0.000100
S23	1.00	0.000681	0.237236	0.000906	0.043970	0.000100	0.000100	0.353565
S24	1.00	0.000100	0.245570	0.001878	0.107560	0.000100	0.000100	0.175448
S25	0.83	0.000100	0.387852	0.000100	0.031845	0.005403	0.161038	0.320121
S26	1.00	0.000100	0.000100	0.002883	0.000100	0.000100	0.504708	0.200536
S27	1.00	0.000105	0.000100	0.002217	0.104356	0.000100	0.000100	0.000100
S28	0.97	0.000797	0.182719	0.000100	0.060273	0.000100	0.150410	0.102022
S29	0.79	0.000100	0.289422	0.000885	0.022828	0.000100	0.268423	0.209789
S30	1.00	0.000100	0.295136	0.000100	0.160754	0.000100	0.144203	0.000100
S31	0.45	0.000661	0.298195	0.000100	0.000100	0.000100	0.732478	0.151544
S32	0.98	0.000100	0.316281	0.000100	0.000100	0.000100	0.478465	0.209817
S33	1.00	0.001483	0.000100	0.000100	0.039616	0.000100	0.000100	0.259387
S34	0.99	0.000822	0.232256	0.000100	0.017506	0.041103	0.248311	0.150631
S35	0.71	0.000100	0.350881	0.000100	0.000100	0.042709	0.246702	0.261140
S36	0.70	0.001702	0.185033	0.000100	0.008427	0.246116	0.000100	0.000100
S37	1.00	0.001579	0.136910	0.000100	0.117579	0.000100	0.000100	0.116743
S38	0.95	0.000523	0.313624	0.000100	0.025511	0.040098	0.120500	0.245089
S39	0.99	0.000832	0.318273	0.000100	0.042972	0.065075	0.000100	0.262378
S40	0.95	0.001705	0.212490	0.000100	0.000100	0.263489	0.000100	0.026215
S41	0.91	0.000608	0.392601	0.000807	0.045178	0.043592	0.000100	0.386872
S42	0.78	0.000480	0.422576	0.000100	0.062896	0.019748	0.000100	0.383408
S43	0.80	0.002153	0.000100	0.000100	0.000100	0.117303	0.000100	0.236472
S44	0.98	0.000100	0.231171	0.001119	0.038539	0.000100	0.159776	0.175003
S45	1.00	0.000422	0.346639	0.000100	0.038956	0.000100	0.214986	0.274829
S46	1.00	0.000725	0.000100	0.001466	0.021035	0.000100	0.000100	0.310646
S47	0.90	0.000662	0.075771	0.001927	0.000100	0.168699	0.000100	0.000100
S48	0.76	0.002049	0.014018	0.000100	0.055986	0.000100	0.000100	0.361302
S49	1.00	0.001674	0.181970	0.000100	0.008223	0.242197	0.000100	0.000100
S50	0.76	0.000100	0.311194	0.001044	0.000100	0.035306	0.162080	0.289645
S51	1.00	0.000188	0.384995	0.000100	0.000100	0.235469	0.347675	0.018785
S52	0.84	0.000789	0.271425	0.000100	0.000100	0.000100	0.426992	0.289930
S53	1.00	0.002400	0.283836	0.000100	0.001291	0.384034	0.000100	0.000100
S54	0.77	0.000861	0.205359	0.000100	0.000100	0.058911	0.218157	0.161621
S55	0.73	0.001423	0.079752	0.000100	0.000100	0.043945	0.101416	0.290798
S56	1.00	0.000100	0.256735	0.001591	0.000100	0.116412	0.489194	0.000100
S57	1.00	0.001957	0.039734	0.000100	0.151836	0.000100	0.000100	0.000100
S58	0.83	0.000668	0.252634	0.000100	0.000100	0.063811	0.120734	0.229881
S59	0.79	0.000100	0.325330	0.000403	0.000100	0.043933	0.224396	0.240846
S60	1.00	0.000100	0.470286	0.000100	0.065601	0.015854	0.000100	0.419035
S61	0.78	0.000100	0.328081	0.000100	0.000100	0.040521	0.230410	0.243958
S62	1.00	0.000655	0.000100	0.001478	0.099188	0.000100	0.000100	0.000100
S63	0.87	0.001404	0.164779	0.000745	0.165956	0.000100	0.000100	0.000100
S64	0.87	0.000530	0.245726	0.000100	0.000100	0.000100	0.597116	0.125928
S65	0.96	0.000862	0.197041	0.000100	0.064501	0.000100	0.166135	0.109823
S66	1.00	0.001664	0.000100	0.000100	0.019138	0.000100	0.194212	0.302636
S67	1.00	0.000689	0.262407	0.000781	0.058663	0.022042	0.000100	0.240952
S68	0.78	0.000597	0.268111	0.000999	0.000100	0.058699	0.000100	0.361723
S69	0.62	0.000100	0.497289	0.000100	0.069428	0.016503	0.000100	0.442776
S70	0.82	0.000389	0.338490	0.000100	0.050009	0.016519	0.000100	0.307967
S71	1.00	0.000100	0.185280	0.002189	0.174675	0.000100	0.000100	0.000100
S72	0.72	0.000100	0.305850	0.000100	0.025446	0.005298	0.119898	0.255880
S73	0.85	0.000100	0.278900	0.000100	0.000100	0.070131	0.172620	0.168248
S74	0.98	0.000100	0.299050	0.000100	0.041335	0.011738	0.000100	0.268487

Table 3. Efficiencies and weights by CCR model.

Sch	E(J)	Z(J)
S01	0.67	0.33
S02	0.74	0.26
S03	0.63	0.37
S04	0.81	0.19
S05	0.87	0.13
S06	0.69	0.31
S07	0.76	0.24
S08	0.90	0.10
S09	0.63	0.37
S10	0.71	0.29
S11	0.87	0.13
S12	0.75	0.25
S13	0.34	0.66
S14	0.42	0.58
S15	0.52	0.48
S16	1.00	0.00
S17	0.87	0.13
S18	0.79	0.21
S19	0.56	0.44
S20	0.76	0.24
S21	0.51	0.49
S22	0.78	0.22
S23	0.95	0.05
S24	0.95	0.05
S25	0.83	0.17
S26	0.81	0.19
S27	0.87	0.13
S28	0.90	0.10
S29	0.74	0.26
S30	1.00	0.00
S31	0.36	0.64
S32	0.86	0.14
S33	0.68	0.32
S34	0.98	0.02
S35	0.63	0.37
S36	0.59	0.41
S37	1.00	0.00
S38	0.92	0.08
S39	0.95	0.05
S40	0.74	0.26
S41	0.84	0.16
S42	0.68	0.32
S43	0.49	0.51
S44	0.94	0.06
S45	1.00	0.00
S46	0.81	0.19
S47	0.75	0.25
S48	0.60	0.40
S49	0.72	0.28
S50	0.69	0.31
S51	0.81	0.19
S52	0.80	0.20
S53	0.63	0.37
S54	0.69	0.31
S55	0.56	0.44
S56	0.84	0.16
S57	0.91	0.09
S58	0.81	0.19
S59	0.76	0.24
S60	0.91	0.09
S61	0.74	0.26
S62	0.97	0.03
S63	0.69	0.31
S64	0.80	0.20
S65	0.93	0.07
S66	0.74	0.26
S67	1.00	0.00
S68	0.69	0.31
S69	0.51	0.49
S70	0.75	0.25
S71	1.00	0.00
S72	0.68	0.32
S73	0.76	0.24
S74	0.91	0.09

Table 4. The common set of weights and the corresponding efficiencies.

References

1. Ali, A. I., and L. M. Seiford: Translation Invarience in Data Envelopment Analysis, Operations Research Letters, Vol. 9, No. 5, (1990), 403-405.
2. Banker, R. D., A. Charnes, and W. W. Cooper: Some Methods for Estimating Technical and Scale Inefficiencies in Data Envelopment Analysis, Management Science, Vol. 30, No. 9, (1984), 1078-1092.
3. Charnes, A., W. W. Cooper, A. Y. Lewin, and L. M. Seiford: Data Envelopment Analysis: Theory, Methodology, and Applications, Boston: Kluwer Academic Publishers, (1994).
4. Charnes, A., W. W. Cooper, and E. Rhodes: Measuring the Efficiency of Decision Making Units, European Journal of Operational Research, Vol. 2, No. 6, (1978), 429-444.
5. Charnes, A., W. W. Cooper, R. M. Thrall: Classifying and Characterizing Efficiencies and Inefficiencies in Data Envelopment Analysis, Operations Research Letters, Vol. 5, No. 3, (1986), 105-110.
6. Charnes, A., W. W. Cooper, Wei QL and Huang ZM: Cone-Ratio Data Envelopment Analysis and Multi-Objective Programming, Int. J. Syst. Sci., Vol. 20, (1989), 1099-1118.
7. Dyson, R. G., and E. Thanassoulis: Reducing Weight Flexibility in Data Envelopment Analysis, Journal of Operational Research Society, Vol. 39, (1988), 563-576.
8. Golany, B., and Y. Roll: Alternate Methods of Treating Factor Weights in DEA, OMEGA Int. J. of Mgmt Sci., Vol. 21, No. 1, (1993), 99-109.
9. Hosseinzadeh Lotfi, F., S. Mehrabian and M. R. Alirezaee: Determining the Comparative Efficiency of High Schools; An Application of Data Envelopment Analysis, Journal of Sciences, Islamic Azad University, Vol. 7, No. 23, (1997), 403-420.
10. Roll, Y., W. Cook and B. Golany: Controlling factor Weights in Data Envelopment Analysis, IIE Trans., Vol. 23, No.1, (1991), 2-9.
11. Roll, Y. and B. Golany: Alternate Methods of Treating Factor Weights in DEA, Omega, Vol. 21, No. 1, (1993), 99-109.
12. Thompson, R. G., L. N. Langemeier, C. T. Lee, E. Lee and R. M. Thrall: The Role of Multiplier Bounds in Efficiency Analysis With Application to Kansas Farming, J. Economet., Vol. 46, (1990), 93-108.
13. Thompson, R. G., W. Lesso, R. R. Pettit, C. Samson, L. H. Sterm, F. D. Singleton Jr., B. A. Smith and R. M. Thrall: Systems Task Force Comparative Site Evaluations for the Texas SSC Project: Vol. E and Appendices, Woodlands, TX: Hoston Area Research Center, (1985).
14. Thompson, R. G., G. Russell, F. D. Singleton Jr., R. M. Thrall and A. Barton: Comparative Site Evaluation for Locating a High-Energy Physics Lab in Texas, Interface, Vol. 16, No. 6, (1986), 35-49.

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Periodic Solutions of Quasilinear Equations with Discontinuous Perturbations

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Abstract

The present study deals with the existence of periodic solutions of equation $u_t - \Delta_p u = F(t,u)$, *where* $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ *is the p-Laplacean, with Dirichlet boundary condition, and when F undergoes abrupt changes depending on u. An abstract result generalizing this situation is also presented.*

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1 Introduction

Monotone operators, in particular the ones that are subdifferentials of convex functions, like the p-Laplacean, appear in several situations. For instance, in LeTallec [6], a problem modelling the behaviour of viscoelastic materials, which includes the p-Laplacean, is studied.

These nonlinear operators may be accompanied by other nonlinear terms or operators that may complicate the analysis of the equations and the prediction of the behaviour of their solutions. This is specially true when these acompanying operators correspond to situations in which it is possible to impart energy into the system by means of the interaction between the material and external actions (by changes in the external or internal conditions, for instance).

This means in particular that the overall operator associated to the problem is no longer always monotone and not even always dissipative. Thus, due to this possible imbalance between the gain and the dissipation of energy, it is not clear that, when acted upon periodic external forces, the internal dissipation will be enough to guarantee that the system also responds in a periodic way. In other words, there is no guarantee that the corresponding mathematical problem will have a time periodic solution.

If, in addition, the system is submitted to some kind of abrupt change, implying in some sort of discontinuity in the characteristics of the problem, the question of the existence of periodic solution is even harder to answer.

In this paper we intend to address this question and furnish a partial answer to it by giving sufficient conditions for the existence of periodic solutions of problems of form:

$$\begin{cases} u_t + Au = F(t,u), \\ u(0) = u(T), \end{cases} \tag{1.1}$$

where A is a monotone operator having certain properties. In particular, this could be a nonlinear quasi parabolic boundary value problem in $\Omega \subset \mathbb{R}^N$, $N \geq 1$. $(t,u) \mapsto F(t,u)$ is a certain operator, which may have discontinuities in u, and which are not necessarily either monotone or bounded in any region of the space of u variable. On the other hand, the absence of such conditions will be compensated by the requirement of certain restriction in the growth of the nonmonotone perturbation F with respect to u. To cope with the discontinuity, the corresponding solution will be understood in the sense of differential inclusions.

Thus, to simplify the notation and ease the exposition, conditions for the existence of solutions of (1.1) will be obtained by initially studying in detail the following model problem:

$$\begin{cases} u_t(t,x) - \Delta_p u(t,x) \in F(u)(t,x), \quad (t,x) \in (0,T) \times \Omega, \\ u(t,x)|_{\partial\Omega} = 0, \quad t \in (0,T), \\ u(0,x) = u(T,x), \quad x \in \Omega. \end{cases} \tag{1.2}$$

In this problem Ω is a bounded and regular open set in $\mathbb{R}^N$, $N \geq 1$; $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p \geq 2$, is the p-Laplacean operator; R is a positive constant. Moreover, the set-valued mapping F is defined as

$$F(u)(t,x) = \begin{cases} \{f_1(t,x,u(t,x))\} & \text{if } u(t,x) < R, \\ \{f_2(t,x,u(t,x))\} & \text{if } u(t,x) > R, \\ \mathcal{A}_{12} & \text{if } u(t,x) = R. \end{cases} \tag{1.3}$$

where the set $\mathcal{A}_{12}$ is defined as

$$\mathcal{A}_{12} = \{\sigma f_1(t,x,u(t,x)) + (1-\sigma) f_2(t,x,u(t,x)); \sigma \in [0,1]\}.$$

Here, for each $i = 1, 2$, $f_i : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function, continuous in its third variable and satisfying

$$|f_i(t,x,\mu)| \leq m_i(t,x)\,|\mu|^{s_i} + h_i(t,x), \tag{1.4}$$

with $m_i \in L^\infty([0,T] \times \Omega)$, $s_i \geq 0$, and $h_i \in L^{p'}(0,T; L^{q'}(\Omega))$, p' and q' are the respecting conjugates of numbers p and q satisfying $1 - N/p > -N/q$.

An approach based on Boldrini and Crema [2], will be used to give, in Section 3, sufficient conditions for the existence of the weak periodic solutions for the problem (1.2)-(1.3).

Theorem 3.1 shows the existence of periodic solution for the subcritical case, i.e, for $0 \leq s_i < p-1$, $i = 1, 2$. In Theorems 3.2 and 3.3, we analyze the critical and supercritical cases, respectivelly, and we obtain periodic solutions since $\sup_{(t,x)\in(0,T)\times\Omega} m(t,x)$ is small enough and either $p < N$ and $p - 1 \leq s_i < p - 1 + 2p/N$ or $p \geq N$ and $p - 1 \leq s_i < p + 1$.

Based on the analysis of this problem, in Section 4 we will state a result containing conditions guaranteeing the existence of solutions of the abstract problem (1.1) for classes of operators A including the p-Laplacean operator as a particular case.

The results of this paper should be compared with previous results of similar problems. Thus we remark that in [8], Ôtani presents sufficient conditions for the existence of strong periodic solutions of problems in a class that includes equations like (1.2).

However, his results require more regularity and a certain smallness condition on h_i. This implies that the amplitude of the periodic solutions in Ôtani [8] are also small (in suitable norms). On the other hand, in our case h_i can be less regular (where a weak solution is obtained), and no smallness condition on h_i is imposed instead, a smallness condition on $|m|_\infty$ is imposed. Thus, our solutions (weak or strong) are found in a large ball and, in fact, it is easy to prove that when h_i is large, our solutions are also large (in suitable norms). Consequently our solutions are different from the ones obtained by the specialization of Ôtani's results.

Recently Avgerinos and Papageorgiou [1] have showed the existence of periodic solutions of equations involving monotone nonlinear parabolic operators perturbed by a discontinuous term of form $F(t,u)(x) = -a(t,x,u) + f(u)$, where $x \in \Omega \subset \mathbb{R}^N$, $N \leq 3$. In this term, f is a function of bounded variation, and $u \to a(t,x,u)$ is a bounded lipschitzian monotone function in $u \in [\phi(t,x), \psi(t,x)]$, where ϕ and ψ are respectively upper and lower solutions of the problem. We observe that for the autonomous case, i.e., when a does not depends on t, it is easy to show the existence of lower and upper solutions, which in general will be constant functions.This is no longer true for the nonautonomous case. For reference see also Kandilakis and Papageorgiou [4].

Boldrini and Crema established in [2] enough conditions to obtain periodic solutions to problems which undergo sudden time-dependent changes in their intrinsic behavior.

It should also be mentioned that investigations concerning the problem involving the p-Laplacean perturbed by a maximal monotonic operator, with periodic boundary conditions, were made by Yamada in [10]. The existence of periodic solutions was proved, but we remark that the problem considered in [10] is always dissipative as opposed to the one considered here.

Kawohl and Ruhl in [5] treated a problem involving the Laplacean, with monotonic boundary conditions changing abruptly in time, perturbed by other monotonic operators. Again the problem considered in [5] is dissipative.

Finally, it should be observed that for simplicity of exposition, all results in this paper will be stated with perturbations with just one discontinuity. However, as it is easy to see from the proofs, the results are still true for perturbations with a finite number of discontinuities.

2 Preliminaries

For the sake of fixing notations and easing the reading, in this section we will recall results to be used later on.

Let V be a Banach Reflexive space and V' be its topological dual. Let H be a Hilbert space such that $V \subset H \subset V'$, with continuous and dense inclusions. We will denote by $|\cdot|_V$, $|\cdot|_{V'}$, and $|\cdot|_H$ respectively the norms of V, V' and H; by $(\ ,\ )_{V',V}$ we denote the duality pairing between V' and V, and by $(\ ,\ )_H$ the inner product of H is represented.

For $T > 0$, we consider the Banach space $L^p(0,T;V) = \{u : (0,T) \to V; \int_0^T |u(t)|_V^p\ dt < \infty\}$, with norm $|u|_{L^pV} = (\int_0^T |u(t)|_V^p\ dt)^{1/p}$. Let p' be defined by $1/p + 1/p' = 1$ and denote $u_t = \dfrac{du}{dt}$; we recall that the Banach space $\{u \in L^p(0,T;V);\ u_t \in L^{p'}(0,T;V')\}$, with norm given by $|u|_{L^pV} + |u_t|_{L^{p'}V'}$ is a subspace of $C([0,T];H)$, the class of continuous functions defined on $[0,T]$ with values in H. Moreover, for u and v in this space, there holds

$$\int_0^T (u_t(t), v(t))_{V'V} + (v_t(t), u(t))_{V'V} dt = (u(T), v(T))_H - (u(0), v(0))_H. \quad (2.1)$$

For a proof of these results, see Lions [7], pp. 156 and 321.

The following compactness criterion given by Aubin-Lions (see Lions [7], p. 58, Theorem 5.1 and Strauss [9], p. 34, Theorem 2) will be necessary:

Lemma 2.1 *Let X, Y and Z be three Banach spaces, X and Z be reflexive, such that $X \subset Y \subset Z$ with continuous inclusions. Moreover, assume that the inclusion $X \subset Y$ is compact and that $1 \leq p \leq \infty$ and $1 < q \leq \infty$. Then, the space $\{u \in L^p(0,T;X);\ u_t \in L^q(0,T;Z)\}$ is compactly included in $L^p(0,T;Y)$.*

$L^q(\Omega)$ will denote the usual Banach space of real valued function defined on Ω, with norm $|u|_q = (\int_\Omega |u(x)|^q dx)^{1/q}$. When $q = 2$, $(\cdot,\cdot)_2$ denotes the usual inner product in the Hilbert space $L^2(\Omega)$.

The norm of $u \in W_0^{1,p}(\Omega)$, which is the usual Sobolev space, will be denoted by

$$|u|_{1,p} = \left(\int_\Omega |\nabla u(x)|^p dx\right)^{1/p}.$$

Here we only recall that if $p \geq 2$, then $W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega)$, with continuous (compact) and dense inclusions.

To rigorously define the p-Laplacean, we first consider the functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$, defined for any $u \in W_0^{1,p}(\Omega)$ by $\Phi(u) = \frac{1}{p}|u|_{1,p}^p$.

Concerning this functional, it is known that it is of class C^1 and that, for any u and v belonging to $W_0^{1,p}(\Omega)$, we have:

$$(\Phi'(u), v)_{W^{-1,p'}, W_0^{1,p}} = \int_\Omega |\nabla u(x)|^{p-2} \langle \nabla u(x), \nabla v(x)\rangle dx.$$

When $\Delta_p u \equiv \operatorname{div}\,(|\nabla u|^{p-2}\nabla u) \in L^2(\Omega)$, we have

$$(\Phi'(u), v)_{W^{-1,p'}, W_0^{1,p}} = -\int_\Omega \Delta_p u(x) \cdot v(x) dx$$

Thus, from now on we will denote Φ' by $-\Delta_p$, and, for simplicity of notation, the duality map between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$ just by $(\cdot,\cdot)$.

It is also known that $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ has the following properties:

(i) *Strong Monotonicity:* there is $\alpha > 0$ such that for any $u, v \in W_0^{1,p}(\Omega)$ it holds

$$((-\Delta_p)u - (-\Delta_p)v, u - v) \geq \alpha|u - v|_{1,p}^p. \tag{2.2}$$

(ii) *Hemicontinuity:* for any $\lambda \in \mathbb{R}$ and any $u, v, w \in W_0^{1,p}(\Omega)$, the following function is continuous:

$$\lambda \to (-\Delta_p(u + \lambda w), v). \tag{2.3}$$

(iii) *Coercivity:* for any $u \in W_0^{1,p}(\Omega)$, it holds that

$$(-\Delta_p u, u) = |u|_{1,p}^p \tag{2.4}$$

(iv) *Boundness:* for any $u \in W_0^{1,p}(\Omega)$, it is true that

$$|-\Delta_p u|_{W^{-1,p'}} \leq |u|_{1,p}^{p-1} \tag{2.5}$$

However, the meaning of weak and strong solutions need to be clarified:

Definitions:
(i) Let $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $u_0 \in L^2(\Omega)$. A function u is a *weak solution* of

$$\begin{cases} u_t - \Delta_p u = f, \\ u|_{\partial\Omega} = 0, \\ u(0) = u_0, \end{cases} \tag{2.6}$$

when $u \in W = \{w \in L^p(0,T;W_0^{1,p}(\Omega)); w_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))\}$, u satisfies (2.6) in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $u(0) = u_0$ (which makes sense because $W \subset C([0,T];L^2(\Omega))$).
(ii) In the case where $f \in L^2(0,T;L^2(\Omega))$, u is called a *strong solution* of (2.6) if it is a weak solution , $u_t \in L^2(0,T;L^2(\Omega))$, and (2.6) is satisfied in $L^2(0,T;L^2(\Omega))$.

3 Existence Results for the Model Problem

The proof of existence of periodic solutions of problem (1.2) will be accomplished by firstly establishing the existence of a solution u_ε, for $\varepsilon > 0$ sufficiently small, of the approximate problem:

$$\begin{cases} (u_\varepsilon)_t - \Delta_p(u_\varepsilon) = F_\varepsilon(u_\varepsilon), \\ u_\varepsilon|_{\partial\Omega} = 0 \\ u_\varepsilon(0) = u_\varepsilon(T) \end{cases} \tag{3.1}$$

where

$$F_\varepsilon(u)(t,x) = \begin{cases} f_1(t,x,u(t,x)) & \text{if } u(t,x) < R, \\ f_2(t,x,u(t,x)) & \text{if } u(t,x) > R + \varepsilon, \\ \left[1 - \left(\dfrac{u(t,x) - R}{\varepsilon}\right)\right] f_1(t,x,u(t,x)) & \\ \qquad + \left(\dfrac{u(t,x)R}{\varepsilon}\right) f_2(t,x,u(t,x)) & \text{if } u(t,x) \in [R, R+\varepsilon] \end{cases} \tag{3.2}$$

and then by showing the existence of a sequence $\{u_{\varepsilon_i}\}_{i\in N}$, of solutions of (3.1)-(3.2), which by its turn converges to a solution of (1.2)-(1.3).

According to the value of $s = \max\{s_1, s_2\}$, as compared to the value of $p-1$, there are differences in the argument and conditions to be imposed, so the exposition will be split into three instances: the subcritical ($s < p-1$), the critical ($s = p-1$) and the supercritical ($s > p-1$) cases.

Theorem 3.1 (Subcritical Case) *Let $2 \leq p \leq q < \infty$ satisfy $1 - N/p > -N/q$, and let the functions in (1.4) be such that for each $i = 1, 2$, $0 \leq s_i < p-1$, $m_i \in L^\infty([0,T] \times \Omega)$ and $h_i \in L^{p'}(0,T; L^{q'}(\Omega))$, with p' and q' being the conjugates of p and q, respectively. Then problem (1.2)-(1.3) has a weak periodic solution.*

Proof: By our hypotheses on f_1, f_2, see(1.4), and by the definition of F_ε, it is easy to verify that

$$|F_\varepsilon(u)(t,x)| \leq |m|_\infty |u(t,x)|^s + h(t,x) \tag{3.3}$$

with, $0 \leq s = \max\{s_1, s_2\} < p-1$, $|m|_\infty = \max\{|m_1|_\infty, |m_2|_\infty\}$, with $|m_i|_\infty = |m_i|_{L^\infty([0,T]\times\Omega)}$, and $h(t,x) = \max_{i=1,2}\{h_i(t,x)\} + 1$.

Moreover, since $p \leq q$, with the help of the Lebesgue Convergence Theorem, we conclude that

$$F_\varepsilon : L^p(0,T; L^q(\Omega)) \to L^{p'}(0,T; L^{q'}(\Omega)) \tag{3.4}$$

is a continuous map such that for any $\varepsilon \in (0,R)$ and $u \in L^p(0,T; L^q(\Omega))$ we find nonnegative constants $a = |m|_\infty$ and b such that

$$|F_\varepsilon(u)|_{L^{p'}L^{q'}} \leq a|u|^s_{L^pL^q} + b. \tag{3.5}$$

Then it is possible to apply Theorem 4.1 (i) in Boldrini and Crema [2] and conclude that, for each $\varepsilon \in (0,R)$ there is a weak solution $u_\varepsilon \in W$ of (3.1)-(3.2). Here the space W is defined as

$$W = \left\{ w \in L^p(0,T; W_0^{1,p}(\Omega))\,;\, w_t \in L^{p'}(0,T; W^{1,p'}(\Omega)) \right\}.$$

The next step is to look for uniform estimates in the norm of W for the sequence $\{u_\varepsilon\}_{\varepsilon\in(0,R)}$. This will allow showing that there is a subsequence $\{u_{\varepsilon_i}\}_{i\in\mathbb{N}}$ which converges in some sense to a solution of (1.2)-(1.3). For that, we observe that uniform estimates for u_ε in $L^p(0,T; {W_0}^{1,p}(\Omega))$ are easily obtained using the boundness of F_ε, given by (3.5), and the usual energy estimates. With the help of these estimates, we can use the equation in (3.1) and (2.5) to conclude the uniform boundness of $(u_\varepsilon)_t$ in $L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Therefore, $\{u_\varepsilon\}_{\varepsilon\in(0,R)}$ is a bounded set of W .

Since by Lemma 2.1, W is compactly embedded in $L^p(0,T;L^q(\Omega))$ for $1-N/p > -N/q$, we can extract a sequence $\{u_{\varepsilon_i}\}_{i\in\mathbb{N}}$ and find $u \in W \subset L^p(0,T;L^q(\Omega))$ such that when $i\to\infty$ then

$$u_{\varepsilon_i} \rightharpoonup u \quad \text{in} \quad W\,, \tag{3.6}$$

$$u_{\varepsilon_i} \to u \quad \text{in} \quad L^p(0,T,L^q(\Omega))\,. \tag{3.7}$$

Here $\rightharpoonup$ denotes weak convergence, while $\to$ denotes convergence in norm.

Moreover, $\{F_\varepsilon(u_\varepsilon)\}_{\varepsilon\in(0,R)}$ is bounded in $L^{p'}(0,T;L^{q'}(\Omega))$ and $\{\Delta_p u_\varepsilon\}_{\varepsilon\in(0,R)}$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. Thus, there are $\xi\in L^{p'}(0,T;L^{q'}(\Omega))$ and $\eta\in L^{p'}(0,T;W^{-1,p'}(\Omega))$ such that along a subsequence

$$-\Delta_p u_{\varepsilon_i} \rightharpoonup \eta \quad \text{in} \quad L^{p'}(0,T;W^{-1,p'}(\Omega))\,, \tag{3.8}$$

$$F_{\varepsilon_i}(u_{\varepsilon_i}) \rightharpoonup \xi \quad \text{in} \quad L^{p'}(0,T;L^{q'}(\Omega))\,. \tag{3.9}$$

Then, taking the weak limit in (3.1), we conclude that u satisfies

$$u_t+\eta=\xi.$$

Now let us verify that $u(0)=u(T)$. Note that if $v\in W_0^{1,p}(\Omega)$ and $\varphi\in C_0^\infty([0,T])$ we have $v\varphi\in L^p(0,T;W_0^{1,p}(\Omega))$. Let us take φ such that $\varphi(0)=\varphi(T)$; then, using (2.1) and the fact of $u_\varepsilon(0)=u_\varepsilon(T)$ we conclude that

$$\int_0^T\Big(\frac{d}{dt}u_\varepsilon(t),v\Big)\varphi(t)dt=-\int_0^T(u_\varepsilon(t),v)\varphi'(t)dt.$$

Passing to the limit in the last equality and using the density of $W_0^{1,p}(\Omega)$ in $L^2(\Omega)$ we conclude that $u(0)=u(T)$. Hence,

$$\begin{cases} u_t+\eta=\xi\,, \\ u(0)=u(T)\,. \end{cases} \tag{3.10}$$

Now we must show that $\eta=-\Delta_p u$ and $\xi(t,x)\in F(u(t,x))$ a.e. in $(0,T)\times\Omega$. To prove the first equality, let us fix an arbitrary element $v\in L^p(0,T;W_0^{1,p}(\Omega))$ and denote

$$S_i:=\int_0^T(-\Delta_p u_{\varepsilon_i}(t)+\Delta_p v(t),u_{\varepsilon_i}(t)-v(t))dt$$

where $S_i \geq 0$ by (2.2).

Due to the fact that u_{ε_i} is solution of (3.1), by using (3.5), (3.6) and (3.7), we conclude that $S_i \to S$, as $i \to \infty$, where

$$S = \int_0^T (\eta(t) + \Delta_p v(t), u(t) - v(t))dt \geq 0 \ .$$

Replacing v with $u - \lambda\omega$ in S, for $\lambda > 0$ and $\omega \in L^p(0,T; W_0^{1,p}(\Omega))$ and after taking the limit when $\lambda \to 0$ we can use the hemicontinuity of Δ_p, (2.3), to conclude that for any ω

$$\int_0^T (\eta(t) + \Delta_p u(t), \omega(t)) \geq 0.$$

Hence $\eta = -\Delta_p u$ and, therefore,

$$\left\{ \begin{array}{l} u_t - \Delta_p(u) = \xi, \\ u(0) = u(T). \end{array} \right. \tag{3.11}$$

Finally let us show that for a.e. in $(0,T)\times\Omega$, we have $\xi(t,x) \in F(u)(t,x)$, which is given by (1.3).

By (3.7), we have in particular that $\{u_{\varepsilon_i}\}$ converges in $L^p((0,T)\times\Omega)$ for $p \leq q$. Thus, there is a new subsequence, still denoted by $\left\{u_{\varepsilon_i}\right\}$, such that

$$u_{\varepsilon_i}(t,x) \to u(t,x) \quad \text{ a.e. } \quad (0,t)\times\Omega \ , \tag{3.12}$$

In particular this convergence occurs for any measurable subset of $(0,T)\times\Omega$. Then, let us consider the following measurable sets:

$$\begin{array}{rcl} I &=& \{(t,x) \in (0,T)\times\Omega\,;\ u(t,x) < R\} \\ J &=& \{(t,x) \in (0,T)\times\Omega\,;\ u(t,x) > R\} \\ K &=& \{(t,x) \in (0,T)\times\Omega\,;\ u(t,x) = R\} \ . \end{array}$$

Firstly we will show that for a.e. I we have that

$$F_{\varepsilon_i}(u_{\varepsilon_i})(t,x) \to f_1(t,x,u(t,x))\,. \tag{3.13}$$

This is easy to prove because if $(t,x) \in I$ then $u(t,x) < R$ and from (3.12), for i large enough, we also have $u_{\varepsilon_i}(t,x) < R$. Consequently, $F_{\varepsilon_i}(u_{\varepsilon_i})(t,x) =$

$f_1(t,x,u_{\varepsilon_i}(t,x))$. Thus (3.13) follows from the continuity of f_1 in its third variable.

Now using Lebesgue Convergence Theorem, with the help of (3.3), we conclude that

$$F_{\varepsilon_i}(u_{\varepsilon_i}) \to f_1(\cdot,\cdot,u) \quad \text{in} \quad L^{p'}(I).$$

However, $L^{p'}((0,T)\times\Omega) = L^{p'}(0,T;L^{p'}(\Omega)) \subset L^{p'}(0,T;L^{q'}(\Omega))$. Then, by (3.9), we obtain

$$\xi(t,x) = f_1(t,x,u(t,x)) \in F(u)(t,x) \;\; \text{a.e.} \;\; \text{in } I\ .$$

Analogously we conclude that

$$\xi(t,x) = f_2(t,x,u(t,x)) \in F(u)(t,x) \;\; \text{a.e.} \;\; \text{in } J\ .$$

Now, we will show that when $(t,x)\in K$, $\xi(t,x)$ is a pontual convex combination of $f_1(t,x,u(t,x))$ and $f_2(t,x;u(t,x))$.

Recall that, by continuity and boundness of f_1, f_2 and f_ε, we have that if

$$u_{\varepsilon_i}(t,x) \to u(t,x) \quad \text{a.e.} \;\; \text{in} \quad K\ ,$$

then

$$F_{\varepsilon_i}(u_{\varepsilon_i}) \rightharpoonup \xi \quad \text{in} \;\; L^{p'}(0,T;L^{q'}(\Omega)) \text{ (in particular weakly in } L^{q'}(K)),$$

$$f_k(\cdot,\cdot,u_{\varepsilon_i}) \to f_k(\cdot,\cdot,u) \quad \text{in} \;\; L^{p'}(0,T;L^{q'}(\Omega)),\ \text{for } k=1,2, \tag{3.14}$$

$$f_k(t,x,u_{\varepsilon_i}(t,x)) \to f_k(t,x,u(t,x)) \quad \text{a.e.} \;\; \text{in} \;\; (0,T)\times\Omega\,. \tag{3.15}$$

By the weak convergence of $F_{\varepsilon_i}(u_{\varepsilon_i})$, we find a sequence of convex combinations of these elements, strongly converging to ξ , i.e., there are functions

$$\xi_i = \sum_{k=1}^{k_i} a_k^{(i)}\, F_{\varepsilon_k}(u_{\varepsilon_k}) \;\; \text{with} \;\; \sum_{k=1}^{k_i} a_k^{(i)} = 1\,,\; 0 \le a_k^{(i)} \le 1 \tag{3.16}$$

such that

$$\xi_i \to \xi \quad \text{strongly in} \quad L^{q'}(K) \tag{3.17}$$

With the help of these facts, a special sequence will be constructed as follows.

Let us take $(t,x) \in K$ such that (3.15) is satisfied. Then, for $m=1$, there is $N(1)>0$ such that for $i \geq N(1)$ and $j=1,2$, we have

$$|f_j(t,x,u_\varepsilon(t,x)) - f_j(t,x,u(t,x))| < 1 \ .$$

Now, as in (3.17), there is $k(1) \geq N(1)$ and $\xi_{i,1}(t,x)$ such that

$$\xi_{i,1}(t,x) = \sum_{k=N(1)}^{k(1)} a_k^{(1)} \, F_{\varepsilon_k}(u_{\varepsilon_k})(t,x)$$

with $a_k^{(1)}$ as in (3.16), and

$$|\xi_{i,1}(t,x) - \xi(t,x)| < 1 \ .$$

Now, for $m=2$ there is $N(2) > k(1)$ such that for $i \geq N(2)$ and $j=1,2$, we have

$$\Big|f_j(t,x,u_{\varepsilon_i}(t,x)) - f_j(t,x,u(t,x))\Big| < \frac{1}{2} \ .$$

Again as in (3.17), there is $k(2) \geq N(2)$ and $\xi_{i,2}(t,x)$ such that

$$\xi_{i,2}(t,x) = \sum_{k=N(2)}^{k(2)} a_k^{(2)} \, F_{\varepsilon_k}(u_{\varepsilon_k})(t,x) \ ,$$

with $a_k^{(2)}$ as in (3.16), and

$$|\xi_{1,2}(t,x) - \xi(t,x)| < \frac{1}{2} \ .$$

Successively, given $m \in \mathbb{N}$, there is $N(m) > k(m-1)$ such that for $i \geq N(m)$ and $j=1,2$ it holds that

$$\Big|f_j(t,x,u_{\varepsilon_i}(t,x)) - f_j(t,x,u(t,x))\Big| < \frac{1}{m} \ . \tag{3.18}$$

Moreover, there is $k(m) \geq N(m)$ and $\xi_{i,m}(t,x)$ such that

$$\xi_{i,m}(t,x) = \sum_{k=N(m)}^{k(m)} a_k^{(m)} \, F_{\varepsilon_k}(u_{\varepsilon_k})(t,x) \ ,$$

with $a_k^{(m)}$ as in (3.16), and

$$|\xi_{i,m}(t,x) - \xi(t,x)| < \frac{1}{m} \ . \tag{3.19}$$

Now recall that by the definition of F_ε there is $0 \leq \alpha_k(t,x) \leq 1$ such that

$$\xi_{i,m}(t,x) = \sum_{k=N(m)}^{k(m)} a_k^{(m)} \left[\alpha_k(t,x) f_1(t,x,u_{\varepsilon_k}(t,x)) + (1-\alpha_k(t,x)) f_2(t,x,u_{\varepsilon_k}(t,x))\right],$$

which implies that $\xi_{i,m}(t,x)$ is also a convex combination of $f_1(t,x,u_{\varepsilon_k}(t,x))$ and $f_2(t,x,u_{\varepsilon_k}(t,x))$. Changing u_{ε_k} by u in $\xi_{i,m}(t,x)$ we can define

$$w_m(t,x) = \sum_{k=N(m)}^{k(m)} a_k^{(m)} \left[\alpha_k(t,x) f_1(t,x,u(t,x)) + (1-\alpha_k(t,x)) f_2(t,x,u(t,x))\right] \ .$$

Then it follows from (3.18) that

$$|w_m(t,x) - \xi_{i,m}(t,x)| < \frac{1}{m} \cdot \sum_{k=N(m)}^{k(m)} a_k^{(m)} = \frac{1}{m} \ .$$

From this and (3.19) we conclude that

$$w_m(t,x) \to \xi(t,x) \ , \qquad \text{a.e. in } \ K \ .$$

However,

$$w_m(t,x) \in \sum_{(t,x)} = \{\sigma\, f_1(t,x,u(t,x)) + (1-\sigma) f_2(t,x,u(t,x));\ \sigma \in [0,1]\},$$

which is a closed set. Therefore, there is $\sigma(t,x) \in [0,1]$ such that

$$\xi(t,x) = \sigma(t,x) f_1(t,x,u(t,x)) + (1-\sigma(t,x)) f_2(t,x,u(t,x)) \in F(u)(t,x),$$

for a.e. $(t,x) \in K$, which concludes the proof. $\square$

Now let us cosider the critical case $s = p - 1$, for which we have the following result:

Theorem 3.2 (Critical Case) *Let $2 \leq p \leq q < \infty$ satisfy $1 - N/p > -N/q$, and let the functions in (1.4) be such that for each $i = 1, 2$, $0 \leq \max\{s_1, s_2\} = p-1$, $m_i \in L^\infty([0,T] \times \Omega)$ and $h_i \in L^{p'}(0,T; L^{q'}(\Omega))$, with p' and q' being the conjugates of p and q, respectively. Then when $|m|_\infty = \max\{|m_1|_\infty, |m_2|_\infty\}$ is small enough, problem (1.2)-(1.3) has a weak periodic solution.*

Proof: It is enough to proceed as in the previous theorem, with the remark that constants $a = |m|_\infty = \max\{|m_1|_\infty, |m_2|_\infty\}$ and b in estimate (3.5) are independent of ε. Then, Theorem 4.1 (ii) in [2] guarantees that when $|m|_\infty$ is small enough, independently of the value of $\varepsilon \in (0, R)$, (3.1)-(3.2) has a solution u_ε, for each ε. The same theorem also furnishes that $\{u_\varepsilon\}$ is uniformly bounded in W.

The rest of the proof follows exactly the same arguments as in the proof of Theorem 3.1 and we conclude that problem (1.2)-(1.3) has a weak periodic solution. $\square$

Our next result will provide sufficient conditions for the existence of time periodic solutions of (1.2)-(1.3) when the growth-rate of one of f_i (given by s_i in (1.4)) is supercritical, that is, it is bigger than $p - 1$.

Before stating and proving it, it is necessary to recall the following result of [2].

Lemma 3.1 *Let $f \in L^{p'}(0,T; W^{-1,p'}(\Omega)$ and v be the solution of*

$$\begin{cases} v_t - \Delta_p v = f, \\ v|_{\partial\Omega} = 0, \\ v(0) = v(T). \end{cases}$$

Then $v \in L^\infty(0, T, L^2(\Omega))$ and satisfies, for some $c > 0$ independent of f,

$$|v|^2_{L^\infty L^2} \leq c \left(|f|^{2/(p-1)}_{L^{p'} W^{-1,p'}} + |f|^{p'}_{L^{p'} W^{-1,p'}} \right) \tag{3.20}$$

We also have the following:

Lemma 3.2 *Problem (3.1)-(3.2) has a weak solution if $|f_i(t,x,\mu)| \leq |m|_\infty |\mu|^{s_i} + |h(t,x)|$ with $h_i \in L^{p'}(0,T; L^{q'}(\Omega))$, $|m|_\infty$ small enough and either*

$$p - 1 \leq s < p - 1 + 2p/N \quad \textit{for} \quad N > p \ , \tag{3.21}$$

or

$$p-1 \leq s < p+1 \quad \mathit{for} \quad N \leq p \ , \tag{3.22}$$

Proof: By the hypotheses on f_i we conclude that $|F_\varepsilon(u)(t,x)| \leq |m|_\infty |u|^s + |h(t,x)|$ where $s = max\{s_1, s_2\}$ and $|h(t,x)| = max|h_i(t,x)|$. Then by Theorem 4.1 of [2], the result follows. □

We briefly sketch here the proof of Theorem 4.1 of [2] because certain of its details and notations will be refered to in the arguments that follow.

Theorem 4.1 of [2] is proved by showing that there are numbers $k, r \geq 1$ and $0 < R = R(|m|_\infty \, , \, s \, , \, |h|_{L^{p'}W^{-1,p'}})$ such that the solution operator S defined by

$$\begin{array}{rcl} S : \overline{B_R(0)} \subset L^k(0,T;L^r(\Omega)) & \longrightarrow & \overline{B_R(0)} \subset L^k(0,T;L^r(\Omega)) \ , \\ v & \longmapsto & S(v) = u. \end{array}$$

where u is the solution of

$$\begin{cases} u_t - \Delta_p u = f(\cdot,\cdot,v) \\ u|_{\partial\Omega} = 0 \\ u(0) = u(T), \end{cases} \tag{3.23}$$

has a fixed point u with $|u|_{L^kL^r} \leq R$. Moreover k and r are obtained by interpolation of $L^p(0,T;L^q(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$, with $1 - N/p > -N/q$. They are such that:

$$\begin{array}{rcl} W \subset X & = & L^p(0,T;L^q(\Omega)) \cap L^\infty(0,T,L^2(\Omega)) \subset L^k(0,T,L^r(\Omega)) \\ & \subset & L^{p's}(0,T;L^{q's}(\Omega)) \end{array} \tag{3.24}$$

$$|u|_{L^kL^r} \leq |u|^{1-\theta}_{L^\infty L^2} \, |u|^{\theta}_{L^pL^q} \qquad \forall \, u \in X \tag{3.25}$$

$$|f(\cdot,\cdot,u)|_{L^{p'}L^{q'}} \leq c|m|_\infty \, |u|^s_{L^kL^r} + |h|_{L^{p'}W^{-1,p'}} \tag{3.26}$$

with $\theta \in (0,1)$ and $c > 0$.

Now it is possible to state the following result:

Theorem 3.3 (Supercritical Case) *If in (1.4) we have nonegative functions $h_i \in L^{p'}(0,T;L^{q'}(\Omega))$, $i = 1,2$, $|m|_\infty$ is small enough and $s = \max\{s_1, s_2\}$ satisfying either (3.21) or (3.22), then (1.2)-(1.3) has a weak solution.*

Proof: As in Theorem 3.1, we will construct a solution for (1.2), (1.3) with the help of the aproximate problem (3.1),(3.2). Remember that according to (3.3), F_ε has uniform estimates in $\varepsilon \in (0, R)$. Then by Lemma 3.2 we conclude that problem (3.1),(3.2) has a solution u_ε satisfying:

$$|u_\varepsilon|_{L^k L^r} \leq R \qquad \forall\, \varepsilon \in (0, R)\,. \tag{3.27}$$

This fact and (3.26) give us

$$|F_\varepsilon(u_\varepsilon)|_{L^{p'} L^{q'}} \leq c|m|_\infty R^s + |h|_{L^{p'} L^{q'}} = \overline{R} \qquad \forall\, \varepsilon \in (0, R)\,. \tag{3.28}$$

Then, using energy estimates and recalling the inclusions $W_0^{1,p}(\Omega) \subset L^q(\Omega) \subset L^2(\Omega) \subset L^{q'}(\Omega) \subset W^{-1,p'}(\Omega)$, we conclude, as in Theorem 3.1 that u_ε is bounded in W.

Still following the proof of Theorem 3.1 we conclude that there is a subsequence $\left\{u_{\varepsilon_i}\right\}_{i\in\mathbb{N}}$, $u \in W$ and $\xi \in L^{p'}(0, T; L^{q'}(\Omega))$ such that $u_{\varepsilon_i} \to u$, in the sense of (3.6) and (3.7). Moreover, u satisfies:

$$\begin{cases} u_t - \Delta_p u = \xi\ , \\ u|_{\partial\Omega} = 0 \\ u(0) = u(T)\,. \end{cases}$$

However, to conclude that $\xi(t, x) \in F(u)(t, x)$ a.e. in $(0, T) \times \Omega$, we need to show that $u_{\varepsilon_i} \to u$ in $L^k(0, T, L^r(\Omega))$ and then follow the proof of Theorem 3.1 .

To obtain this convergence observe that by using (3.24) and (3.25), u and u_{ε_i} belong to $W \cap L^\infty(0, T, L^2(\Omega)) \subset L^k(0, T, L^r(\Omega))$ and there is $\theta \in (0, 1)$ such that

$$|u_{\varepsilon_i} - u|_{L^k L^r} \leq |u_{\varepsilon_i} - u|^{1-\theta}_{L^\infty L^2}\, |u_{\varepsilon_i} - u|^{\theta}_{L^p L^q}\ . \tag{3.29}$$

Thus, we only need to show that $\left\{|u_{\varepsilon_i} - u|_{L^\infty L^2}\right\}_{i\in\mathbb{N}}$ is bounded.

However, according to Lemma 3.1 and (3.28) we have the estimate

$$\begin{aligned} \left|u_{\varepsilon_i}\right|^2_{L^\infty L^2} &\leq c\left(\left|F_{\varepsilon_i}(u_{\varepsilon_i})\right|^{2/(p-1)}_{L^{p'} W^{-1,p'}} + \left|F_{\varepsilon_i}(u_{\varepsilon_i})\right|^{p'}_{L^{p'} W^{-1,p'}}\right) \\ &\leq \tilde{c}\left(\left|F_{\varepsilon_i}(u_{\varepsilon_i})\right|^{2/(p-1)}_{L^{p'} L^{q'}} + \left|F_{\varepsilon_i}(u_{\varepsilon_i})\right|^{p'}_{L^{p'} L^{q'}}\right) \\ &\leq \tilde{c}\left(\overline{R}^{2/(p-1)} + \overline{R}^{p'}\right)\ , \quad \text{for } \ i \in \mathbb{N}\ , \end{aligned}$$

which gives us the boundness of $(u_{\varepsilon_i} - u)$ in $L^\infty(0,T,L^2(\Omega))$. Thus following the proof of Theorem 3.1, from (3.17) on, we conclude the required result. $\Box$

Remark:(Strong Solutions) If we take $h_i \in L^2(0,T;L^2(\Omega))$ and either $0 \le s = max\{s_1, s_2\} \le p/2$ or $|m|_\infty$ small enough and $p/2 \le s \le q/2$, with q so that $1 - N/p > -N/q$, then strong solutions of (1.2)-(1.3) are obtained.

To prove this result, firstly we use Theorem 6.1 of [2] to show the existence of a bounded sequence of solutions u_ε of (3.1)-(3.2) such that $u_\varepsilon \in \tilde{W} = \{w \in L^p(0,T;W_0^{1,p}(\Omega));\ w_t \in L^2(0,T;L^2(\Omega))\}$. We then follow the proof of the previous theorem to conclude the required result.

4 An Abstract Result

Following the ideas in the proofs of the theorems of the previous section, it is easy to obtain the following abstract result.

Theorem 4.1 *Let V and X be reflexive Banach spaces, with V' and X' being their respective topological duals, and H being a Hilbert space such that $V \subset X \subset H \subset X' \subset V'$, with continuous and dense inclusions and $V \subset X$ compactly. Let also $A : V \to V'$ be a monotone and hemicontinuous operator for which there are constants $\alpha_1, \alpha_2 > 0$, $\beta, \gamma_1, \gamma_2 \ge 0$ and $p \ge 2$ such that for any $u, v \in V$ the following holds:*

$$(Au - Av\,,\, u - v)_{V',V} \ge \alpha_1 |u - v|_V^p\ ,$$

$$(Au\,,\, u)_{V',V} \ge \alpha_2 |u|_V^p - \beta\ ,$$

$$|A(u)|_{V'} \ge \gamma_1 |u|_V^{p-1} + \gamma_2\ .$$

Also consider a set-valued operator $F(t,\cdot) : X \to X'$, with $t \in (0,T)$ defined by

$$F(t,u) = \begin{cases} \{F_1(t,u)\} & \text{if } |u|_X < R, \\ \{F_2(t,u)\} & \text{if } |u|_X > R, \\ \{\sigma F_1(t,u) + (1 - \sigma F_2(t,u);\ \sigma \in [0,1]\} & \text{if } |u|_X = R, \end{cases} \tag{4.1}$$

where for each $j = 1, 2$, $F_j(t,\cdot) : X \to X'$ is a continuous and bounded operator satisfying, for some positive functions $m \in L^\infty(0,T)$ and $h_j \in L^{p'}(0,T)$ and any $u \in X$, the following

$$|F_j(t,u)|_{X'} \le m(t)|u|_X^{s_j} + h_j(t)\ .$$

Then, if either $0 \leq s = \max\{s_1, s_2\} < p-1$ *or* $s = p-1$ *and* $|m|_\infty$ *is small enough, there is* $u \in \{\omega \in L^p(0,T;V);\, \omega_t \in L^{p'}(0,T;V')\}$ *and* $\xi \in L^{p'}(0,T;X')$ *such that*

$$\begin{cases} u_t + Au = \xi, \\ u(0) = u(T), \end{cases} \tag{4.2}$$

where

$$\xi(t) \in F(t, u(t)) \quad \text{a.e} \quad \text{in } (0,T).$$

(Here A *is naturally seeing as an operator from* $L^p(0,T;V)$ *to* $L^{p'}(0,T;V')$*)*

Remarks: Observe that in this last theorem the abstract result corresponding to the supercritical case of the previous section was not stated . The reason for this is that in Theorem 3.3 either conditions on $s = \max\{s_1, s_2\}$ are directly related to Sobolev functional spaces on a set Ω; more specifically, to the dimension N where $\Omega \subset \mathbb{R}^N$. If we allow the spaces be concrete Sobolev spaces, in Theorem 4.1 for instance, $V = W_0^{1,p}(\Omega)$, $H = L^2(\Omega)$ and $X = L^r(\Omega)$ (with r as in (3.25)), s satisfies either (3.21) or (3.22), and $|m|_\infty$ is small enough, then (4.2) has a weak periodic solution for any operator A as described above. That is, for such concrete Sobolev spaces a partially abstract version of Theorem 3.3 is true .

In addition to $-\Delta_p$ with Dirichlet boundary condition the following operators are examples to which the results in Theorem 4.1 (and in the previous remark) apply.

Example 1: $Au = -\mathrm{div}\,(a(|\nabla u|^p)\,|\nabla u|^{p-2}\,\nabla u)$ with $D(A) = W_0^{1,p}(\Omega)$, $p \geq 2$, $a \in C^1(\mathbb{R})$, $a' \geq 0$ and $\delta\,\sigma^{p-1} \leq a(\sigma^p)\sigma^{p-1} \leq \alpha\,\sigma^{p-1} + \beta$ where α, β and δ are positive constants, $\beta \geq 2$.
Example 2: $Au = -\lambda\Delta_p u + \lambda_2\Delta_{p_2} u - \lambda_1\Delta_{p_1} u$ with $DA = W_0^{1,p}(\Omega)$, $p > p_2 > p_1 \geq 2$, $p = 2p_2 - p_1$, $\lambda\,, \lambda_1\,, \lambda_2 > 0$, $\lambda_2^2 \leq 4\,\lambda\,\lambda_1$ and $[\lambda_2(p_2-1)]^2 - 4\,\lambda\,\lambda_1(p-1)(p_1-1) \leq 0$.
Example 3: $Au = \sum_{i=1}^{N} \dfrac{\partial}{\partial x_i}\left(\left|\dfrac{\partial u}{\partial x_i}\right|^{p-2} \dfrac{\partial u}{\partial x_i}\right)$ with $D(A) = W_0^{1,p}(\Omega)$, $p \geq 2$.

Example 4: $Bu = Au + \lambda|u|^{p-2}u$, $\lambda > 0$ with A any of the operator described above and $D(B) = \left\{u \in W^{1,p}(\Omega);\, \dfrac{\partial u}{\partial \eta} = 0\right\}$.

References

[1] E.P. Avgerinos, N.S. Papageorgiou, On the Existence of Extremal Periodic Solutions for Nonlinear Parabolic Problems with Discontinuities. Yokohama Math. J. 45 (1998).no.1,39-60.

[2] J.L. Boldrini, J. Crema, On Forced Periodic Solutions of Superlinear Problems, Eletronic J.D.E.,vol.1998 (1998),no.14,pp.1-18; URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu; ftp (login:ftp) 147.26.103.110 or 129.120.3.113 .

[3] H. Brezis, Operateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert, Notes of Mathematics 50, North Holland Mathematics Studies, 1973.

[4] D.A.Kandilakis, N.S.Papageorgiou, Nonlinear Periodic Parabolic Problems with Nonmonotone Discontinuities, Proceedings of Edinburg Mathematical Society(1998), 41, 117-132.

[5] E. Kawohl, R. Rühl, Periodic Solutions of Nonlinear Heat Equations under Descontinuous Boundary Conditions, Lecture Notes in Math., 1017, Springer Verlag, 1990.

[6] P. LeTallec, Numerical Analysis of ViscoelasticProblems, RMA 15, Masson, Springer Verlag, 1990.

[7] J.L. Lions, Quelques Méthodes de Resolution de Problèmes aux Limites Non Lineaires, Dunot Gauthier-Villars, Paris, 1969.

[8] M. Ôtani, Nonmonotone Perturbations for Nonlinear Parabolic Equations Associated with Subdifferential Operators, Periodic Problems, J.D.E. 54,(1984), 248-273.

[9] W.A. Strauss, The Energy Method in Nonlinear Partial Differential Equations, Notas de Matemática No. 47, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brazil, 1969.

[10] Y. Yamada, Periodic Solutions of certain Nonlinear Parabolic Equations in Domains with Periodically Moving Boundaries, Nagoya Math. J. 70 (1978), 111-123.

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A Generalization of 2-Periodic Trigonometric Interpolation

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Abstract

It is well-known that the interpolation theory plays an important role in many fields of computer vision, especially in surface reconstruction and Image processing. In this paper, we introduce a new kind of 2-period interpolation of functions with period 2π *at the nodes* $x_k = x_{k,n} = k\pi/n, \quad (k = 0, 1, \cdots, 2n-1)$. *We find out the necessary and sufficient conditions for regularity of this new interpolation problem. Moreover, a closed form expression for the interpolation polynomial is given. Our interpolation is of more practical significance. Our results provide the theoretical basis for using our interpolation in practical problems.*

A.M.S. Classification Codes: 65
Key Words: Interpolation, Trigonometric polynomial, Regularity, Computer vision

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1 Introduction

It is well-known that the interpolation theory plays an important role in many fields of computer vision, especially in surface reconstruction and Image processing. In many practical issues of computer vision, we do not know if the interpolated function is differentiable or we only know the values of the interpolated function at nodes. We often use divided difference to replace derivative because divided difference is the descretization of the derivative of functions. In this paper, we introduce a new kind of 2-periodic interpolation of functions with period 2π at the nodes $x_k = x_{k,n} = k\pi/n, \quad (k = 0, 1, \cdots, 2n-1)$. That is, we would replace the conditions of derivatives for 2-periodic interpolation by the those of differences at the nodes. Similar idea was used by Jiang[1] for Birkhoff-type interpolation.

For $f \in C_{2\pi}$ and $0 < h < \pi/n$, we define

$$\delta_h f(x) = \delta_h^1 f(x) = f(x+h) - f(x-h),$$
$$\delta_h^m f(x) = \delta_h(\delta_h^{m-1} f(x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(x + (m-2k)h), \quad m \geq 2.$$

We shall say that $t(x) \in \mathcal{T}_n$, if $t(x)$ is a polynomial of the form

$$t_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

If

$$t_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx) + a_n \cos(nx + \varepsilon\pi/2),$$
$$(\varepsilon = 0 \quad \text{or} \quad 1),$$

we shall say that $t_n(x) \in \mathcal{T}_{n,\varepsilon}(\varepsilon = 0 \quad \text{or} \quad 1)$.

Our problems are

P_1: For any two given sets of complex numbers $\{\alpha_k\}_0^{n-1}$ and $\{\beta_k\}_0^{n-1}$, whether or not there exists a unique trigonometric polynomial $t_n(x) \in \mathcal{T}_{n,\varepsilon}(\varepsilon = 0 \quad \text{or} \quad 1)$ satisfying the conditions

$$t_n(x_{2k}) = \alpha_k, \quad \delta^m(t_n)(x_{2k+1})/(2h)^m = \beta_k, \quad (k = 0, 1, \cdots, n-1).$$

P_2: If the answer of Problem P_1 is positive, usually, we say the interpolation problem is regular. Find the necessary and sufficient conditions on m and n such that Problem P_1 is regular.

P_3: Find the fundamental polynomials of the interpolation when Problem P_1 is regular.

The rest of this paper is organized as follows. Section 2 describes some lemmas needed for the subsequent parts. Section 3 is devoted to the regularity of this new interpolation. Section 4 will give the fundamental polynomials of the new interpolation. Final section is the conclusion and the future works.

2 Lemmas

In order to prove our main results we need the following lemmas. Among them, Lemma 2.1 and Lemma 2.2 come from Sharma and Sun [3]

Lemma 2.1 ([6, Lemma 2.1]) *If*

$$K_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), \tag{1}$$

then

(1) $K_n(x_{2k}) = \delta_{0,k}, (k = 0, 1, \cdots, n-1)$ *if and only if*

$$\begin{cases} \frac{1}{2}a_0 + a_n = \frac{1}{n} \\ a_k + a_{n-k} = \frac{2}{n} \\ b_k - b_{n-k} = 0 \\ k = 1, \cdots, n-1. \end{cases}$$

(2) $K_n(x_{2k}) = 0, (k = 0, 1, \cdots, n-1)$ *if and only if*

$$\begin{cases} \frac{1}{2}a_0 + a_n = 0, \\ a_k + a_{n-k} = 0, \\ b_k - b_{n-k} = 0, \\ k = 1, \cdots, n-1. \end{cases}$$

Proof It is an easy consequence of the following two known identities.

$$\frac{\sin \frac{nx}{2}}{n \sin \frac{x}{2}} = \frac{1}{n}\left[1 + 2\sum_{j=1}^{(n-1)/2} \cos jx\right], \qquad n \quad \text{odd},$$

and

$$\frac{\cos \frac{x}{2} \sin \frac{nx}{2}}{n \sin \frac{x}{2}} = \frac{1}{n}\left[1 + 2\sum_{j=1}^{(n-2)/2} \cos jx + \cos\frac{nx}{2}\right], \qquad n \quad \text{even}.$$

Lemma 2.2 ([5, Lemma 2.2]) *If* $K_n(x)$ *is given by (1) , then*

(1) $K_n(x_{2k+1}) = \delta_{0,k}, (k = 0, 1, \cdots, n-1)$ *if and only if*

$$\begin{cases} \frac{1}{2}a_0 - a_n = \frac{1}{n} \\ a_k - a_{n-k} = \frac{2}{n}\cos jx_1 \\ b_k + b_{n-k} = \frac{2}{n}\sin jx_1 \\ k = 1, \cdots, n-1. \end{cases}$$

(2) $K_n(x_{2k+1}) = 0, (k = 0, 1, \cdots, n-1)$ *if and only if*

$$\begin{cases} \frac{1}{2}a_0 - a_n = 0, \\ a_k - a_{n-k} = 0, \\ b_k + b_{n-k} = 0, \\ k = 1, \cdots, n-1. \end{cases}$$

The proof depends upon using Lemma 2.1 after the observation that

$$\begin{aligned}
&K_n(x_{2k+1}) = K_n(x_{2k} + \tfrac{\pi}{n}) \\
&= \tfrac{a_0}{2} - a_n + \textstyle\sum_{j=1}^{n-1}(P_j \cos jx_{2k} + Q_j \sin jx_{2k}), \\
&(k = 0, 1, \cdots, n-1).
\end{aligned}$$

where

$$P_j = a_j \cos jx_1 + b_j \sin jx_1,$$

and

$$Q_j = b_j \cos jx_1 - a_j \sin jx_1.$$

Lemma 2.3 *For $m = 1, 2, \cdots,$ we have*

$$\begin{aligned}
\delta^m \cos jx &= 2^m \sin^m jh \cos(jx + \frac{m\pi}{2}), \\
\delta^m \sin jx &= 2^m \sin^m jh \sin(jx + \frac{m\pi}{2}).
\end{aligned}$$

Proof It is easy to prove it by induction.

3 Regularity

Theorem 3.1 *(1) For $\varepsilon = 0$, Problem P_1 is regular if and only if m is even.*
(2) For $\varepsilon = 1$, Problem P_1 is regular if and only if m and n are both odd.

Proof We only prove part (1) because the proof of part (2) is similar. Let $Q_n(x) \in \mathcal{T}_{n,0}$ have the form

$$Q_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1}(a_k \cos kx + b_k \sin kx) + a_n \cos nx,$$

and let

$$\begin{cases} Q_n(x_{2k}) = 0 \\ \delta^m(Q_n)(x_{2k+1}) = 0. \\ k = 0, 1, \cdots, n-1. \end{cases}$$

Taking $K_n(x) = Q_n(x)$ by the part (2) of Lemma 2.1, we have

$$\begin{cases} \frac{1}{2}a_0 + a_n = 0 \\ a_j + a_{n-j} = 0 \\ b_j - b_{n-j} = 0 \\ j = 1, \cdots, n-1. \end{cases} \tag{2}$$

Applying Lemma 2.3, we have

$$\begin{aligned}
\delta^m(Q_n)(x) \quad = \quad & \sum_{j=1}^{n-1}(a_j 2^m \sin^m jh \cos(jx + \frac{m\pi}{2}) + b_j 2^m \sin^m jh \sin(jx + \frac{m\pi}{2})) \\
& + a_n 2^m \sin^m nh \cos(nx + \frac{m\pi}{2})
\end{aligned}$$

If $m = 2s$, we have

$$\begin{aligned}\delta^m(Q_n)(x) &= \sum_{j=1}^{n-1}(a_j(-1)^{m/2}2^m \sin^m jh \cos jx + b_j(-1)^{m/2}2^m \sin^m jh \sin jx) \\ &\quad + a_n(-1)^{m/2}2^m \sin^m nh \cos nx\end{aligned}$$

Taking $K_n(x) = \delta^m(Q_n)(x)$ by the part (2) of Lemma 2.2, we have

$$\begin{cases} a_n(-1)^{m/2}2^m \sin^m nh = 0 \\ a_j(-1)^{m/2}2^m \sin^m jh - a_{n-j}(-1)^{m/2}2^m \sin^m (n-j)h = 0 \\ b_j(-1)^{m/2}2^m \sin^m jh + b_{n-j}(-1)^{m/2}2^m \sin^m (n-j)h = 0 \\ j = 1, 2, \cdots, n-1. \end{cases}$$

That is

$$\begin{cases} a_n \sin^m nh = 0 \\ a_j \sin^m jh - a_{n-j} \sin^m (n-j)h = 0 \\ b_j \sin^m jh + b_{n-j} \sin^m (n-j)h = 0 \\ j = 1, 2, \cdots, n-1. \end{cases} \tag{3}$$

From the first equations (2) and (3) we have $a_0 = a_n = 0$.
From other equations of (2) and (3) we have

$$\begin{cases} a_j(\sin^m jh + \sin^m (n-j)h) = 0 \\ b_j(\sin^m jh + \sin^m (n-j)h) = 0 \\ j = 1, 2, \cdots, n-1. \end{cases} \tag{4}$$

From $m = 2s$ we have

$$\sin^m jh + \sin^m (n-j)h > 0 \quad (j = 1, 2, \cdots, n-1).$$

Hence $a_j = b_j = 0 \quad (j = 1, 2, \cdots, n-1)$. This implies that Problem P_1 is regular.
If $m = 2s+1$, we have

$$\begin{aligned}\delta^m(Q_n)(x) &= \sum_{j=1}^{n-1}(b_j(-1)^{m/2}2^m \sin^m jh \cos jx - a_j(-1)^{m/2}2^m \sin^m jh \sin jx) \\ &\quad - a_n(-1)^{m/2}2^m \sin^m nh \sin nx\end{aligned}$$

Taking $K_n(x) = \delta^m(Q_n)(x)$ by the part (2) of Lemma 2.2, we have

$$\begin{cases} b_j \sin^m jh - b_{n-j} \sin^m (n-j)h = 0 \\ a_j \sin^m jh + a_{n-j} \sin^m (n-j)h = 0 \\ j = 1, 2, \cdots, n-1. \end{cases} \tag{5}$$

It is clear that equation systems (2) and (5) does not have a unique solution. Hence, Problem P_1 is not regular. This completes the proof of Theorem 3.1.

4 Fundamental Polynomials of Interpolation

If we denote the fundamental polynomials of $(0, \delta^m)$ interpolation by $r_{j,\varepsilon}(x)$ and $\rho_{j,\varepsilon}(x)$, respectively, then it is clear that

$$r_{j,\varepsilon}(x) = r_{0,\varepsilon}(x - x_j), \quad \rho_{j,\varepsilon}(x) = \rho_{0,\varepsilon}(x - x_j), j = 1, 2, \cdots, n-1.$$

We come to give the explicit forms of the fundamental polynomials $r_{0,\varepsilon}(x)$ and $\rho_{0,\varepsilon}(x)$. They determined by the following conditions.

$$r_{0,\varepsilon}(x_{2k}) = \delta_{0,k}, \quad \delta^m(r_{0,\varepsilon})(x_{2k+1})/(2h)^m = 0. \tag{6}$$

$$\rho_{0,\varepsilon}(x_{2k}) = 0, \quad \delta^m(\rho_{0,\varepsilon})(x_{2k+1})/(2h)^m = \delta_{0,k}. \tag{7}$$

$k = 1, 2, \cdots, n-1.$

Theorem 4.1 *(1) If $\varepsilon = 0$ and m is even, then the fundamental polynomials are given by*

$$\begin{cases} r_{0,0} = \frac{1}{n} + \frac{2}{n}\sum_{j=1}^{n-1} \frac{\sin^m(n-j)h\cos jx}{\sin^m(n-j)h + \sin^m jh} \\ \rho_{0,0} = \frac{2(-1)^{m/2}h^m}{n}\left\{-\frac{1}{2\sin^m nh} + \sum_{j=1}^{n-1} \frac{\cos jx_1 \cos jx + \sin jx_1 \sin jx}{\sin^m(n-j)h + \sin^m jh} + \frac{\sin nx}{\sin^m nh}\right\} \end{cases} \tag{8}$$

(2) If $\varepsilon = 1$ and m and n are odd, then the fundamental polynomials are given by

$$\begin{cases} r_{0,1} = \frac{1}{n} + \frac{2}{n}\sum_{j=1}^{n-1} \frac{\sin^m(n-j)h\cos jx}{\sin^m(n-j)h - \sin^m jh} \\ \rho_{0,1} = \frac{2(-1)^{m/2}h^m}{n}\left\{\sum_{j=1}^{n-1} \frac{\cos jx_1 \cos jx + \sin jx_1 \sin jx}{\sin^m(n-j)h - \sin^m jh} - \frac{\sin nx}{2\sin^m nh}\right\} \end{cases} \tag{9}$$

Proof We only prove part (2) because the proof of part (1) is similar. For $\varepsilon = 1$, we denote

$$Q_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1}(a_k \cos kx + b_k \sin kx) + b_n \sin nx. \tag{10}$$

Since m is odd, we have

$$\begin{aligned} \delta^m(Q_n)(x) &= \sum_{j=1}^{n-1}(b_j(-1)^{m/2}2^m \sin^m jh \cos jx - a_j(-1)^{m/2}2^m \sin^m jh \sin jx) \\ &\quad + b_n(-1)^{m/2}2^m \sin^m nh \cos nx \end{aligned}$$

If $Q_n(x)$ satisfies conditions (7), then taking $K_n(x) = Q_n(x)$ by the part (1) of Lemma 2.1 and $K_n(x) = \delta^m(Q_n)(x)$ by the part (2) of Lemma 2.2 respectively, we have

$$\begin{cases} \frac{1}{2}a_0 = \frac{1}{n} \\ a_j + a_{n-j} = \frac{2}{n} \\ b_j - b_{n-j} = 0 \\ j = 1, \cdots, n-1, \end{cases} \tag{11}$$

and

$$\begin{cases} b_n \sin^m nh = 0 \\ b_j \sin^m jh - b_{n-j} \sin^m (n-j)h = 0 \\ a_j sin^m jh + a_{n-j} \sin^m (n-j)h = 0 \\ j = 1, 2, \cdots, n-1. \end{cases} \tag{12}$$

From the first equations (1) and (2) we have $a_0 = 2/n$ and $b_n = 0$.

From other equations of (4) and (7) we have

$$\begin{cases} a_j (\sin^m (n-j)h - \sin^m jh) = \frac{2}{n} \sin^m (n-j)h \\ b_j (sin^m jh - \sin^m (n-j)h) = 0 \\ j = 1, 2, \cdots, n-1. \end{cases} \tag{13}$$

Since n is odd, $\sin^m (n-j)h - \sin^m jh \neq 0$. Hence, from (14) we have

$$\begin{cases} a_j = \frac{2 \sin^m (n-j)h}{n(\sin^m (n-j)h - \sin^m jh)} \\ b_j = 0 \\ j = 1, 2, \cdots, n-1. \end{cases}$$

This implies the first equality of (10).

If $Q_n(x)$ satisfies conditions (8), then taking $K_n(x) = Q_n(x)$ by the part (2) of Lemma 2.1 and $K_n(x) = \delta^m (Q_n)(x)/(2h)^m$ by the part (1) of Lemma 2.2 respectively, we have

$$\begin{cases} \frac{1}{2} a_0 = 0 \\ a_j + a_{n-j} = 0 \\ b_j - b_{n-j} = 0 \\ j = 1, \cdots, n-1, \end{cases} \tag{14}$$

and

$$\begin{cases} -b_n (-1)^{m/2} 2^m \sin^m nh/(2h)^m = \frac{1}{n} \\ b_j (-1)^{m/2} 2^m \sin^m jh/(2h)^m - b_{n-j} (-1)^{m/2} 2^m \sin^m (n-j)h/(2h)^m = \frac{2}{n} \cos jx_1 \\ a_j (-1)^{m/2} 2^m \sin^m jh/(2h)^m + a_{n-j} (-1)^{m/2} 2^m \sin^m (n-j)h/(2h)^m = \frac{2}{n} \sin jx_1 \\ j = 1, 2, \cdots, n-1. \end{cases} \tag{15}$$

From (15) and (16) we have

$$a_0 = 0, \quad b_n = -\frac{(-1)^{m/2} h^m}{n \sin^m nh} \tag{16}$$

and

$$\begin{cases} a_j = \frac{2(-1)^{m/2} h^m \sin jx_1}{n(\sin^m (n-j)h - \sin^m jh)} \\ b_j = \frac{2(-1)^{m/2} h^m \cos jx_1}{n(\sin^m (n-j)h - \sin^m jh)} \\ j = 1, 2, \cdots, n-1. \end{cases}$$

This implies the Second equality of (10).

5 Conclusion

It is well-known that interpolation theory plays an important role in many fields of computer vision, especially in surface reconstruction. In many practical issues of computer vision, we do not know if the interpolated function is differentiable or we only know the values of the interpolated function at nodes. From computational view, divided difference is a natural candidate for replacing derivative because divided difference is the descretization of the derivative of functions. In this paper, we would replace the conditions of derivatives for Hermite interpolation and Birkhoff interpolation by those of symmetric differences at the nodes. First, we introduce a new kind of 2-period interpolation of functions with period 2π. Second, we find out the necessary and sufficient conditions for regularity of this new interpolation problem. Moreover, a closed form expression for the interpolation polynomial is given. Our interpolation is of more practical significance. Our results provide the theoretical basis for using our interpolation in practical problems. Here we only consider the interpolation of 2π-periodic functions at the nodes $x_k = x_{k,n} = 2k\pi/n, \quad (k = 0, 1, \cdots, n-1)$. In future, we will try to get the convergence of this new interpolation. We would also like to extend our idea to algebraic polynomial interpolation case and multivariate interpolation cases.

References

[1] Tianzi Jiang, On a New kind of Birkhoff Type Trigonometric Interpolation, BIT, 35(4)(1995),540-547.

[2] Yongping Liu, On the Trigonometric interpolation and the entire interpolation, Approx. Theory and its Appl.,**6**:4(1990), 85-106.

[3] A. Sharma, J. Szabados, R. S. Varga, 2-periodic lacunary trigonometric interpolation: the $(0; M)$ case, "Constructive theory of functions", (Varna, 1987), 420–427, Bulgar. Acad. Sci., Sofia, 1988.

[4] A. Sharma, J. Szabados, R. S. Varga, Some 2π-periodic trigonometric interpolation problems on equidistant nodes, Analysis 11 (1991), no. 2-3, 165–190.

[5] A. Sharma, J. Szabados, R. S. Varga, Some 2-periodic trigonometric interpolation problems on equidistant nodes (II): Convergence, Studia Sci. Math. Hungar. 29 (1994), no. 3-4, 415–432.

[6] A. Sharma, Xie Hua Sun, A 2-periodic trigonometric interpolation problem, Approx. Theory Appl. 8 (1992), no. 4, 1–16.

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Fast Multiplication of Small Numbers

Jan-Christoph Puchta

Abstract

We give an algorithm to multiply two n-digit numbers with running time $O\left(\frac{n^2}{\log n}\right)$. Different from asymptotically better algorithms this algorithm outnumbers the standard algorithm allready for 11-digit numbers.

Subject Classification Codes: 68Q25
Keywords: Multiplication of Integers, Algorithm

Let f be a function with the following two properties: If $f(n)$ is known, $f(2n)$ can be computed at cost a, and if $f(n)$ and $f(m)$ are known, $f(n+m)$ can be computed at cost b, where obviously $a \leq b$. Then one can compute $f(n)$ recursively by computing $f\left(\lfloor \frac{n}{2} \rfloor\right)$, and then computing $f(n)$ with cost a resp. $a+b$ depending on whether n is even or odd. The expected running time to compute $f(n)$ is $a \log_2 n + \frac{b}{2} \log_2 n$. We will refer to this procedure as the standard algorithm.

If a and b are large compared to the cost of bitoperations, planing the computation can reduce the total costs, e.g. a^n can be computed with $(1+o(1)) \log_2 n$ multiplications instead of $(\frac{3}{2}+o(1)) \log_2 n$ multiplications by exploiting addition chains[2]. The aim of this note is to show that similar methods can be used to improve easier computations, too. The most important example is certainly the multiplication of integers.

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Choose an integer $k \geq 2$ and compute $f(a)$ for all odd integers $< 2^k$. To compute $f(n)$ for odd n, compute $f\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right)$. If n is odd, compute $f\left(2^k \left\lfloor \frac{n}{2^k} \right\rfloor\right)$ with cost $k \cdot a$, then $f(n)$ with additional cost b. For even n compute $\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$, then $f(n)$ with cost a.

Proposition 1

The total cost of the algorithm described above is $< a \log_2 n + \frac{b}{k} \log_2 n + 2^{k-1} b + a$. *Set* $f(n) = n \cdot m$ *and assume that* n *and* m *have* d *binary digits. Then the total cost of the algorithm above is* $\ll \frac{d^2}{\log d}$. *Moreover with* $k = 2$ *the expected running time of the algorithm above is less then the running time of the standard algorithm if* n *has more then 11 binary digits.*

Proof:

1. During the computation of $f(n)$ we have to do $\log_2 n)$ shifts of cost a, and at most $\frac{\log_2 n}{k}$ additions of cost b. Finally we can compute $f(n)$ for all $n \leq 2^k$ at total cost $2^{k-1} b + a$ by first computing $f(2)$ at cost a, then $f(2k+1)$ out of $f(2k-1)$ at cost b. □

2. For this special choice of f, b is $\sim \log_2 m$, however a is neglectable. With $k = \frac{1}{2} \log_2 d$ and the estimate in 1. we obtain total costs of $\frac{2d^2}{\log_2 d} + d^{3/2} \ll \frac{d^2}{\log_2 d}$. Now for $k = 2$ a saving of b is achieved for every pair of consecutive 1's in the binary representation of n. On the other hand we have to compute $f(3)$ with cost b. Thus in average a saving is achieved as soon as the expected number of consecutive pairs of 1's in the binary representation of n is > 2. Now a simple computation shows that this number equals $\frac{d}{6} + \frac{1}{6} + \frac{(-1)^{d-1}}{3 \cdot 2^{d-1}}$, which for $d \geq 11$ is indead > 2. □

Note that there is an algorithms by Karatsuba and Ofman with running time $O(d^{1.585})$ and even an algorithm with running time $O(n \log n \log \log n)$ by Schonhagen and Strasser[1], however, the constants implied by the O-symbols are so large that these algorithms are useless for computations with numbers with less than 1000 binary digits.

References

[1] A. V. Aho, J. E. Hopcroft, J. D. Ullman, *The design and Analysis of Computer Algorithms*, Reading, Mass., Addison – Wesley 1974

[2] D. E. Knuth, *The Art of Computer Programming, Volume 2: Seminumerical Algorithms*, Reading, Mass., Addison – Wesley 1969

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On the Solution of an Integral Equation Appearing in Displacement Problems

M.H. Fahmy, M.A. Abdou and M.A. Darwish

Abstract

In this paper we are interested in solving an integral equation appearing in the study of two kinds of displacement problems. A simple way is used to obtain a solution of this equation.

AMS Mathematics Subject Classification : 45B05
Key Words: Fredholm Integral Equations, Displacement Problems

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1. Introduction

It is known, in general, that a Fredholm integral equation of the first kind is difficult to solve. However, in this paper we present a simple way to solve an integral equation of the first kind arising in the study of two kinds of displacement problems. Let us consider the Fredholm integral equations of the first kind

$$\int_{-a}^{a} k(x,\ y)\ \phi(y)\ dy = \pi\ f(x). \tag{1.1}$$

Here, $f(x)$ and $k(x,\ y)$ are given functions, and $k(x,\ y)$ is the kernel of the integral equation (1.1) which may have a singularity while the unknown function $\phi(x)$ is to be determined.

Mkhitarian and Abdou, [6], applied Krein's method to obtain the basic formula of the potential function when $k(y) = -\ln(|y|)$ and $f(x) \in C^1[-a, a]$. Also, Popov in his work [9] solved Eq. (1.1) under certain conditions on the kernel using orthogonal polynomials method. The crucial achievement in Eq.(1.1) is Muskhelishvili's result, [7], for solving singular integral equations. Abdou [1] established Eq.(1.1) from three-dimensional contact problems in the theory of elasticity with potential kernel where the coefficients bed of the compressible materials are neglected. More recently, Fahmy and others [3] obtained the solution of Eq.(1.1) with Gauss's hypergeometric kernel in the half-plane using Abel's theorem.

In this paper, we consider the integral equation

$$-\int_{-a}^{a} K\left(\frac{y-x}{\lambda}\right) \phi(y)\ dy = \pi\ f(x), \quad |x| < a, \tag{1.2}$$

under the conditions

$$\int_{-a}^{a} \phi(y)\ dy = P < \infty, \tag{1.3}$$

and

$$K(\xi) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tanh\ \eta}{\eta} e^{i\eta\xi}\ d\eta. \tag{1.4}$$

Here, the given function f is assumed to be sufficiently smooth in order to guarantee the existence and uniqueness of a solution $\phi \in C[-a, a]$ (see [8]) and λ is a parameter.

We are interested in solving Eq.(1.2), under the conditions (1.3) and (1.4), since it arises in the study of two kinds of displacement problems, see Section 3. In [2], the solution of Eqs.(1.2)- (1.4) is given by using Krein's method.

In the following section, we present a simple way to solve the integral equations (1.2)-(1.4).

2. Solution of the problem

Firstly, the second condition, (1.4), may be written in the form, [10, p. 32],

$$K(\xi) = -\ln\left|\tanh\frac{\pi\xi}{4}\right|. \tag{2.1}$$

If $\lambda \to \infty$ and ξ is sufficiently small, i.e., $\tanh\,\theta \approx \theta$, we have that

$$\ln\left|\tanh\frac{\pi(y-x)}{4\lambda}\right| = \ln|y-x| - c,\ c = \ln\frac{4\lambda}{\pi}. \tag{2.2}$$

Then Eq.(1.2) becomes

$$\int_{-a}^{a} [c - \ln|y-x|]\,\phi(y)\,dy = -\pi f(x), \tag{2.3}$$

where $\phi(y)$ satisfies (1.3). Secondly, rewrite Eq.(2.3) in the form

$$\int_{-a}^{a} \ln|y-x|\;\phi(y)\,dy = B + \pi f(x), \tag{2.4}$$

where

$$B = c\int_{-a}^{a}\phi(y)\,dy. \tag{2.5}$$

But the solution of Eq.(2.4) is given by, [5, pp. 209],

$$\begin{aligned}(2.6)\ \phi(x) \;=\;& \frac{1}{\pi}\int_{-a}^{a}\sqrt{\frac{a^2-y^2}{a^2-x^2}}\,\frac{f'(y)}{y-x}\,dy +\\ &+\frac{1}{\pi\,\ln(a/2)\sqrt{a^2-x^2}}\left\{B+\int_{-a}^{a}\frac{f(y)}{\sqrt{a^2-y^2}}\,dy\right\}.\end{aligned}$$

Now, it remains to find the unknown constant B in Eq.(2.6). Substitute Eq.(2.6) in Eq.(2.5) we obtain

$$B = \frac{c}{\ln(a/2) - c}\int_{-a}^{a}\frac{f(y)}{\sqrt{a^2-y^2}}\,dy, \tag{2.7}$$

where we used the fact that

$$\int_{-\alpha}^{\alpha} \frac{du}{\sqrt{\alpha^2 - u^2}} = \pi$$

and, from the complex analysis, the fact that

$$\int_{-\alpha}^{\alpha} \frac{1}{\sqrt{\alpha^2 - u^2}} \, \frac{du}{v - u} = 0, \quad |v| < 1.$$

Finally, substituting the value of B from (2.7) into (2.6) we have the the solution of Eq.(1.2) under the conditions (1.3) and (1.4) as

$$(2.8)\quad \phi(x) = \frac{1}{\pi\sqrt{a^2 - x^2}} \int_{-a}^{a} \sqrt{a^2 - y^2} \frac{f'(y)}{y - x} \, dy + \frac{P}{\pi \sqrt{a^2 - x^2}}, \quad |x| < a$$

and

$$(2.9)\qquad P = \frac{1}{\ln(a/2) - c} \int_{-a}^{a} \frac{f(y)}{\sqrt{a^2 - y^2}} \, dy.$$

This result is in accordance with the results obtained in [2, 6].

3. Discussion

The displacement problems of anti-plane deformation of an infinite rigid strip with width $2a$, putting on an elastic layer of thickness h, can be formulated as, see [2] and and reference therein,

$$(3.1)\qquad -\int_{-a}^{a} \phi(\tau) \, \ln \left| \tanh \, \frac{\pi \, (\tau - x)}{4h} \right| \, d\tau = D,$$

where $\phi(\tau)$ is the unknown displacement stress and D is the displacement stress.

The problems of infinite rigid strip, with width $2a$ immersed in a viscous liquid layer of thickness h, where the strip has a velocity resulting from the impulsive force $V = V_0 \, e^{-i\omega t}$. Such problems are represented by the Fredholm integral equation, see [3] and reference therein,

$$(3.2)\qquad -\int_{-a}^{a} \phi(\tau) \, \ln \left| \tanh \, \frac{\pi \, (\tau - x)}{4h} \right| \, d\tau = \pi\mu \, V,$$

where V_0 is the constant velocity, ω is the angular velocity resulting from rotating the strip about $z - axis$ and μ is the viscosity coefficient.

The depth of the liquid or the thickness of the elastic material becomes infinite as $h \to \infty$.

References

[1] M.A. Abdou, *Fredholm integral equation of the second kind with potential kernel*, J. Comput. Appl. Math. **72 (1)** (1996) 161-167.

[2] M.A. Abdou and N.Y. Ezzeldin, *Krein's method with certain singular kernel for solving the integral equation of the first kind*, Periodica Mathematica Hungarica **28**(2) (1994) 143-149.

[3] M.H. Fahmy, M.A. Abdou and M.A. Darwish, *Integral equations and potential-theoretic type integrals of orthogonal polynomials*, J. Comput. Appl. Math. **106 (2)** (1999), 245-254.

[4] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products* Fifth Edition (Academic Press, Inc., 1994).

[5] R.P. Kanwal, *Linear Integral Equations*, Birkhäuser, Boston, 1996.

[6] S.M. Mkhitarian and M.A. Abdou, *On different methods for solving the Fredholm integral equation of the first kind with logarithmic kernel*, Dakl. Acad, Nauk. Armenia **90** (1989) 1-10.

[7] N.I. Muskhelishvili, *Singular Integral Equations* , Dover Publ., New York, 1992 (Russian edition: Moscow, 1946).

[8] A.D. Polyanin and A.V. Manzhirov, *Hand Book of Integral Equations*, CRC Press, 1998.

[9] G.Ya. Popov, *On a method of solving axisymmetric contact problems of elasticity theory*, PMM **25**(1) (1961).

[10] G. Ya. Popov, *Contact Problems for a Linearly Deformable Base* (Kiev, Odessa, 1982).

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