Volume 1 July–December 1995 ISSN 1083-0464

SOUTHWEST JOURNAL OF PURE AND APPLIED MATHEMATICS

CAMERON UNIVERSITY
DEPARTMENT OF MATHEMATICAL SCIENCES
LAWTON, OK 73505, U.S.A

Southwest Journal
of
Pure and Applied Mathematics

Editor in Chief: Ioannis Argyros
Managing Editor: Mohammad Tabatabai
Technical Editor: Gary Huckabay

Editorial Office

SWJPAM
Cameron University
Department of Mathematical Sciences
Lawton, OK 73505, U.S.A.

Publication Program

2 issues per year
July, December

Information for Authors

Scope. SWJPAM is an electronic journal devoted to all aspects of Pure and Applied mathematics, and related topics. Authoritative expository and survey articles on subjects of special interest are also welcomed.

Since SWJPAM aims to serve not only the specialists, but also the user of mathematics (promoting, in such a way, a strong interaction between them), particular emphasis is placed on applications. Thus, SWJPAM publishes papers which, although not directly contributing to mathematics, contain, however, significant results (from any field) obtained by making an essential and well stressed use of the methods and results of mathematics.

Submission of Manuscripts. Since SWJPAM is a completely electronic journal, all manuscripts must be submitted in either AMSTEX, LaTEX, TEX, or post script form. Papers for publication should be submitted to ftp://rattler.cameron.edu/SWJPAM_incoming by anonymous ftp. The articles must be placed in the 'incoming' directory. The submission should include the title of the paper, the author's name and affiliation, and a short abstract which summarizes the paper; a list of keywords and a classification according to the Mathematical Reviews' Subject Classification (1991). Authors are informed by e-mail of referee communication.

Key Words. Key words should be comprehensive and subject specific. It is not necessary to list the subject area of the journal's coverage as a key word. Four key words should be sufficient to cover the major subjects off a given paper, although more can always be supplied if the author deems it necessary.

 (continued on inside back cover)

SOUTHWEST JOURNAL
OF
PURE AND APPLIED MATHEMATICS
VOLUME 1, JULY–DECEMBER 1995

CONTENTS

ISSN 1083-0464

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Vol. 01, 1995, pp. 1–12.
Submitted: March 8, 1995. Published: June 20, 1995.

PERTURBED NEWTON-LIKE METHODS AND NONDIFFERENTIABLE OPERATOR EQUATIONS ON BANACH SPACES WITH A CONVERGENCE STRUCTURE

Ioannis K. Argyros

Abstract. In this study, we use perturbed Newton-like methods to find solutions of nonlinear, nondifferentiable operator equations on Banach spaces with a convergence structure. This technique involves the introduction of a generalized norm as an operator from a linear space into a partially ordered Banach space. In this way the metric properties of the examined problem can be analyzed more precisely. Moreover, this approach allows us to derive from the same theorem, on the one hand, semi-local results of Kantorovich-type, and on the other hand, global results based on monotonicity considerations. Furthermore, we show that special cases of our results reduce to the corresponding ones already in the literature. Finally, our results are used to solve integral equations that cannot be solved with existing methods.

(1991) A.M.S. (MOS) Subject Classification Codes. 65J15, 65B05, 47D15, 47H17, 49D15.

Key Words and Phrases. Banach space, perturbed Newton-like methods, nondifferentiable operator.

I. INTRODUCTION

In this study, we are concerned with approximating a solution x^* of the nonlinear operator equation

$$F(x) + Q(x) = 0, \tag{1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in X, and Q is a nondifferentiable nonlinear operator with the same domain and values in X.

We introduce the perturbed Newton-like method

$$x_{n+1} = x_n + A(x_n)^*[-(F(x_n) + Q(x_n))] - z_n, \quad x_0 = 0 \quad (n \geq 0) \tag{2}$$

to approximate a solution x^* of equation (1). Here $A(x_n)^*$ $(n \geq 0)$ denotes a linear operator which is an approximation for $F'(x_n)^{-1}$ $(n \geq 0)$.

The importance of studying perturbed Newton-like methods comes from the fact that many commonly used variants of Newton's method can be considered procedures of this type. Indeed, approximation (2) characterizes any iterative process in which the corrections are taken as approximate solutions of the Newton equations. In particular for $A(x) = F'(x)$, $Q(x) = 0$ and $z_n = 0$ $(n \geq 0)$ we obtain the ordinary Newton-Kantorovich method [4].

The notion of a Banach space with a convergence structure was used in the elegant paper in [5] (see also [1], [2], [3], [8]) to solve equation (1), when $A(x) = F'(x)$, $Q(x) = 0$ for all $x \in D$ and $z_n = 0$ for all $n \geq 0$. However, there are many interesting real life applications already in the literature, where

Department of Mathematics, Cameron University, Lawton, OK 73505, U.S.A.
E-mail Address: ioannisa@cua.cameron.edu

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

equation (1) contains a nondifferentiable term. See for example the applications at the end of this study. Moreover, since the Newton-like iterates can rarely be computed exactly, suitable choices of the points z_n will "correct" the sequence x_n and force it to converge to a solution x^* of equation (1). The case when $Q(x) = 0$ for all $x \in D$ and $F'(x_n)^* = F'(x_n)^{-1}$ $(n \geq 0)$ has already been considered but on a Banach space without generalized structure [3], [4], [7], [8], [9].

As in [6], we provide semi-local results of Kantorovich-type and global results based on monotonicity considerations from the same general theorem. Moreover, we show that our results can be reduced to the one obtained in [6], when $Q(x) = 0$ for all $x \in D$ and $z_n = 0$ $(n \geq 0)$, and furthermore to the ones obtained in [1], [2], [3], [5], [6], [7], [8] by further relaxing the requirements on X.

Finally, our results apply to solve a nonlinear integral equation involving a nondifferentiable term that cannot be solved with existing methods.

II. PRELIMINARIES

We will need the definitions:

Definition 1. The triple (X, V, E) is a Banach space with a convergence structure if

(C_1) $(X, \|\cdot\|)$ is a real Banach space;

(C_2) $(V, C, \|\ \|_{\mathrm{v}})$ is a real Banach space which is partially ordered by the closed convex cone C; the norm $\|\ \|_{\mathrm{v}}$ is assumed to be monotone on C;

(C_3) E is a closed convex cone in $X \times V$ satisfying $\{0\} \times C \subseteq E \subseteq X \times C$;

(C_4) the operator $|\ | : D \to C$ is well defined:

$$|x| = \inf\{q \in C | (x, q) \in E\}$$

for

$$x \in D = \{x \in X | \exists q \in C : (x, q) \in E\};$$

and

(C_5) for all $x \in D$ $\quad \|x\| \leq \big\| |x| \big\|_{\mathrm{v}}$.

The set

$$U(a) = \{x \in X | (x, a) \in E\}$$

defines a sort of generalized neighborhood of zero.

Definition 2. An operator $L \in C^1(V_1 \to V)$ defined on an open subset V_1 of an ordered Banach space V is order convex on $[a, b] \subseteq V_1$ if

$$c, d \in [a, b],\ c \leq d \Rightarrow L'(d) - L'(c) \in L_+(V),$$

where for $m \geq 0$

$$L_+(V^m) = \{L \in L(V^m) | 0 \leq x_i \Rightarrow 0 \leq L(x_1, x_2, \ldots, x_m)\}$$

and $L(V^m)$ denotes the space of m-linear, symmetric, bounded operators on V.

Definition 3. The set of bounds for an operator $H \in L(X^m)$ is defined to be

$$B(H) = \{L \in L_+(V^m) | (x_i, q_i) \in E \Rightarrow (H(x_1, \ldots, x_m), L(q_1, \ldots, q_m)) \in E\}$$

Definition 4. Let $H \in L(X)$ and $y \in X$ be given, then

$$H^*(y) = z \Leftrightarrow z = T^\infty(0) = \lim_{n\to\infty} T^n(0),$$
$$T(x) = (I - H)(x) + y \Leftrightarrow z = \sum_{i=0}^{\infty} (I - H)^i y,$$

if this limit exists.

We will also need the Lemmas [2], [3], [5], [6]:

Lemma 1. Let $L \in L_+(V)$ and $a, q \in C$ be given such that:

$$L(q) + a \leq q \text{ and } L^n(q) \to 0 \text{ as } n \to \infty. \tag{3}$$

Then the operator

$$(I - L)^* : [0, a] \to [0, a]$$

is well defined and continuous.

The following is a generalization of Banach's lemma [3], [5], [6].

Lemma 2. Let $H \in L(X)$, $L \in B(H)$, $y \in D$ and $q \in C$ be such that

$$L(q) + |y| \leq q \text{ and } L^n(q) \to 0 \text{ as } n \to \infty.$$

Then the point $x = (I - H)^*(y)$ is well defined, $x \in S$ and

$$|x| \leq (I - L)^* |y| \leq q.$$

Moreover, the sequence

$$b_{n+1} = L(b_n) + |y|, \quad b_0 = 0$$

is well defined and

$$b_{n+1} \leq q, \quad \lim_{n \to \infty} b_n = b = (I - L)^* |y| \leq q.$$

III. CONVERGENCE ANALYSIS

Let K, L, M, M_1 be operators such that $L \in C^1(V_1 \to V)$, $K, M_1 \in L_+(V)$, $M \in C(V_1 \to V)$, and x_n be points in D. It is convenient for us to define the sequences $c_n, d_n, a_n, b_n \quad (n \geq 0)$ by

$$d_{n+1} = (L + K + M + M_1)(d_n) + L'(|x_n|)c_n, \quad d_0 = 0, \tag{4}$$

$$c_n = |x_{n+1} - x_n|, \tag{5}$$

$$a_n = (L + K + M + M_1)^n(a) \text{ for some } a \in C, \tag{6}$$

$$b_n = (L + K + M + M_1)^n(0), \tag{7}$$

and the point b by

$$b = (L + K + M + M_1)^\infty(0). \tag{8}$$

We can now state and prove the main result of this section.

Theorem 1. Let X be a Banach space with convergence structure (X, V, E) with $V = (V, C, \|\ \|_{\mathrm{v}})$, an operator $F \in C^1(D \to X)$ with $D \subseteq X$, an operator $Q \in C(D \to X)$, an operator $A(x) \in L(X)$, operators $L, L_1 \in C^1(V_1 \to V)$ with $V_1 \subseteq V$, operators $K, M_1(V_1 \to V)$ with $K, M_1 \in L_+(V)$, $M \in C(V_1 \to V)$, a point $a \in C$, and a null sequence $\{z_n\} \in D$ such that the following conditions are satisfied:

(C_6) the inclusions $U(a) \subseteq D$ and $[0, a] \subseteq V_1$ are true;

(C_7) L is order-convex on $[0, a]$ and satisfies

$$K + L'(|x| + |y|) - L'(|x|) \in B(A(x) - F'(x + y))$$

for all $x, y \in U(a)$ with $|x| + |y| \leq a$;

(C_8) L_1 is order-convex on $[0, a]$ and satisfies

$$L_1'(|x|) - L_1'(0) \in B(A(0) - A(x))$$

for all $x, y \in U(a)$ with $|x| + |y| \leq a$;

(C_9) the estimate $L_1' \leq L'$ is true;

(C$_{10}$) M satisfies the conditions

$$0 \leq (Q(x) - Q(x+y), M(|x| + |y|) - M(|x|) \in E$$

for all $x, y \in U(a)$ with $|x| + |y| \leq a$,

$$M(w_1) - M(w_2) \leq M(w_3) - M(w_4) \text{ and } M(w) \geq 0$$

for all $w, w_1, w_2, w_3, w_4 \in [0, a]$ with $w_1 \leq w_3$ and $w_2 \leq w_4$;

(C$_{11}$) M_1, x_n, z_n satisfy the inequality

$$0 \leq (A(x_n)(z_n) - A(x_{n-1})(z_{n-1}), M_1(d_n - d_{n-1})) \in E \text{ for all } n \geq 1;$$

(C$_{12}$) $L_1'(0) \in B(I - A(0))$ and $\big(-(F(0) + Q(0) + A(0)(z_0)), L(0) + M(0) + M_1(0)\big) \in E$;

(C$_{13}$) $(L + K + M + M_1)(a) \leq a$ with $0 \leq L + K + M + M_1$;

and

(C$_{14}$) $(M + M_1 + L'(a) + K)^n a \to 0$ as $n \to \infty$.

Then

(i) The sequence $(x_n, d_n) \in (X \times V)^N$ is well defined, remains in E^N, is monotone and satisfies for all $n \geq 0$

$$d_n \leq b$$

where d_n, b are given by (4) and (8) respectively and b is the smallest fixed point of $L + K + M + M_1$ in $[0, a]$.

(ii) Moreover, the iteration $\{x_n\}$ $n \geq 0$ generated by (2) converges to a solution $x^* \in U(b)$ of the equation $F(x) + Q(x) = 0$, which is unique in $U(a)$.

(iii) Furthermore, the following estimates are true for all $n \geq 0$:

$$b_n \leq d_n \leq b, \tag{9}$$

$$b_n \leq a_n, \tag{10}$$

$$|x_{n+1} - x_n| \leq d_{n+1} - d_n, \tag{11}$$

$$|x_n - x^*| \leq b - d_n \tag{12}$$

and

$$|x_n - x^*| \leq a_n - b_n \text{ for } M_1 = 0 \text{ and } z_n = 0 \quad (n \geq 0) \tag{13}$$

where d_n, a_n, b_n and b are given by (4), (6), (7) and (8) respectively.

Proof. We first note that b replacing a also satisfies the conditions of the theorem. Using conditions (C$_9$) and (C$_{10}$) we obtain

$$\begin{aligned}
|I - A(0)|(b) + |-(F(0) + Q(0) + A(0)(z_0)| &\leq L_1'(0)(b) + L(0) + M(0) + M_1(0) \\
&\leq L_1'(0)(b - 0) + L(0) + M(0) + M_1(0) \\
&\leq (L_1'(0) + M + M_1)(b - 0) + L(0) + M(0) + M_1(0) \\
&\leq L(b) - L(0) + M(b) - M(0) + M_1(b) - M_1(0) + L(0) + M(0) + M_1(0) \\
&\leq (L + K + M + M_1)(b) = b.
\end{aligned}$$

Hence, by Lemma 2, x_1 is well defined and $(x_1, b) \in E$. We also get

$$x_1 = (I - A(0))(x_1) + \big[-\big(F(0) + Q(0) + A(0)(z_0)\big)\big]$$

$\Longrightarrow$

$$|x_1| \leq L'(0)|x_1| + M(0) + L(0) + M_1(0) = d_1$$

and by the order convexity of L

$$\begin{aligned}
d_1 &= L'(0)|x_1| + (M + M_1 + L)(0) \le L'(0)b + (M + M_1 + L)(0) \\
&\le \big(L'(0) + M + M_1\big)(b - 0) + (M + M_1 + L)(0) \\
&\le L(b) - L(0) + M(b) - M(0) + M_1(b) - M_1(0) + M(0) + M_1(0) + L(0) \\
&\le (M + M_1 + L + K)(b) = b.
\end{aligned}$$

That is, we get $|x_1 - x_0| \le d_1 - d_0$ or

$$0 \le (x_0, d_0) \le (x_1, d_1).$$

We now assume that

$$0 \le (x_{k-1}, d_{k-1}) \le (x_k, d_k) \text{ for } k = 1, 2, \ldots, n \text{ and } d_k \le b.$$

We need to find a bound for $I - A(x_n)$. We will show that $L_1'(|x_n|) \in B(I - A(x_n))$.

This fact follows from (C_7), (C_{12}) and the estimate

$$\begin{aligned}
|I - A(x_n)| &\le |I - A(0)| + |A(0) - A(x_n)| \\
&\le L_1'(0) + L_1'(0 + |x_n|) - L_1'(0) = L_1'(|x_n|).
\end{aligned}$$

Using (2) we obtain the approximation

$$\begin{aligned}
-\big[F(x_n) + Q(x_n) + A(x_n)(z_n)\big] &= -F(x_n) - Q(x_n) - A(x_n)(z_n) \\
&\quad + A(x_{n-1})(x_n - x_{n-1}) + F(x_{n-1}) + Q(x_{n-1}) + A(x_{n-1})(z_{n-1}).
\end{aligned}$$

Hence, by (C_7), (C_{10}), (C_{11}) and the induction hypotheses, we obtain in turn

$$\begin{aligned}
&|-F(x_n) + F(x_{n-1}) + A(x_{n-1})(x_n - x_{n-1})| + |Q(x_{n-1}) - Q(x_n)| + |A(x_n)(z_n) - A(x_{n-1})(z_{n-1})| \\
&\le \int_0^1 [L'(|x_{n-1}| + tc_{n-1}) - L'(|x_{n-1}|) + K]\, c_{n-1}\, dt + M(|x_{n-1}| + |x_n - x_{n-1}|) - M(|x_{n-1}|) \\
&\qquad + M_1(d_n - d_{n-1}) \\
&\le L(|x_{n-1}| + c_{n-1}) - L(|x_{n-1}|) - L'(|x_{n-1}|)c_{n-1} + Kc_{n-1} + M(|x_{n-1}| + |x_n - x_{n-1}|) - M(|x_{n-1}|) \\
&\qquad + M_1(d_n - d_{n-1}) \\
&\le L(d_{n-1} + d_n - d_{n-1}) - L(d_{n-1}) - L'(|x_{n-1}|)c_{n-1} + K(d_n - d_{n-1}) \\
&\qquad + M(d_{n-1} + d_n - d_{n-1}) - M(d_{n-1}) + M_1(d_n - d_{n-1}) = (M + M_1 + L + K)(d_n) - d_n.
\end{aligned} \tag{14}$$

We can now obtain that

$$\begin{aligned}
&L_1'(|x_n|)(b - d_n) + |-(F(x_n) + Q(x_n) + A(x_n)(z_n))| + d_n \\
&\le L'(d_n)(b - d_n) + L(d_n) + M(d_n) + M_1(d_n) + K(d_n) - d_n + d_n \\
&\le (L'(d_n) + M + M_1 + K)(b - d_n) + (L + K + M + M_1)(d_n) \\
&\le K(b - d_n) + L(b) - L(d_n) + M(b) - M(d_n) + M_1(b) - M_1(d_n) + (L + K + M + M_1)(d_n) \\
&= (M + M_1 + L + K)(b) = b.
\end{aligned}$$

That is, x_{n+1} is also well defined by Lemma 2 and $c_n \le b - d_n$. Hence d_{n+1} is well defined too and we obtain:

$$\begin{aligned}
d_{n+1} &\le (M + M_1 + L + K)(d_n) + L'(d_n)(b - d_n) \le (M + M_1 + L + K)(d_n) \\
&\quad + (L'(d_n) + M + M_1)(b - d_n) \\
&\le (M + M_1 + L + K)(d_n) + L(b) - L(d_n) + M(b) - M(d_n) + M_1(b) - M_1(d_n) + K(b - d_n) \\
&= (M + M_1 + L + K)(b) = b.
\end{aligned}$$

The monotonicity $(x_n, d_n) \le (x_{n+1}, d_{n+1})$ can be derived from

$$\begin{aligned} c_n + d_n &\le L'(|x_n|)c_n + |-(F(x_n) + Q(x_n) + A(x_n)(z_n))| + d_n \\ &\le L'(|x_n|)c_n + (M + M_1 + L + K)d_n = d_{n+1}. \end{aligned}$$

The induction has now been completed. We need to show that

$$b_n \le d_n \text{ for all } n \ge 1.$$

For $n = 1$ and from the definitions of b_n, d_n

$$b_1 = (L'(0) + K + M + M_1)^1(0) \le d_1.$$

Assume that

$$b_k \le d_k \text{ for } k = 1, 2, \ldots, n.$$

Then, we obtain in turn

$$\begin{aligned} b_{n+1} = (L'(0) + K + M + M_1)^{n+1}(0) &= (L'(0) + K + M + M_1)(L'(0) + K + M + M_1)^n(0) \\ &\le (L'(0) + K + M + M_1)(d_n) \\ &\le L'(0)(d_n) + (M + K + M_1)(d_n) \\ &\le L(d_n) - L(0) + (M + K + M_1)(d_n) \\ &\le (L + K + M + M_1)d_n \le d_n \le d_{n+1}. \end{aligned}$$

Since $d_n \le b$, we have $b_n \le d_n \le b$. We also need to show that

$$0 \le a_n - b_n \le (L'(a) + K + M + M_1)^n(a) \qquad (n \ge 1). \tag{15}$$

For $n = 1$, we get

$$\begin{aligned} a_1 - b_1 &= (L + K + M + M_1)(a) - (L + K + M + M_1)(0) = (L(a) - L(0)) + (M(a) - M(0)) \\ &\quad + K(a - 0) + (M_1(a) - M_1(0)) = \int_0^1 [L'(ta) + K + M + M_1](a)\, dt \\ &\le \int_0^1 (L'(a) + K + M + M_1)(a)\, dt = (L'(a) + K + M + M_1)^1(a), \end{aligned}$$

which shows (15) for $n = 1$. Let us now assume that (15) is true for $k = 1, 2, \ldots, n$, then we can obtain in turn

$$\begin{aligned} a_{n+1} - b_{n+1} &= (L + K + M + M_1)^{n+1}(a) - (L + K + M + M_1)^n(0) \\ &= (L + K + M + M_1)\big[(L + K + M + M_1)^n(a)\big] - (L + K + M + M_1) \\ &\qquad \cdot \big[(L + K + M + M_1)^n(0)\big] \\ &= L\big[(L + K + M + M_1)^n(a)\big] - L\big[(L + K + M + M_1)^n(0)\big] \\ &\quad + (M + K + M_1)\big[(L + K + M + M_1)^n(a)\big] - (M + K + M_1)\big[(L + K + M + M_1)^n(0)\big] \\ &= \int_0^1 L'\big[(L + K + M + M_1)^n(0) + t\big((L + K + M + M_1)^n(a) - (L + K + M + M_1)^n(0)\big)\big] \\ &\qquad \cdot \big[(L + K + M + M_1)^n(a) - (L + K + M + M_1)^n(0)\big]\, dt \\ &\quad + (M + K + M_1)\big[(L + K + M + M_1)^n(a) - (L + K + M + M_1)^n(0)\big] \\ &\le \int_0^1 L'\big((L + K + M + M_1)^n(a)\big)(L'(a) + K + M + M_1)^n(a)\, dt \\ &\qquad + (M + K + M_1)\big((L'(a) + K + M + M_1)^n(a)\big) \\ &\le L'(a)(L'(a) + K + M + M_1)^n(a) + (M + K + M_1)(L'(a) + K + M + M_1)(a) \\ &= (L'(a) + K + M + M_1)(a). \end{aligned}$$

By condition (C_{14}) and the above, we deduce that the sequence $\{b_n\}$ $n \geq 0$ is Cauchy in a Banach space C, and as such, it converges to some $b = (L+K+M+M_1)^{\infty}(0)$. From $(L+K+M+M_1)(b) = (L+K+M+M_1)\big(\lim_{n\to\infty}(L+K+M+M_1)^n(0)\big) = \lim_{n\to\infty}(L+K+M+M_1)^{n+1}(0) = b$, we obtain

$$(L+K+M+M_1)(b) = b \leq a,$$

which makes b smaller than any solution of the inequality

$$(L+K+M+M_1)(p) \leq p. \qquad \text{(see also [2], [3], [5]).}$$

It also follows that the sequence $\{x_n\}$ $n \geq 0$ is Cauchy in X, and as such, it converges to some $x^* \in U(b)$. By letting $n \to \infty$ in (14) and using the hypotheses that $\lim_{n\to\infty} z_n = 0$, we deduce that x^* is a solution of the equation $F(x) + Q(x) = 0$.

To show uniqueness, let us assume that there exists another solution y^* of the equation $F(x)+Q(x) = 0$ in $U(a)$. Then, exactly as in [1], [2], [3], [5], [6], by considering the modified Newton-process

$$x_{n+1} = x_n - (F(x_n) + Q(x_n)),$$

we can show that this sequence converges, under the hypotheses of the theorem. Moreover, as above, we can easily show (see also [1], [2], [3], [5]) that

$$|y^* - x_n| \leq a_n - b_n \text{ for } M_1 = 0 \text{ and } z_n = 0 \ (n \geq 0),$$

from which follows that $x_n \to y^*$ as $n \to \infty$. Finally, estimates (12) and (13) are obtained by using standard majorization techniques [3], [4], [9].

That completes the proof of the theorem.

We will now introduce results on a posteriori estimates. It is convenient to define the operator

$$R_n(q) = (I - L'(|x_n|))^* S_n(q) + c_n$$

where

$$S_n(q) = (L+K+M+M_1)(|x_n|+q) - (L+K+M+M_1)(|x_n|) - L'(|x_n|)(q)$$

and the interval

$$I_n = [0, a - |x_n|].$$

It can easily be seen that the operators S_n are monotone on I_n. Moreover, the operators $R_n : [0, a-d_n] \to [0, a-d_n]$ are well defined and monotone. This fact follows from lemma 1 and the scheme

$$d_n + c_n \leq d_{n+1} \Rightarrow R(a) - d_{n+1} \leq a - d_n - c_n \Rightarrow S_n(a-d_n) + L'(|x_n|)(a - d_n - c_n) \leq a - d_n - c_n \quad (n \geq 0).$$

Then, exactly as in [6], we can show:

Proposition. The following implications are true:

(i) if $q \in I_n$ satisfy $R_n(q) \leq q$, then

$$c_n \leq R_n(q) = p \leq q$$

and

$$R_{n+1}(p - c_n) \leq p - c_n \text{ for all } n \geq 0;$$

(ii) under the hypotheses of Theorem 1, let $q_n \in I_n$ be a solution of $R_n(q) \leq q$, then

$$|x^* - x_m| \leq a_m \qquad (m \geq n)$$

where

$$a_n = q_n \text{ and } a_{m+1} = R_m(a_m) - c_m;$$

and

(iii) under the hypotheses of Theorem 1, any solution $q \in I_n$ of $R_n(q) \leq q$ is such that

$$|x^* - x_n| \leq R_n^{\infty}(0) \leq q.$$

Remark 1.

(a) The results obtained in Theorem 1 and the proposition reduce immediately to the corresponding ones in [6] (Theorem 5 and Lemmas 10–12) when $A(x) = F'(x_n)^{-1}$, $Q(x) = 0$ for all $x \in D$ and $z_n = 0$ for all $n \geq 0$ (M, M_1, K will be 0 then and $L = L_1$).

(b) As in [2], [3], [5], we can also show that if conditions (C_6)–(C_{12}) are satisfied and there exists $t \in (0,1)$ such that $(L + K + M + M_1)(a) \leq ta$, then there exists $a_1 \in [0, ta]$ satisfying conditions (C_6)–(C_{14}). The solution $x^* \in U(a_1)$ is unique in $U(a)$.

(c) From the approximation

$$A(x_n)(z_n) - A(x_{n-1})(z_{n-1}) = (A(x_n)(z_n) - z_n) + (-A(x_{n-1})(z_{n-1}) + z_{n-1}) + I(z_n - z_{n-1})$$

we observe that (C_{11}) will be true if

$$M_1 = 2L_0'(b) + I \text{ with } L_0 = L_1 \text{ or } L$$

and z_n are chosen so that

$$|z_n| + |z_{n-1}| + |z_n - z_{n-1}| \leq d_n - d_{n-1} \qquad (n \geq 1).$$

(d) Another choice for M_1, z_n can be $M_1 = |\varepsilon| I$ and $z_n = z_{n-1} + \varepsilon_n(x_n - x_{n-1})\,(n \geq 1)$ and $|\varepsilon_n| \leq |\varepsilon| \quad (n \geq 0)$ provided that $F'(x) = I$ for all $x \in D$. It can then easily be seen that (C_{11}) is satisfied. The sequence z_n must still be chosen to be null. At the end of this paper, in Part V, we have given examples to cover this case. Several other choices are also possible.

(e) From the proof of the theorem, it can easily be seen that condition (C_9) can be replaced by the weaker condition (which is what we actually need)

$$L_1'(d_n)(b - d_n) \leq L(b) - L(d_n) \quad (n \geq 0).$$

This condition however is not easy to verify in general.

(f) It can easily be seen that the conclusion of Theorem 1 remains valid if in the hypotheses L_1 replaces L everywhere except in condition (C_7).

(g) Condition (C_7) can be replaced by the set of conditions

$$K + L_2'(|x| + |y|) - L_2'(|x|) \in B(F'(x) - F'(x+y))$$

and

$$L_3'(|x|) - L_3'(|0|) \in B(A(x) - F'(x))$$

for all $x, y \in [0, a]$ with $|x| + |y| \leq a$ as long as we can find L such that

$$L'(|x| + |y|) = L_3'(|x|) + L_2'(|x| + |y|) + K \text{ and } L'(|x|) = L_2'(|x|) + L_3'(|0|),$$

for all $x, y \in [0, a]$ with $|x| + |y| \leq a$. The operators $L, L_2, L_3 \in C^1(V_1 \to V)$ defined here must also be order convex.

(h) It can easily be seen from the proof of theorem 1 that the results obtained in theorem 1 remain valid if (C_{11}) is replaced by the weaker condition:

$$0 \leq (A(x_n)(z_n) - A(x_{n-1})(z_{n-1}), M_2|x_n - x_{n-1}|) \in E \quad (n \geq 1)$$

for some $M_2 \in L_+(V)$. This condition seems to be more helpful since it relates z_n with x_n and not an unknown sequence depending on M_1. Under this condition the term $M_1(b - b_n)$ appearing in the Proposition can be replaced by $M_2|x_n - x_{n-1}|$ $(n \geq 1)$. Other choices involving M_2 are possible.

(i) Define the residuals $r_n = -A(x_n)(z_n)$ $(n \geq 0)$ and set $\delta_n = x_{n+1} - x_n$ $(n \geq 0)$. Then from the approximation

$$r_n = [(I - A(x_n)) - I]\,(z_n)$$

we obtain

$$|r_n| \leq (\,L_1'(|x_n|) + I\,)\,|z_n|$$

which shows that $r_n \to 0$ as $n \to \infty$ if $z_n \to 0$ as $n \to \infty$. Consequently, the conclusions of theorem 1 remain true for the system

$$x_{n+1} - x_n = \delta_n, \quad A(x_n)\delta_n = -(F(x_n) + Q(x_n)) + r_n \quad (n \geq 0).$$

(j) (C_{11}) is equivalent to the condition that $(A(x_n)(z_n), M_1(d_n)) \in E^N$ $(n \geq 1)$ is an increasing sequence. Similar observations can be made for (C_{10}).

IV. THE MONOTONE CASE

Let $A \in L(X \to X)$ be a given operator. Define the operators $P, T(D \to X)$ by

$$P(x) = AT(x + u)$$
$$T(x) = G(x) + R(x), \quad P(x) = F(x) + Q(x)$$

and

$$F(x) = AG(x + u), \quad Q(x) = AR(x + u),$$

where G, R are as F, Q respectively. We deduce immediately that under the hypotheses of Theorem 1, the zero x^* of P is a zero of AT also, if $u = 0$.

We will now provide a monotonicity result to find a zero x^* of AT. The space X is assumed to be partially ordered and satisfies the conditions for V given in (C_1)–(C_5). Moreover, we set $X = V,\ D = C^2$ so that $|\ |$ turns out to be I.

We can now state and prove the main result of this section.

Theorem 2. Let V be a partially ordered Banach space satisfying conditions (C_1)–(C_5), Y be a Banach space, $G \in C^1(D \to Y)$, $R \in C(D \to Y)$ with $D \subseteq V$, $A, A(x) \in L(X \to V)$, $K, M_1(D \to V)$ with $K, M_1 \in L_+(V)$, $M \in C(V_1 \to V)$ and $u, v \in V$ such that:

(C_{15}) $[u, v] \subseteq D$;

(C_{16}) $2I - AG'(u) - A(v - u) + M + K + M_1 \in L_+(V)$;

(C_{17}) for all $w, w_1, w_2 \in [u, v] : w_1 \leq w_2 \Rightarrow AG'(u + w) \leq I$,

$$A(w_2) \leq A(w_1)$$
$$A(w_2) \leq AG'(w_1 + u) + K$$
$$A(w_1) - A(w_2) + AG'(u + w_1) - AG'(u + w_2) \geq 0$$

and

$$-(Q(0) + A(0)(z_0)) \leq (M + K + M_1)(0);$$

(C_{18}) $AT(u) + A(u)(z_0) \leq 0$, $AT(v) + A(v)(z_0) \geq 0$ and $v - u \leq \int_0^{v-u} A(z)dz + AG(v) - (M + K + M_1)(v - u)$;

(C_{19}) condition (C_{10}) is satisfied;

(C_{20}) condition (C_{11}) is satisfied; and

(C_{21}) $(2I - AG'(v) - A(v - u) + M + K + M_1)^n(v - u) \to 0$ as $n \to \infty$.

Then the Newton sequence

$$y_0 = u, \quad y_{n+1} = y_n + (A(y_n))^*[-AT(y_n)] - z_n \qquad (n \geq 0)$$

is well defined for all $n \geq 0$, monotone and converges to a unique zero x^* of AT in $[u, v]$.

Proof. The proof follows immediately by setting $P(x) = AT(x+u)$, $L_1(e) = e - \int_0^e A(z)dz$, $L_2(e) = e - F(e)$, $L = L_1 + L_2$ and $a = v - u$.

Remark 2.

(a) By setting $R(x) = 0$ for all $x \in D$ and $z_n = 0 \quad (n \geq 0)$, we immediately deduce that our results reduce to the ones obtained in [6, thm 13].

(b) If M, R are linear operators, then condition (C_{19}) will certainly be satisfied if we set $M(e) = -Q(e)$.

(c) The operators L, L_1, L_2 can be chosen in general as $L_1(e) = z_1 + R_1(e) - \int_0^e A(z)dz$, $L_2 = z_2 + R_2(e) - F(e)$ and $L = L_1 + L_2$ with $z_1, z_2 \in [u, v]$ and $R_1, R_2 \in L(D \to Y)$. Conditions similar to the ones in Theorem 2 can then be easily produced for the above choices of L, L_1, L_2. For example, conditions (C_{16}) and (C_{21}) will become

$$R_1 + R_2 + K - AG'(u) - A(v-u) + M_1 + M \in L_+(V)$$

and

$$(R_1 + R_2 - AG'(u) - A(v-u) + M_1 + K + M)^n(v-u) \to 0 \text{ as } n \to \infty$$

respectively.

V. APPLICATIONS

We will complete this study with two examples that show how to choose L, M, M_1, z_n in practical applications. For simplicity, we assume that $A(x) = F'(x)$ for all $x \in D$.

Example 1. We discuss the case of a real Banach space with norm $\|\ \|$. Assume for simplicity that $F'(0) = I$ and there exists a monotone operator

$$f : [0, a] \to IR$$

such that

$$\|F''(x)\| \leq f(\|x\|) \text{ for all } x \in U(a)$$

and a continuous nondecreasing function g on $[0, r]$, $r \leq a$ such that

$$\|Q(x) - Q(y)\| \leq g(r)\|x - y\| \tag{16}$$

for all $x, y \in U(\frac{r}{2})$.

We showed in [3], (see also [9]) that (16) implies that

$$\|Q(x+\ell) - Q(x)\| \leq h(r + \|\ell\|) - h(r), \quad x \in U(a), \quad \|\ell\| \leq a - r, \tag{17}$$

where

$$h(r) = \int_0^r g(t)\, dt.$$

Conversely, it is not hard to see that we may assume, without loss of generality, that the function h and all functions $h(r+t) - h(r)$ are monotone in r. Hence, we may assume that $h(r)$ is convex and hence differentiable from the right. Then, as in [9], we show that (17) implies (16) and $g(r) = h'(r+0)$. Hence, we can set

$$L_1(q) = L(q) = \|F(0) + Q(0)\| + \int_0^q ds \int_0^s f(t)\, dt \tag{18}$$

and

$$M(q) = \int_0^q g(t)\, dt. \tag{19}$$

In Remark 1 (c) and (d) we have already provided some choices for M_1, z_n. Here, however, for simplicity let us choose $z_n = 0 \quad (n \geq 0)$ and $M_1 = 0$.

Then condition (C_{13}) will be true if

$$\frac{1}{2} f(a)a^2 - (1 - g(a))a + \|F(0) + Q(0)\| \leq 0. \tag{20}$$

If we set $Q = 0$ and $g = 0$, (20) is true if $\|F(0)\| f(a) \leq \frac{1}{2}$, which is a well known condition due to Kantorovich [4, Ch. 18]. If $Q \neq 0$, condition (20) is the same condition with the one found in [3], [9] for the Zincenko iteration.

In the example that follows, we show that our results can apply to solve nonlinear integral equations involving a nondifferentiable term, whereas the results obtained in [6] (or [5], [7], [8]) cannot apply.

Example 2. Let $X = V = C[0,1]$, and consider the integral equation

$$x(t) = \int_0^1 k(t, s, x(s))\, ds \text{ on } X, \tag{21}$$

where the kernel $k(t, s, x(s))$ with $(t, s) \in [0,1] \times [0,1]$ is a nondifferentiable operator on X. Consider (21) in the form (1) where $F, Q : X \to X$ are given by

$$F(x)(t) = Ix(t)$$

and

$$Q(x)(t) = -\int_0^1 k(t, s, x(s))\, ds.$$

The operator $|\ |$ is defined by considering the sup-norm. We assume that V is equipped with natural partial ordering, and there exists $\alpha,\ a \in [0, +\infty)$, and a real function $\alpha(t, s)$ such that

$$\|k(t, s, x) - k(t, s, y)\| \leq \alpha(t, s)\|x - y\|$$

for all $t, s \in [0,1],\ x, y \in U(\frac{a}{2})$, and

$$\alpha \geq \sup_{t \in [0,1]} \int_0^1 \alpha(t, s)\, ds.$$

Define the real functions h, f, g on $[0, a]$ by $h(r) = \alpha r$, $f(r) = 0$ and $g(r) = \alpha$ for all $r \in [0, a]$. By choosing L, M, M_1 as in (18) and (19) and Remark 1 (d) respectively, and by setting $K = 0$, we can easily see that the conditions (C_1)–(C_{12}) of Theorem 1 are satisfied. In particular, condition (C_{13}) becomes

$$(1 - \alpha - |\epsilon|)a - \|Q(0)\| \geq 0 \tag{22}$$

which is true in the following cases: if $0 \leq \alpha < 1 - |\epsilon|$, choose $a \geq \beta = \frac{\|Q(0)\|}{1 - \alpha - |\epsilon|}$; if $\alpha = 1 - |\epsilon|$ and $Q(0) = 0$, choose $a \geq 0$; if $\alpha > 1 - |\epsilon|$ and $Q(0) = 0$, choose $a = 0$. If in (22) strict inequality is valid, then there exists a solution a^* of equation (22) satisfying condition (C_{14}). Note that if we choose $\alpha \in (0, 1 - |\epsilon|)$, $a \in (\beta, +\infty)$ and $\epsilon \in (-1, 1)$, condition (22) is valid as a strict inequality. Finally, we remark that the results obtained in [6] (or [5], [7], [8]) cannot apply here to solve equation (21), since Q is nondifferentiable on X and the z_n's are not necessarily zero.

CONCLUSION

In this study, we used perturbed Newton-like methods to find solutions of nonlinear, nondifferentiable operator equations on Banach spaces with a generalized structure. This technique involves the introduction of a generalized norm as an operator from a linear space into a partially ordered Banach space. This way the metric properties of the examined problem have been shown to be analyzed more precisely. Convergence results and error estimates have been improved compared with the real norm theory. Moreover, this approach also allowed us to derive, on the one hand, semi-local results of Kantorovich-type and, on the other hand, global results base on monotonicity considerations, from the same general theorem. All our convergence theorems were derived from metric estimates. Furthermore, we show that special cases of our results reduce to the corresponding ones already in the literature [1], [2], [5], [6], [7]. Finally, we used our results to solve applied problems that cannot be solved with existing methods.

REFERENCES

[1] Argyros, I.K. The secant method in generalized Banach spaces, Appl. Math. Comp. 39, (1990), 111–121.

[2] ______________. Newton-like methods in generalized Banach spaces, Functiones et Approximatio Commentarii Mathematici, XXII, (1993), 13–20.

[3] Argyros, I.K. and Szidarovszky, F. The Theory and Applications of iteration methods, C.R.C. Press, Inc. Boca Raton Florida, U.S.A. 1993.

[4] Kantorovich, L.V. and Akilov, G.P. Functional Analysis in normed spaces, Academic Press, New York, 1978.

[5] Meyer, P.W. Newton's method in generalized Banach spaces, Numer. Funct. Anal. Optimiz. 9 (3 and 4), (1987), 249–259.

[6] ______________. A unifying theorem for Newton's method, Numer. Funct. Anal. Optimiz. 13 (5 and 6), (1992), 463–473.

[7] Pandian, M.C. A convergence test and component-wise error estimates for Newton-type methods, SIAM J. Numer. Anal. 22, (1985), 779–799.

[8] Vandergraft, J.A. Newton's method for convex operators in partially ordered spaces, SIAM J. Numer. Anal. 4 (1967), 406–432.

[9] Zabrejko, P.P. and Nguen, D.F. The majorant method in the theory of Newton-Kantorovich approximations and the Ptak error estimates, Numer. Funct. Anal. Optimiz. 9 (5 and 6), (1987), 671–684.

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu
ISSN 1083-0464
Vol. 01, 1995, pp. 13–16.
Submitted: March 16, 1995. Published: June 30, 1995.

DIFFERENTIATED SHIFT-INVARIANT MULTIVARIATE INTEGRAL OPERATORS

George A. Anastassiou

Abstract. In a recent paper [1] the author, together with H. Gonska, introduced some wavelet-type, very general multivariate integral operators over $\mathbb{R}^d$, $d \geq 1$, and studied their various properties such as shift-invariance, global smoothness preservation and optimality, convergence to the unit, differentiation, and preservation of continuous multivariate probability distribution functions. These operators are introduced through a convolution-like iteration of another very general multivariate operator with a general multivariate scaling type function. In this paper the author gives sufficient conditions, so that the *partial derivatives of the above described multivariate operators* possess most of the above listed nice properties of their originals. In particular, a sufficient condition is given so that a corresponding "global smoothness preservation" sharp multivariate inequality is attained. The corresponding convergence of differentiated operators to the differentiated unit operator is given with rates of convergence.

(1991) A.M.S. (MOS) Subject Classification Codes. 60, 62, 28.

Key Words and Phrases. Wavelet integral operators, Probability distribution functions.

Main Results.

Let $X := C_U(\mathbb{R}^d)$, $d \geq 1$ be the space of uniformly continuous functions from $\mathbb{R}^d$ into $\mathbb{R}$.

For $f \in X$, define the first order modulus of continuity of f by

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in\mathbb{R}^d \\ \|x-y\|\leq\delta}} |f(x)-f(y)|, \delta > 0,$$

where $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^d$. It is known [1] that $\omega_1(f,\delta) < \infty$, $\forall\delta > 0$. Here $C(\mathbb{R}^d)$ denotes the space of continuous functions from $\mathbb{R}^d$ into $\mathbb{R}$. Let $\{\ell_k\}_{k\in\mathbb{Z}}$ be a sequence of linear operators from $C(\mathbb{R}^d)$ into itself such that

$$\ell_k(f,x) = \ell_0(f(2^{-k}\cdot),x), \quad \text{all } x \in \mathbb{R}^d,\ f \in C(\mathbb{R}^d). \tag{1}$$

Assume that

$$\ell_0 : C^{(\rho)}(\mathbb{R}^d) \to C^{(\rho)}(\mathbb{R}^d), \quad \rho \in \mathbb{N} \quad \text{be fixed}, \tag{2}$$

where $C^{(\rho)}(\mathbb{R}^d)$ is the space of ρ-times continuously differentiable functions.

In this article we consider *only* $f \in C^{(\rho)}(\mathbb{R}^d)$ such that $\frac{\partial^\rho f}{\partial x_{i_1}^{j_1}\cdots\partial x_{i_m}^{j_m}} \in X$, where fixed $i_1,\dots,i_m \in \{1,\dots,d\}$ be such that $i_1 < \cdots < i_m$, and fixed $j_1,\dots,j_m \in \mathbb{N}$ be such that $\sum_{r=1}^m j_r = \rho$.

From now on we will denote by

$$f^\partial := \frac{\partial^\rho f}{\partial x_{i_1}^{j_1}\cdots\partial x_{i_m}^{j_m}}, \quad f \in C^{(\rho)}(\mathbb{R}^d), \tag{3}$$

Department of Mathematics, Memphis State University, Memphis, TN
Email: anastasg@ kermes.msci.memst.edu

where the mixed partial is defined as above.

Let φ be a bounded real valued function of compact support $\subseteq \times_{i=1}^{d}[-a_i, a_i]$, $a_i > 0$. We assume that $\varphi \geq 0$, φ is Lebesgue measurable and

$$\underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{d-\text{fold}} \varphi(x_1 - u_1, x_2 - u_2, \ldots, x_d - u_d)\, du_1 \ldots du_d = 1,$$

$$\text{all } (x_1, \ldots, x_d) \in \mathbb{R}^d \tag{4}$$

((4) same as

$$\int_{\mathbb{R}^d} \varphi(x - u)\, du = 1, \text{all } x \in \mathbb{R}^d;$$

where $x = (x_1, \ldots, x_d)$, $u = (u_1, \ldots, u_d)$). One can easily prove that

$$\int_{\mathbb{R}^d} \varphi(u) du = 1. \tag{5}$$

Example. For $i = 1, \ldots, d$ take the characteristic function

$$\varphi_i(x_i) := \chi_{[-\frac{1}{2}, \frac{1}{2})}(x_i) = \begin{cases} 1, & x_i \in \left[-\frac{1}{2}, \frac{1}{2}\right) \\ 0, & \text{else.} \end{cases}$$

Define

$$\varphi^*(x) := \prod_{i=1}^{d} \varphi_i(x_i), \quad \text{all } x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Note that φ^* is bounded, supp $\varphi^* \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]^d$, $\varphi^* \geq 0$ and φ^* is Lebesgue measurable. Also

$$\underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{d-\text{fold}} \prod_{i=1}^{d} \varphi_i(x_i - u_i)\, du_1 \ldots du_d = 1,$$

i.e.,

$$\int_{\mathbb{R}^d} \varphi^*(x - u) du = 1, \quad \text{all } x \in \mathbb{R}^d.$$

Let $\{\mathcal{L}_k\}_{k \in \mathbb{Z}}$ be the sequence of linear operators acting on $C^{(\rho)}(\mathbb{R}^d)$ and defined as follows

$$\mathcal{L}_0(f, x) := \int_{\mathbb{R}^d} \ell_0(f, u) \varphi(x - u)\, du \tag{6}$$

and

$$\mathcal{L}_k(f, x) := \mathcal{L}_0(f(2^{-k} \cdot); 2^k x), \tag{7}$$

all $x \in \mathbb{R}^d$, $k \in \mathbb{Z}$. I.e.,

$$\mathcal{L}_k(f, x) = \int_{\mathbb{R}^d} \ell_k(f, u) \varphi(2^k x - u)\, du. \tag{8}$$

For our convenience we shall use the notation

$$\int_{-\vec{a}}^{\vec{a}} \bullet = \int_{\times_{i=1}^{d}[-a_i, a_i]} \bullet. \tag{9}$$

Furthermore, we obtain that

$$\begin{aligned} \mathcal{L}_k^{\partial}(f, x) &= 2^{k\rho} \cdot \int_{-\vec{a}}^{\vec{a}} \ell_k^{\theta}(f, 2^k x - u) \varphi(u)\, du \\ &= 2^{k\rho} \cdot \int_{\mathbb{R}^d} \ell_k^{\partial}(f, 2^k x - u) \varphi(u)\, du \\ &= 2^{k\rho} \cdot \int_{\mathbb{R}^d} \ell_k^{\partial}(f, u) \varphi(2^k x - u)\, du, \text{all } x \in \mathbb{R}^d, \ k \in \mathbb{Z}. \end{aligned} \tag{10}$$

Assumption. Take $f \in C^{(\rho)}(\mathbb{R}^d)$, $\rho \in \mathbb{N}$ fixed, such that $f^\partial \in X$. For fixed $\alpha > 0$ we assume that

$$\sup_{\substack{u,y \in \mathbb{R}^d \\ \|u-y\|_\infty \le \alpha}} |\ell_0^\partial(f,u) - f^\partial(y)| \le \omega_{1,\infty}\left(f^\partial, \frac{m\alpha+n}{2^r}\right), \tag{11}$$

is true for f as above, $m \in \mathbb{N}$, $n \in \mathbb{Z}_+$, $r \in \mathbb{Z}$, where $\omega_{1,\infty}$ is the modulus of continuity ω_1 defined with respect to $\|\cdot\|_\infty$.

Here we present our results without proofs due to their length. We intend to show proofs elsewhere.

Definition 1. Let $f_\alpha(\cdot) := f(\cdot + \alpha)$, $\alpha \in \mathbb{R}^d$ and ϕ be an operator. If $\phi(f_\alpha) = (\phi f)_\alpha$, then ϕ is called a *shift invariant operator.*

Proposition 1. *Assume that*

$$\ell_0^\partial\left(f(2^{-k}\cdot + \alpha); 2^k u\right) = \ell_0^\partial\left(f(2^{-k}\cdot); 2^k(u+\alpha)\right), \tag{12}$$

all $k \in \mathbb{Z}$, $\alpha \in \mathbb{R}^d$ fixed, all $u \in \mathbb{R}^d$; any $f \in C^{(\rho)}(\mathbb{R}^d)$; $\rho \in \mathbb{N}$ be given. Then $\mathcal{L}_k^\partial$, any $k \in \mathbb{Z}$ is a shift invariant operator.

Next we establish the property of "*global smoothness preservation*" of operators $\mathcal{L}_k^\partial$, $k \in \mathbb{Z}$.

Theorem 1. *For any $f \in C^{(\rho)}(\mathbb{R}^d)$, such that $f^\partial \in X$, $\rho \in \mathbb{N}$ fixed, and any $u \in \mathbb{R}^d$ assume that*

$$|\ell_0^\partial(f, x-u) - \ell_0^\partial(f, y-u)| \le \omega_1(f^\partial, \|x-y\|), \quad \textit{all } x, y \in \mathbb{R}^d, \tag{13}$$

where $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^d$. Then

$$\omega_1((\mathcal{L}_k f)^\partial, \delta) \le \omega_1(f^\partial, \delta), \tag{14}$$

any $\delta > 0$.

Optimality of (14) follows.

Theorem 2. *Let $i_1, \ldots, i_m \in \{1, \ldots, d\}$ be such that $i_1 < \cdots < i_m$, and $j_1, \ldots, j_m \in \mathbb{N}$ be such that $\sum_{r=1}^m j_r = \rho$. Consider $g_{j_1}(x) := \frac{x^{j_1+1}}{(j_1+1)!}$, and $g_{j_r}(x) := \frac{x^{j_r}}{j_r!}$, $r = 2, \ldots, m$; all $x \in \mathbb{R}$. Denote by $f^\partial := \frac{\partial^\rho f}{\partial x_{i_1}^{j_1} \ldots \partial x_{i_m}^{j_m}}$, for any $f \in C^{(\rho)}(\mathbb{R}^d)$. Denote also by $pr_{i_r} : \mathbb{R}^d \ni (x_1, \ldots, x_d) \to x_{i_r}$, $r = 1, \ldots, m$ the projection onto the i_r coordinate. For all $x, y, u \in \mathbb{R}^d$ assume that*

$$\begin{aligned}
&\left(\ell_0\left(\left(\sum_{r=1}^m g_{j_r} \circ pr_{i_r}\right)(2^{-k}\cdot)\right)\right)^\partial (2^k x - u) \\
&- \left(\ell_0\left(\left(\prod_{r=1}^m g_{j_r} \circ pr_{i_r}\right)(2^{-k}\cdot)\right)\right)^\partial (2^k y - u) = 2^{-k\rho} \cdot (x_{i_1} - y_{i_1}).
\end{aligned} \tag{15}$$

Then

$$\omega_1\left(\left(\mathcal{L}_k\left(\prod_{r=1}^m g_{j_r} \circ pr_{i_r}\right)\right)^\partial; \delta\right) = \omega_1\left(\left(\prod_{r=1}^m g_{j_r} \circ pr_{i_r}\right)^\partial; \delta\right), \tag{16}$$

for any $\delta > 0$; $k \in \mathbb{Z}$. That is establishing that (14) is sharp!

The convergence of $\frac{\partial^\rho \mathcal{L}_k}{\partial x_{i_1}^{j_1} \ldots \partial x_{i_m}^{j_m}} (\sum_{r=1}^m j_r = \rho)$ to $\frac{\partial^\rho}{\partial x_{i_1}^{j_1} \ldots \partial x_{i_m}^{j_m}}$ operator, as $k \to +\infty$, with rates given in the following.

Theorem 3. *For $f \in C^{(\rho)}(\mathbb{R}^d)$, $d \ge 1$, such that $f^\partial \in C_U(\mathbb{R}^d)$, under the assumption* (11)*, it holds*

$$|\mathcal{L}_k^\partial(f, x) - f^\partial(x)| \le \omega_{1,\infty}\left(f^\partial, \frac{ma+n}{2^{k+r}}\right), \tag{17}$$

where $m \in \mathbb{N}$, $n \in \mathbb{Z}_+$, $k, r \in \mathbb{Z}$, and $a := \max(a_i)$, $i = 1, \ldots, d$.

Again, due to their length we intend to present applications of the above results elsewhere.

References

1. G.A. Anastassiou and H.H. Gonska, *On some shift-invariant integral operators, multivariate case*, Proceedings of International Conference in Approximation Theory and Probability, Santa Barbara, California, May 20-22, 1993 (1994), Plenum, New York, 41–64.
2. H. Bauer, *Maß-und Integrationstheorie* (1990), de Gruyter, Berlin.
3. C. Cottin and H.H. Gonska, *Simultaneous approximation and global smoothness preservation*, Proceedings of the Second International Conference in Functional Analysis and Approximation Theory Acquafredda di Maratea-Potenza-Italy, 1992. *Supplemento ai Rendiconti del Circolo Matematico di Palermo*, Serie II, Numero 33, (1993), 259–279..
4. A.N. Shiryayev, *Probability*, Springer-Verlag, New York, Berlin, 1984.

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Vol. No. 01, 1995, pp. 17–26
Submitted: June 11, 1995. Published: September 5, 1995.

STOCHASTIC CALCULUS AND INITIAL VALUE ANALYSIS ON THE ONE DIMENSIONAL DIFFUSIONS

Dražen Pantić

Abstract. Stochastic calculus and initial value analysis are used for complete description of distribution type of diffusion processes with Lipschitz coefficients. Sufficient conditions so that the solutions of stochastic differential equations posses absolutely continuous one dimensional distribution are given. For stochastic differential equations with uniformly elliptic coefficients probability density is investigated in detail. Also, the distribution of inverse process is given.

(1991) A.M.S. Subject Classification Codes. 60H07, 60H10.

Key Words and Phrases. Malliavin calculus, solutions of stochastic differential equations, point distributions.

INTRODUCTION

Let $\{X_{t,x}, t \geq 0\}, x \in \mathbb{R}$, be solution of SDE

$$X_{t,x} - x = \int_0^t a(X_{s,x})dB_s + \int_0^t b(X_{s,x})ds \tag{1}$$

where coefficients a and b have bounded and continuous first order derivatives $(a, b \in \mathbb{C}_b^1)$. Under those conditions for $p > 1$ there exists L_p derivative $DX_{t,x}$ of $X_{t,x}$ w.r.t. initial condition x, so that, as in [2, §8],

$$DX_{t,x} - 1 = \int_0^t a'(X_{s,x})DX_{s,x}dB_s + \int_0^t b'(X_{s,x})DX_{s,x}\, ds. \tag{2}$$

Using Ito formula, $DX_{t,x}$ could be computed explicitly as

$$DX_{t,x} = \exp\Big\{\int_0^t a'(X_{s,x})dB_s + \int_0^t (b'(X_{s,x}) - \frac{1}{2}a'(X_{s,x}))^2 ds\Big\}. \tag{3}$$

The boundeness of derivatives implies that, for $p > 1$, $DX_{t,x}$ as well as its inverse $(DX_{t,x})^{-1}$ are in L_p, [6], while $(DX_{t,x})^{-1}$ satisfies following SDE, [10, Lemma (3.7)]:

$$(DX_{t,x})^{-1} - 1 = -\int_0^t a'(X_{s,x})(DX_{s,x})^{-1}dB_s + \int_0^t \big(a'(X_{s,x})^2 - b'(X_{s,x})\big)(DX_{s,x})^{-1}\, ds.$$

Department of Mathematics, University of Belgrade, Studentski trg 16, 11000 Beograd, Yugoslavia
E-mail Address: epantidr@ubbg.etf.bg.ac.yu

Also, for a Skorohod integrable process $\boldsymbol{u} = \{u_s, 0 \leq s \leq T\}$ a directional derivative $D_u X_{t,x}$ of functional $X_{t,x}$ could be expressed in terms of derivatives $\{DX_{t,x}, t \geq 0\}$ in the following way, [12],

$$D_u X_{t,x} = DX_{t,x} \int_0^t (DX_{s,x})^{-1} a(X_{s,x})\, u_s\, ds. \tag{4}$$

First we shall treat the general case. Defining process $\underline{\boldsymbol{u}} = \{\underline{u}_s, 0 \leq s \leq t\}$ as

$$\underline{u}_s = a(X_{s,x}) DX_{s,x}$$

we have

$$D_{\underline{u}} X_{t,x} = DX_{t,x} \int_0^t a(X_{s,x})^2\, ds. \tag{5}$$

Process $\underline{\boldsymbol{u}}$ is Ito integrable, so for a continuously differentiable function with compact support f it follows, by integration by parts formula, [5], and relation (5), that

$$\mathbb{E}\{f'(X_{t,x}) DX_{t,x} \int_0^t a(X_{s,x})^2 ds\} = \mathbb{E}\{D_{\underline{\boldsymbol{u}}}(f(X_{t,x}))\} = \mathbb{E}\{f(X_{t,x}) \int_0^t a(X_{s,x}) DX_{s,x} dB_s\}. \tag{6}$$

Knowing that $DX_{t,x}$ is a.s. positive, we have to find conditions so that $\int_0^t a(X_{s,x})^2 ds$ is a.s. positive in order to apply next Lemma.

Lemma 1. *Let X be a random variable such that there exists integrable and almost sure positive random variable Y and a constant $C > 0$ so that for each $f \in \mathbb{C}_b^1$ we have*

$$\mid \mathbb{E}\{f'(X) \cdot Y\} \mid \leq C \sup\{\mid f(x) \mid : x \in \mathbb{R}\}.$$

Then probability law generated by the random variable X is absolutely continuous with respect to Lebesgue measure.

Proof. Let us define positive bounded linear functional on the space of the continuously differentiable functions with compact support with

$$L(f) = \mathbb{E}\{f(X)Y\} = \mathbb{E}\{f(X)\mathbb{E}\{Y \mid X\}\} = \int f(x)\mathbb{E}\{Y \mid X = x\} G(dx)$$

where G is probability measure generated by X. Then, we have

$$\mid \int f'(x)\mathbb{E}\{Y \mid X = x\} G(dx) \mid \leq C \sup\{\mid f(x) \mid : x \in \mathbb{R}\}$$

so using [4, §1] it follows that there exists density g so that

$$\int f(x)\mathbb{E}\{Y \mid X = x\} dG(x) = \int f(x) g(x)\, dx.$$

Clearly, the previous relation could be extended to the space of bounded measurable functions, so for a Borel set B we have

$$\mathbb{E}\{\mathbb{I}\{X \in B\} Y\} = \int \mathbb{I}\{x \in B\} g(x)\, dx.$$

If B is of Lebesgue measure zero then

$$\mathbb{E}\{\mathbb{I}\{X \in B\} Y\} = 0.$$

Knowing that $Y > 0$ almost sure one easily gets that $\mathbb{P}\{X \in B\} = 0$, i.e. probability law generated by the random variable X is absolutely continuous with respect to Lebesgue measure. □

GENERAL CASE

In the next theorem we shall completely describe the first moment τ from which $\int_0^t a(X_{s,x})^2 ds$ is a.s. positive, as well as the behavior of diffusion process (1) up to that moment.

Theorem 1. *Let $\{X_{t,x}, t \geq 0\}, x \in \mathbb{R}$, be solution of SDE (1) with coefficients in $\mathbb{C}_1^b$, and $\{Y_{t,x}, t \geq 0\}$ deterministic process defined by*

$$Y_{t,x} - x = \int_0^t b(Y_{s,x})\, ds.$$

Define

$$\tau_x = \inf\{t \geq 0 : | a(X_{t,x}) | > 0\} \text{ and } D_x = \inf\{t \geq 0 : | a(Y_{t,x}) | > 0\}.$$

Then, with probability one, $\tau_x = D_x$ and $X_{t,x} = Y_{t,x}$ for $t \leq D_x$.

Proof. Assume first that $0 < D_x < \infty$, and let $0 < t < D_x$. Then

$$X_{t,x} - Y_{t,x} = \int_0^t \big(a(X_{s,x}) - a(Y_{s,x})\big)\, dB_s + \int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big) ds, \quad \text{and}$$

$$\begin{aligned}
&\mathbb{E}\{(X_{t,x} - Y_{t,x})^2\}^{1/2} \leq \\
&\leq \mathbb{E}\Big\{\Big(\int_0^t \big(a(X_{s,x}) - a(Y_{s,x})\big) dB_s\Big)^2\Big\}^{1/2} + \mathbb{E}\Big\{\Big(\int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big) ds\Big)^2\Big\}^{1/2} \leq \\
&\leq \mathbb{E}\Big\{\int_0^t \big(a(X_{s,x}) - a(Y_{s,x})\big)^2 ds\Big\}^{1/2} + \mathbb{E}\Big\{t \int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big)^2 ds\Big\}^{1/2} \leq \\
&\leq K\Big(\int_0^t \mathbb{E}\{(X_{s,x} - Y_{s,x})^2\} ds\Big)^{1/2} + t^{1/2} K\Big(\int_0^t \mathbb{E}\{(X_{s,x} - Y_{s,x})^2\} ds\Big)^{1/2} \leq \\
&\leq K(1 + D_x^{1/2})\Big(\int_0^t \mathbb{E}\{(X_{s,x} - Y_{s,x})^2\} ds\Big)^{1/2},
\end{aligned}$$

where K is upper bound for first derivatives of a and b. Then, using Gronwall inequality, we can conclude that with probability one $X \equiv Y$ on $[0, D_x]$, and hence $D_x \leq \tau_x$ a.s.

Assume now that $0 < \tau_x < \infty$, and let $0 < t < \tau_x$. As τ_x is optional time the set

$$A = \{\omega : t < \tau_x(\omega)\}$$

is $\mathcal{F}_t$ measurable, where $\{\mathcal{F}_t, t \geq 0\}$ is standard filtration generated by $\{B_t, t \geq 0\}$. Then we have

$$\mathbb{E}\{\int_0^t a(X_{s,x})^2 ds\ \mathbb{I}_A\} = 0.$$

It is obvious that

$$\Big(\int_0^t a(X_{s,x})\, dB_s\Big)^2 \mathbb{I}\{t < \tau_x\} \leq \Big(\int_0^{t\wedge\tau_x} a(X_{s,x})\, dB_s\Big)^2.$$

Having in mind that $t \wedge \tau_x$ is optional time we have

$$\mathbb{E}\Big\{\Big(\int_0^t a(X_{s,x})\, dB_s\Big)^2 \mathbb{I}\{t < \tau_x\}\Big\} \leq \mathbb{E}\Big\{\Big(\int_0^{t\wedge\tau_x} a(X_{s,x})\, dB_s\Big)^2\Big\} = \mathbb{E}\Big\{\int_0^{t\wedge\tau_x} a(X_{s,x})^2\, ds\Big\} = 0,$$

and consequently $\int_0^t a(X_{s,x})dB_s = 0$ a.s. on set A. Hence for each $t > 0$ on the set A we have

$$\begin{aligned}
X_{t,x} - Y_{t,x} &= \int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big)\, ds, \qquad \text{and} \\
\mathbb{E}\{(X_{t,x} - Y_{t,x})^2\, \mathbb{I}\{t < \tau_x\}\} &= \\
&= \mathbb{E}\Big\{ \Big(\int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big)\, ds\Big)^2\, \mathbb{I}\{t < \tau_x\}\Big\} \leq \\
&\leq t\,\mathbb{E}\Big\{ \int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big)^2 ds\, \mathbb{I}\{t < \tau_x\}\Big\} \leq \\
&\leq t\,\mathbb{E}\Big\{ \int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big)^2\, \mathbb{I}\{s < \tau_x\}\, ds\Big\} \leq \\
&\leq tK^2 \int_0^t \mathbb{E}\{\, (X_{s,x} - Y_{s,x})^2\, \mathbb{I}\{s < \tau_x\}\}\, ds
\end{aligned}$$

Let us mark $\Delta_t = \mathbb{E}\{(X_{t,x} - Y_{t,x})^2 \mathbb{I}\{t < \tau_x\}\}$. Then previous relation could be written as

$$\Delta_t \leq tK^2 \int_0^t \Delta_s\, ds,$$

from where it follows that $\Delta_t = 0$ for each $t > 0$. Hence we can conclude that for each $t > 0$

$$\mathbb{P}\{\omega : t < \tau_x(\omega), X_{t,x} \neq Y_{t,x}\} = 0.$$

Using the continuity of paths we have that

$$\mathbb{P}\{\omega : (\exists t > 0)(t < \tau_x(\omega), X_{t,x} \neq Y_{t,x})\} = 0,$$

so with probability one $X \equiv Y$ on $[0, \tau_x]$, i.e. $\tau_x \leq D_x$ a.s. and the proof is complete. $\square$

The following theorem completely describes the type of one dimensional distributions of the solutions of SDE :

Theorem 2. *Under conditions from* Theorem 1. *the distribution of random variable $X_{t,x}$ is either degenerate or absolutely continuous, whether $t \leq D_x$ or $t > D_x$.*

Proof. The first part when $t \leq D_x$ is described in Theorem 1 so let $t > D_x$. Then $\int_0^t a(X_{s,x})^2 ds$ is positive a.s. and from relation (6) and Lemma 1 we can reach our conclusion. $\square$

UNIFORMLY ELLIPTIC CASE

Now, we shall treat an important special case when that coefficient a is uniformly elliptic in sense that

$$\inf\{\mid a(x) \mid : x \in \mathbb{R}\} > 0. \tag{EL}$$

Theorem 3. *Let the diffusion process $\{X_{t,x}, t \geq 0\}$ defined by SDE* (1) *has continuously differentiable coefficients a and b with bounded derivatives and let*

$$\inf\{|a(x)| : x \in \mathbb{R}\} > 0.$$

Then we have :

1° *for each $t > 0$ and $x \in \mathbb{R}$ the probability law of random variable $X_{t,x}$ is absolutely continuous w.r.t. Lebesgue measure with transition density $p_t(x, \cdot)$;*

2° *for any bounded, measurable function* f *and* $t > 0$ *the mapping* $x \to \mathbb{E}\{f(X_{t,x})\}$ *is continuously differentiable and*

$$\frac{\partial}{\partial x}\mathbb{E}\{f(X_{t,x})\} = t^{-1}\,\mathbb{E}\{f(X_{t,x})\int_0^t a(X_{s,x})^{-1}DX_{s,x}dB_s\}; \tag{7}$$

3° *for each* $y \in \mathbb{R}$ *and* $t > 0$ *transition density* $p_t(x,y)$ *is differentiable w.r.t. initial condition* x *and*

$$\frac{\partial}{\partial x}p_t(x,y) = t^{-1}\,\mathbb{E}\{\int_0^t a(X_{s,x})^{-1}DX_{s,x}dB_s|X_{t,x}=y\}\,p_t(x,y). \tag{8}$$

Proof. 1° Taking $\tilde{u}_s = a(X_{s,x})^{-1}DX_{s,x}$, one has, from (4)

$$D_{\tilde{u}}X_{t,x} = DX_{t,x}\,t. \tag{9}$$

Process $\{\tilde{u}\}$ is Ito integrable, so for a continuously differentiable function with compact support f it follows by integration by parts formula and relation (9), that

$$\mathbb{E}\{f'(X_{t,x})DX_{t,x}\,t\} = \mathbb{E}\{D_{\tilde{u}}(f(X_{t,x}))\} = \mathbb{E}\{f(X_{t,x})\int_0^t a(X_{s,x})^{-1}DX_{s,x}dB_s\}. \tag{10}$$

Now, $DX_{t,x}$ is a.s. positive random variable and its inverse $(DX_{t,x})^{-1}$ is in L_p for each $p > 1$. Also the assumptions we have made on coefficient a assure, via a Burgholder theorem, that $\int_0^t a(X_{s,x})^{-1}DX_{s,x}dB_s$ is in L_2, hence by Lemma 1 assertion 1° holds.

2° First, take $f \in \mathbb{C}_b^1$. Then by [2] and relation (10) one has

$$\frac{\partial}{\partial x}\mathbb{E}\{f(X_{t,x})\} = \mathbb{E}\{f'(X_{t,x})DX_{t,x})\} = t^{-1}\mathbb{E}\{f(X_{t,x})\int_0^t a(X_{s,x})^{-1}DX_{s,x}dB_s\}.$$

Now, for $x, y \in \mathbb{R}$ we have

$$\mathbb{E}\{f(X_{t,y})\} - \mathbb{E}\{f(X_{t,x})\} = \int_x^y \mathbb{E}\{f(X_{t,z})\int_0^t a(X_{s,z})^{-1}DX_{s,z}\,dB_s\}\,\frac{dz}{t}. \tag{11}$$

Both sides of relation (11) are continuous linear functionals on the set $\mathcal{M}_b$ of measurable and bounded functions, so we can conclude that (11) is valid for each $f \in \mathcal{M}_b$. Also, for $f \in \mathcal{M}_b$ and $t > 0$ mapping $x \to \mathbb{E}\{f(X_{t,x})\}$ is absolutely continuous.

If we define semigroup $\{P_t, t \geq 0\}$ of operators on the space $\mathcal{M}_b$ of bounded measurable functions as $P_tf(x) = \mathbb{E}\{f(X_{t,x})\}$, then Markov property of diffusion $\{X_{t,x}\}$ yields, for $\delta > 0$

$$P_{t+\delta}f(x) = \mathbb{E}\{P_\delta f(X_{t,x})\}.$$

Substituting t with $t+\delta$ and f with $P_\delta f$, relation (11) reads as

$$P_{t+\delta}f(y) - P_{t+\delta}f(x) = \int_x^y \mathbb{E}\{P_\delta f(X_{t,z})\int_0^t a(X_{s,z})^{-1}DX_{s,z}\,dB_s\}\,\frac{dz}{t}.$$

Then, from continuity of the function $P_\delta f(x)$ it follows that for each $x \in \mathbb{R}$ there exists

$$\lim_{y\to x}\frac{P_{t+\delta}f(y) - P_{t+\delta}f(x)}{y-x}, \quad \text{and}$$

$$\frac{\partial}{\partial x}P_{t+\delta}f(x) = t^{-1}\,\mathbb{E}\{P_\delta(X_{t,x})\int_0^t a(X_{s,x})^{-1}DX_{s,x}dB_s\}$$

hence $\frac{\partial}{\partial x}P_{t+\delta}f(x)$ is continuous function in x.
Now relation (11) yields that, for each x

$$\frac{\partial}{\partial x}P_t f(x) = t^{-1}\,\mathbb{E}\{f(X_{t,x})\int_0^t a(X_{s,x})^{-1}DX_{s,x}\,dB_s\}.$$

3° Let us mark

$$W_{t,x} = t^{-1}\int_0^t a(X_{s,x})^{-1}DX_{s,x}dB_s. \tag{12}$$

Then relation (10) reads as

$$\mathbb{E}\{f'(X_{t,x})DX_{t,x}\} = \mathbb{E}\{f(X_{t,x})W_{t,x}\},$$

or, having in mind 1°,

$$\int_{-\infty}^{\infty} f'(u)\,\mathbb{E}\{DX_{t,x}|X_{t,x}=u\}p_t(x,u)\,du = \int_{-\infty}^{\infty} f(u)\,\mathbb{E}\{W_{t,x}|X_{t,x}=u\}p_t(x,u)\,du.$$

Taking, for $\epsilon>0$ and $y\in\mathbb{R}$,

$$f_\epsilon(x) = \int_{-\infty}^{x} e^{-(t-y)^2/2\epsilon}\frac{dt}{\sqrt{2\pi\epsilon}},$$

we get, using partial integration

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-(u-y)^2/2\epsilon}\,\mathbb{E}\{DX_{t,x}|X_{t,x}=u\}p_t(x,u)\,\frac{du}{\sqrt{2\pi\epsilon}} &=\\ = \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{u} e^{-(t-y)^2/2\epsilon}\frac{dt}{\sqrt{2\pi\epsilon}}\Big)\mathbb{E}\{W_{t,x}|X_{t,x}=u\}p_t(x,u)\,du &=\\ = -\int_{-\infty}^{\infty} e^{-(t-y)^2/2\epsilon}\Big(\int_{-\infty}^{t}\mathbb{E}\{W_{t,x}|X_{t,x}=u\}p_t(x,u)\,du\Big)\frac{dt}{\sqrt{2\pi\epsilon}}&.\end{aligned} \tag{13}$$

¿From relation (11) follows, via [3, Theorem 1.28.], that $\mathbb{E}\{DX_{t,x}|X_{t,x}=y\}p_t(x,y)$ is continuous and bounded function so letting $\epsilon\downarrow 0$ one can conclude that

$$\mathbb{E}\{DX_{t,x}|X_{t,x}=y\}p_t(x,y) = -\int_{-\infty}^{y}\mathbb{E}\{W_{t,x}|X_{t,x}=u\}p_t(x,u)du = -\mathbb{E}\{W_{t,x}\mathbb{I}\{X_{t,x}\le y\}\}. \tag{14}$$

It is obvious that there exist

$$\lim_{y\to\pm\infty}\mathbb{E}\{DX_{t,x}|X_{t,x}=y\}p_t(x,y)$$

and as $\mathbb{E}\{DX_{t,x}|X_{t,x}=\cdot\}p_t(x,\cdot)$ is nonnegative, integrable function it follows that

$$\lim_{y\to\pm\infty}\mathbb{E}\{DX_{t,x}|X_{t,x}=y\}p_t(x,y) = 0.$$

Relation (7) implies that

$$\mathbb{E}\{W_{t,x}\mathbb{I}\{X_{t,x}\le y\}\} = \frac{\partial}{\partial x}\mathbb{E}\{\mathbb{I}\{X_{t,x}\le y\}\} = \frac{\partial}{\partial x}\mathbb{P}\{X_{t,x}\le y\},$$

so relation (14) read as

$$\mathbb{E}\{DX_{t,x}|X_{t,x}=y\}p_t(x,y) = -\frac{\partial}{\partial x}\mathbb{P}\{X_{t,x}\le y\}. \tag{15}$$

Now, as $\mathbb{P}\{X_{t,x} \le y\}$ is continuously differentiable in x, we have for $x_1 < x_2$

$$\begin{aligned}\mathbb{P}\{X_{t,x_2} \le y\} - \mathbb{P}\{X_{t,x_1} \le y\} &= -\int_{x_1}^{x_2} \mathbb{E}\{DX_{t,z}|X_{t,z} = y\}p_t(z,y)\,dz \\ &= \int_{x_1}^{x_2}\int_{-\infty}^{y} \mathbb{E}\{W_{t,z}|X_{t,z} = u\}p_t(z,u)\,dudz,\end{aligned}$$

and by Fubini theorem for almost all u function $\mathbb{E}\{W_{t,\cdot}|X_{t,\cdot} = u\}p_t(\cdot,u)$ is integrable and

$$\int_{x_1}^{x_2}\int_{-\infty}^{y} \mathbb{E}\{W_{t,z}|X_{t,z} = u\}p_t(z,u)\,dudz = \int_{-\infty}^{y}\int_{x_1}^{x_2} \mathbb{E}\{W_{t,z}|X_{t,z} = u\}p_t(z,u)\,dzdu.$$

Finally, as $\mathbb{P}\{X_{t,x} \le y\} = \int_{-\infty}^{y} p_t(x,u)\,du$ we have that, for each $y \in \mathbb{R}$,

$$\int_{-\infty}^{y} p_t(x_2,u)\,du - \int_{-\infty}^{y} p_t(x_1,u)\,du = \int_{-\infty}^{y}\int_{x_1}^{x_2} \mathbb{E}\{W_{t,z}|X_{t,z} = u\}p_t(z,u)\,dzdu$$

so for almost all y we have

$$p_t(x_2,y) - p_t(x_1,y) = \int_{x_1}^{x_2} \mathbb{E}\{W_{t,z}|X_{t,z} = u\}p_t(z,u)dz$$

hence $p_t(x,y)$ is absolutely continuous function in x and

$$\frac{\partial}{\partial x}p_t(x,y) = \mathbb{E}\{W_{t,x}|X_{t,x} = y\}\,p_t(x,y). \quad \square$$

As an appendix to the previous theorem we shall study in detail partial derivative $\frac{\partial}{\partial x}p_t(x,y)$. Precisely, we shall show that the process $t^{-1}\mathbb{E}\{\int_0^t a(X_{s,x})^{-1}DX_{s,x}dB_s|X_{t,x}\}$ is a backwards martingale.

As $\{X_{t,x}, t \ge 0\}$ is a Markov process, for a $f \in \mathcal{M}_b$ and $s < t$ we have

$$\mathbb{E}_x\{f(X_t)\} = \mathbb{E}_x\{\mathbb{E}\{f(X_t)|\mathcal{F}_s\}\} = \mathbb{E}_x\{\mathbb{E}_{X_s}\{f(X_{t-s})\}\},$$

where $\{\mathcal{F}_t, t \ge 0\}$ is standard filtration generated by $\{B_t, t \ge 0\}$, and $\mathbb{E}_x$ is expectation operator generated by initial value $X_0 = x$. Now, 1° of Theorem 3 yields

$$\begin{aligned}&\frac{\partial}{\partial x}\mathbb{E}_x\{f(X_t)\} = \mathbb{E}_x\{f(X_t)W_t\} = \mathbb{E}_x\left\{f(X_t)\mathbb{E}\{W_t|X_t\}\right\} \text{ as well as} \\ &\frac{\partial}{\partial x}\mathbb{E}_x\{f(X_t)\} = \frac{\partial}{\partial x}\mathbb{E}_x\left\{\mathbb{E}_{X_s}\{f(X_{t-s})\}\right\} = \mathbb{E}_x\left\{\mathbb{E}_{X_s}\{f(X_{t-s})\}\,W_s\right\} = \\ &\qquad = \mathbb{E}_x\left\{\mathbb{E}_{X_s}\{f(X_{t-s})\}\,\mathbb{E}\{W_s|X_s\}\right\} = \mathbb{E}_x\left\{f(X_t)\,\mathbb{E}\{W_s|X_s\}\right\},\end{aligned}$$

hence, for $s < t$

$$\mathbb{E}\{W_{t,x}|X_{t,x}\} = \mathbb{E}\{\,\mathbb{E}\{W_{s,x}|X_{s,x}\}\,|X_{t,x}\}.$$

If we mark

$$\tilde{W}_{t,x} = \mathbb{E}\{W_{t,x}|X_{t,x}\}$$

then previous relation could be rewritten as

$$\tilde{W}_{t,x} = \mathbb{E}\{\tilde{W}_{s,x}|X_{s,x}\},\ s < t.$$

Let us define backwards flow of σ-algebras as

$$\overleftarrow{\mathcal{F}}_{t,x} = \sigma\{X_{u,x}, u \ge t\},\ t > 0,\ x \in \mathbb{R},$$

and let for $\phi(x_1, \dots, x_n)$ be n-dimensional bounded Borel function. For $s < t \le t_1 < \dots < t_n$ we have

$$\begin{aligned}
\mathbb{E}_x \{\tilde{W}_{s,x}\phi(X_{t_1,x}, \dots, X_{t_n,x})\} &= \mathbb{E}_x \left\{ \mathbb{E}\{\tilde{W}_{s,x}\, \phi(X_{t_1,x}, \dots, X_{t_n,x}) | \mathcal{F}_t\} \right\} = \\
&= \mathbb{E}_x \left\{ \tilde{W}_{s,x}\, \mathbb{E}\{\phi(X_{t_1,x}, \dots, X_{t_n,x}) | \mathcal{F}_t\} \right\} = \\
&= \mathbb{E}_x \left\{ \tilde{W}_{s,x}\, \mathbb{E}_{X_{t,x}} \{\phi(X_{t_1-t,x}, \dots, X_{t_n-t,x})\} \right\} = \\
&= \mathbb{E}_x \left\{ \mathbb{E}\{\tilde{W}_{s,x} | X_{t,x}\}\, \mathbb{E}_{X_{t,x}} \{\phi(X_{t_1-t,x}, \dots, X_{t_n-t,x})\} \right\} = \\
&= \mathbb{E}_x \left\{ \mathbb{E}\{\tilde{W}_{s,x} | X_{t,x}\}\, \phi(X_{t_1,x}, \dots, X_{t_n,x}) \right\},
\end{aligned}$$

hence

$$\mathbb{E}\{\tilde{W}_{s,x} | \overleftarrow{\mathcal{F}}_{t,x}\} = \mathbb{E}\{\tilde{W}_{s,x} | X_{t,x}\} = \tilde{W}_{t,x}, \quad \text{i.e.}$$

process

$$\tilde{W}_{t,x} = t^{-1}\mathbb{E}\{\int_0^t a(X_{s,x})^{-1} DX_{s,x} dB_s | X_{t,x}\}$$

is a backward martingale, relative to flow of σ-algebras

$$\overleftarrow{\mathcal{F}}_{t,x} = \sigma\{X_{u,x}, u \ge t\},\ t > 0,\ x \in \mathbb{R}.$$

Having in mind the definition of $\tilde{W}_{t,x}$ and $3°$ of Theorem 3, we deduce

$$\int_{-\infty}^{\infty} \left|\frac{\partial}{\partial x} p_t(x, y)\right| dy = \mathbb{E}\{\left|\tilde{W}_{t,x}\right|\} < \infty.$$

Moreover, for $s < t$

$$\begin{aligned}
\int_{-\infty}^{\infty} \left|\frac{\partial}{\partial x} p_t(x, y)\right| dy = \mathbb{E}\{\left|\tilde{W}_{t,x}\right|\} &= \mathbb{E}\{ \left|\mathbb{E}\{\tilde{W}_{s,x} | \overleftarrow{\mathcal{F}}_{t,x}\}\right| \} \\
&\le \mathbb{E}\{ \mathbb{E}\{\left|\tilde{W}_{s,x}\right| | \overleftarrow{\mathcal{F}}_{t,x}\} \} = \mathbb{E}\{ \left|\tilde{W}_{s,x}\right| \} = \\
&= \int_{-\infty}^{\infty} \left|\frac{\partial}{\partial x} p_s(x, y)\right| dy
\end{aligned}$$

We have proved the following Corollary:

Corollary 1. *For each $t > 0$ and $x \in \mathbb{R}$ process*

$$\tilde{W}_{t,x} = t^{-1}\mathbb{E}\{\int_0^t a(X_{s,x})^{-1} DX_{s,x} dB_s | X_{t,x}\}$$

is a backward martingale, relative to flow of σ-algebras

$$\overleftarrow{\mathcal{F}}_{t,x} = \sigma\{X_{u,x}, u \ge t\},\ t > 0,\ x \in \mathbb{R}.$$

Consequently, function

$$\int_{-\infty}^{\infty} \left|\frac{\partial}{\partial x} p_t(x, y)\right| dy$$

is finite and decreasing with t. $\square$

POINT CUT AS SPATIAL HOMEOMORPHISM

It is well known [10, Theorem 2.20] that, for each $t > 0$, the mapping $x \to X_{t,x}$ is a.s. homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$, hence we can define inverse random transformation

$$X_t^{-1}(\cdot) : \mathbb{R} \to \mathbb{R}$$

such that

$$X_t^{-1}(X_{t,x}) = x, \quad x \in \mathbb{R}.$$

In the next theorem we shall investigate the distribution of random variable $X_t^{-1}(y)$, i.e. the random function $X_t^{-1}(\cdot)$ applied to the point y.

Theorem 4. *Let everything be as in* Theorem 3. *Then:*

$1°$ *for each $t > 0$ and $y \in \mathbb{R}$ function $\mathbb{E}\{DX_{t,\cdot}|X_{t,\cdot} = y\}p_t(\cdot, y)$ is integrable and*

$$\int_{-\infty}^{+\infty} \mathbb{E}\{DX_{t,x}|X_{t,x} = y\}p_t(x, y)\,dx = 1; \tag{16}$$

$2°$ *for each $t > 0$ and $x, y \in \mathbb{R}$*

$$\mathbb{P}\{X_t^{-1}(y) \le x\} = \int_{-\infty}^{x} \mathbb{E}\{DX_{t,z}|X_{t,z} = y\}p_t(z, y)\,dz; \tag{17}$$

$3°$ *probability density*

$$\rho_t(y, x) = \mathbb{E}\{DX_{t,x}|X_{t,x} = y\}\,p_t(x, y) \tag{18}$$

of the random variable $X_t^{-1}(y)$ satisfy Chapman-Kolmogorov equations:

$$\rho_{t+s}(y, x) = \int_{-\infty}^{+\infty} \rho_s(y, z)\,\rho_t(z, x)\,dz \tag{19}$$

for $t, s > 0$ and $x, y \in \mathbb{R}$.

Proof. $1°$ As for each $t > 0$, $X_{t,\cdot}$ is a.s. homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$ and $DX_{t,x}$, partial derivative in $L_2(\Omega)$ sense of random variable $X_{t,x}$ w.r.t. x, is a.s. positive, we have

$$\lim_{x\to\pm\infty} X_{t,x} = \pm\infty,$$

and from relation (15) we have, for $x_1 < x_2$

$$\mathbb{P}\{X_{t,x_2} \le y\} - \mathbb{P}\{X_{t,x_1} \le y\} = -\int_{x_1}^{x_2} \mathbb{E}\{DX_{t,z}|X_{t,z} = y\}p_t(z, y)\,dz. \tag{20}$$

Now, letting $x_1 \to -\infty$ and $x_2 \to +\infty$ one can conclude that

$$\int_{-\infty}^{+\infty} \mathbb{E}\{DX_{t,z}|X_{t,z} = y\}p_t(z, y)\,dz = 1,$$

what was to be proved in $1°$.

$2°$ ¿From the fact that a.s. $X_{t,x}$ is homeomorphism and that $X_t^{-1}(y)$ is its inverse we have the following equivalence

$$X_t^{-1}(y) \le x \Leftrightarrow X_{t,x} \ge y,$$

so, using relation (20), with $x_1 = -\infty$ and $x_2 = x$

$$\mathbb{P}\{X_t^{-1}(y) \le x\} = \mathbb{P}\{X_{t,x} \ge y\} = 1 - \mathbb{P}\{X_{t,x} < y\} = \int_{-\infty}^{x} \mathbb{E}\{DX_{t,z}\,|X_{t,z} = y\}p_t(z,y)dz.$$

3° Denote with $\mathbb{P}_x\{\cdot\}$ the probability measure generated by the Markov process $\{X_{t,x}, t \ge 0\}$ starting from x. Then for $t, s > 0$ the Markov property yields

$$\mathbb{P}\{X_{t+s,x} \le y\} = \mathbb{P}_x\{X_{t+s} \le y\} = \int_{-\infty}^{+\infty} \mathbb{P}_z\{X_s \le y\}p_t(x,z)\,dz = \mathbb{E}\{\mathbb{P}_{X_{t,x}}\{X_{s,z} \le y\}\}.$$

Using relation (15) we have

$$\begin{aligned}
\rho_{t+s}(y,x) &= -\frac{\partial}{\partial x}\mathbb{P}\{X_{t+s,x} \le y\} = -\frac{\partial}{\partial x}\mathbb{E}\{\mathbb{P}_{X_{t,x}}\{X_{s,z} \le y\}\} = \\
&= -\mathbb{E}\{\frac{\partial}{\partial x}\mathbb{P}_{X_{t,x}}\{X_{s,z} \le y\}\,DX_{t,x}\} = \\
&= -\mathbb{E}\{-\mathbb{E}_{X_{t,x}}\{DX_s\,|X_s = y\}p_s(X_{t,x},y)\,\mathbb{E}\{D_{t,x}\,|X_{t,x}\}\} = \\
&= \int_{-\infty}^{+\infty} \mathbb{E}_z\{DX_s|X_s = y\}p_s(z,y)\,\mathbb{E}\{D_{t,x}\,|X_{t,x} = z\}p_t(x,z)\,dz = \\
&= \int_{-\infty}^{+\infty} \mathbb{E}\{DX_{s,z}\,|X_{s,z} = y\}p_s(z,y)\,\mathbb{E}\{D_{t,x}\,|X_{t,x} = z\}p_t(x,z)\,dz = \\
&= \int_{-\infty}^{+\infty} \rho_s(y,z)\,\rho_t(z,x)\,dz,
\end{aligned}$$

and the theorem is completely proved. $\square$

References

1. N. Bouleau and F. Hirsch, *Formes de Dirichlet generales et densite des variables aleatoires reelles sur l'espace de Wiener.*, J. Func. Anal. **69** (1986), 229–259.
2. I.I. Gihman and A.V. Skorohod, *Stochastic differential equations*, Naukova Dumka, Kiev, 1968. (Russian)
3. S. Kusuoka and D. Stroock, *Applications of the Malliavin Calculus.* I , Proc of the Taniguchi Inter. Symp. on Stoch. Analysis. Katata, North-Holland, 1982, pp. 271–308.
4. P. Malliavin, *Stochastic calculus of variations and hypoelliptic operators*, Proc. Inter. Symp. on Stoch. Diff. Equations, Kyoto 1976, pp. 195–263.
5. D. Nualart, *Noncausal stochastic integrals and calculus*, Proc. of Workshop on Stoch. Anal. and Rel. Topics. Silivri 1986, 1988, pp. 80–129.
6. D. Nualart and M. Zakai, *Generalized stochastic integrals and Malliavin Calculus*, Probability Theory and Rel. Fields **73** (1986), 255–280.
7. D. Pantić, *Stochastic Calculus on Distorted Brownian Motion*, J.Math.Phys. **29** (1988), 207-210.
8. D. Pantić, *Stochastic Calculus and Schroedinger Semigroups Admitting Ground State*, J. Math. Phys. **31** (1990), 1221-1225.
9. D. Pantić, *Toward the Existence of Density for the Distribution of a Brownian Functional*, submitted for publication.
10. D. W. Stroock, *Topics in stochastic differential equations*, Tata Institute of Fundamental Research, 1982.
11. S. Watanabe, *Lectures on stochastic differential equations and Malliavin Calculus*, Tata Institute of Fundamental Research, 1984.
12. M. Zakai, *The Malliavin Calculus*, Acta Appl. Math **3** (1985), 175–207.

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Vol. 01, 1995, pp. 27–29.
Submitted: July 19, 1995. Published: September 18, 1995.

A LOGICAL ALTERNATIVE TO THE EXISTING POSITIONAL NUMBER SYSTEM

Robert R. Forslund

Abstract. This article introduces an alternative positional number system. The advantages of this alternative system over the existing one are discussed, and an illustration of the use of the system to re-interpret apparent errors in ancient archaeological documents is presented.

(1991) A.M.S. (MOS) Subject Classification Codes. 11, 01.

Key Words and Phrases. positional number system, archaeology, zero, placeholder.

Introduction

To simplify this article, the history of positional number systems will not be discussed, and interested readers can find this information in the references listed. In addition, the discussion is restricted to the set of positive integers; however, the extension to decimal and negative values, if necessary, is easily accomplished.

The existing positive integer number system in a selected base (b), is represented by the following mathematical notation :

$$N = \sum(a_j * b^j) \text{ where } 0 \le a_j \le (b-1),\ b > 1, \text{ and the sum is over } j$$

with $0 \le j \le (n-1)$, and a_j are the n digits of the positive integer N in base b.

For example:

$$10203 = 1 * 10^4 + 0 * 10^3 + 2 * 10^2 + 0 * 10^1 + 3$$

Traditionally we have selected base : $b = 10$, as in the example, and commonly use other bases such as binary ($b = 2$), octal ($b = 8$) and hexadecimal ($b = 16$). Ancient civilizations introduced other bases such as 60 used by the Babylonians [1], and 18 and 20 used by the Mayans [1].

The alternative system

There is a problem with the existing number system. Base 1 is illogical since the digits are equal to zero, and therefore we must select base, $b > 1$ when defining the system. In addition, the digit integers starting at one are concrete concepts, while the concept of zero (nothingness) is abstract. Therefore, I sought a system that did not require the use of zero as a placeholder. As you will see shortly, there is such a system, and the concept of a zero placeholder adds unnecessary complexity.

As a result, I reasoned as follows. First of all, and most importantly, why not use digits from 1 to b rather than 0 to $(b-1)$? In base 10, this simply means introducing a 10th symbol for the digit "ten". Any

Ministry of Natural Resources Centre for Northern Forest Ecosystem Research c/o Lakehead University, 955 Oliver Road, Thunder Bay, Ontario, Canada P7B 5E1

E-mail Address: mjohnsto@sky.Lakehead.ca

symbol will do, however the symbol "**A**" borrowed from the hexadecimal system provides some familiarity. This is the only change to the notation used in the introduction for N : viz : $1 \leq a_j \leq b$ and $b \geq 1$. This alternative system has no need for a placeholder (zero) as I illustrate below. I will use bolded symbols for digits in the alternative system to distinguish between the two systems.

In the alternative system (base **A**), the number 10 is now represented by the digit **A**. The number 20 now becomes : $\mathbf{1A} = \mathbf{1} * \mathbf{A} + \mathbf{A}$, and 90 is **8A**. The number 100 now becomes : $\mathbf{9A} = \mathbf{9} * \mathbf{A} + \mathbf{A}$, and $1000 = \mathbf{99A} = \mathbf{9} * \mathbf{A^2} + \mathbf{9} * \mathbf{A} + \mathbf{A} = 900 + 90 + 10$, and so on. Notice that, with the exception of integers that have digits equal to zero in the existing system, all other numbers remain the same as in the alternative system as long as the digits lie between 1 and $(b-1)$ inclusive. Therefore the smaller the base, the more the differences will be between the two systems. Base 2 for example will remain the same as the existing system only when all digits are 1. An example in base 2 is : $21 = 10101$ in the existing system, and **1221** in the alternative system. In comparison, the number 31 in both systems remains at **11111** in base 2.

Advantages of the alternative system

Base 1 is now permissible and completely logical and is simply a tally system. For example, $7 =$ **1111111** (base 1) in the alternative system. Secondly, for number theorists, the re-arrangement of digits in the existing system causes digits to be lost. For example the reversal of 12300 in the existing system is 321 and two zero digits are "lost". In the alternative system this does not occur. For example, $12300 =$ **1229A**, which is **A9221** upon reversal. As this simple example illustrates, this alternative system lends itself to further research in number theory. In addition, in the existing system, powers of the base are identified by the number of zeros in the number - for example $10000 = 10^4$. Powers are similar in the alternative system and are identified as in this example by $\mathbf{999A} = \mathbf{A^4}$. More generally, in base b, powers $= \ldots(\mathbf{b-1})(\mathbf{b-1})\ldots(\mathbf{b-1})\mathbf{b}$. Notice that fewer digits are often needed to represent the numbers - this too is an advantage, and one of the main reasons positional notation was first introduced. Also, in base 10 for example, instead of having 9 numbers with one digit, 90 numbers with 2 digits, and so on, the alternative system now has the first 10 numbers having a single digit, the next 100 numbers having 2 digits, and so on. This too is simpler since the number of integers with n digits in base b, is b^n rather than $b^n - b^{(n-1)}$ in the existing system.

Application of the alternative system to archaeology

It appears that this alternative system would have been a slightly more logical first step in the development of a number system in ancient times. One possible use of this system might be in the re-interpretation of ancient calculations found in archaeological documents. History tends to identify the introduction of the zero placeholder into the number system, as an advancement in mathematics. However, the alternative system, with its inherent simplicity, illustrates that the introduction of zero may have unnecessarily complicated the use of integers, and numbers in general.

Perhaps by re-interpreting ancient mathematical documents using this alternative system, mathematical archaeologists will reveal that some of the ancient calculations, previously thought to be in error, really were correct. I will illustrate using the commonly used digit symbols in base ten, realizing that ancient documents used other symbols for digits. The ancients may have used 10 symbols (if base 10 was used); however, the two symbols "10" may really have meant **1A** - which is twenty (not ten). In other words, it may be that the zero symbol represented the digit ten - not zero. To detect such discrepancies, ancient documents must contain calculations such as addition. For example : $10 + 11 = 31$ - this appears to be incorrect. However, if the symbol "0" actually represents the digit ten (**A**) and not zero, then the equation is correct and equivalent to $\mathbf{1A} + \mathbf{11} = \mathbf{31} = \mathbf{20} + \mathbf{11}$ in the existing system. Another example : $101 + 9 = 100$. Once again, this appears incorrect, however, if "0" symbol is re-interpreted as "**A**", then this is: $\mathbf{1A1} + \mathbf{9} = \mathbf{1AA}$, which is $201 + 9 = 210$ in the existing system. In base 10, these apparent errors would only exist when the symbol for ten (and interpreted as zero) occurred in the digits of numbers. Larger bases such as 60 used by the Mesopotamians [1], would be even more void of such "errors", and therefore it would be harder to determine if the alternative number system was used. I leave it to archaeologists specializing in the interpretation of these ancient documents to examine this usage of the alternative positional number system.

In summary, the alternative system appears to be a slightly more logical system relative to the existing one. As illustrated, it has immediate application to the field of archaeology. Readers should have little problem finding other fields for the application of this system especially in the theory of numbers. If this article motivates others to explore the possibilities of this alternative system, then it has succeeded.

References

1. Boyer, C.B. and Merzbach, U.T. : A History of Mathematics (Second Edition), John Wiley & Sons Inc., New York, 1991.
2. Hellemans, A. and Bunch, B. : The Timetables of Science - A Chronology of the Most Important People and Events in the History of Science, Simon & Schuster Inc., New York, 1991.
3. Swetz, F.J. : From Five Fingers to Infinity, Open Court, Chicago and La Salle Illinois, 1995.

Acknowledgements

I wish to thank Dr. Mark Johnston and Mr. Luke Dalla Bona at the Centre for Northern Forest Ecosystem Research for providing me with access to the use of electronic communication via the Internet and for their encouragement and comments regarding this paper.

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Vol. 01, 1995, pp. 30-36.
Submitted: May 12, 1995. Published: October 16, 1995.

RESULTS ON CONTROLLING THE RESIDUALS OF PERTURBED NEWTON-LIKE METHODS ON BANACH SPACES WITH A CONVERGENCE STRUCTURE

Ioannis K. Argyros

Abstract. In this study, we provide sufficient conditions for controlling the residuals of perturbed Newton-like methods, in such a way that the convergence of such methods is ensured. In order to achieve this, we apply a theorem due to Kantorovich on a setting of a regular space with a convergence structure.

(1991) A.M.S. (MOS) Subject Classification Codes. 65J15, 65B05, 47D15, 47H17, 49D15.

Key Words and Phrases. Newton-like methods, residuals.

I. INTRODUCTION

In this study, we are concerned with approximating a solution x^* of the nonlinear operator equation

$$F(x) + Q(x) = 0, \tag{1}$$

where F is a Frechet-differentiable operator defined on a convex subset D of a Banach space X with values in X, and Q is a non-differentiable nonlinear operator with the same domain and values in X.

We generate a sequence $\{x_n\}$ $(n \geq 0)$ using the perturbed Newton-like method scheme given by

$$x_{n+1} = x_n + \delta_n \quad (n \geq 0) \tag{2}$$

where the correction δ_n satisfies

$$A(x_n)\delta_n = -(F(x_n) + Q(x_n)) + r_n \quad (n \geq 0) \tag{3}$$

for a suitable $r_n \in X$ $(n \geq 0)$ called residual. The importance of studying perturbed Newton-like methods comes from the fact that variants of Newton's method can be considered as procedures of this type. In fact, the approximations (2) and (3) characterize any iterative process in which the corrections are taken as approximate solutions of the Newton equations. This happens when approximation (2) is solved by any iterative method or when therein the derivative is replaced by a suitable approximation. In [4] we provided sufficient conditions for the convergence of iteration (2) to a solution x^* of equation (1), by assuming that X is a Banach space with a convergence structure (to be precised later).

Here we derive sufficient conditions for controlling the residuals r_n in such a way that the convergence of the sequence $\{x_n\}$ $(n \geq 0)$ to a solution of equation (1) is ensured.

We also refer the reader to [5], [6], [8] and the references there for relevant work, which however is valid on a Banach space X without a convergence structure. The advantages of working on a Banach space with a convergence structure have been explained in some detail in [4], [6], [7].

Department of Mathematics, Cameron University, Lawton, OK 73505, U.S.A.
E-mail Address: ioannisa@cua.cameron.edu

II. PRELIMINARIES

We will need the definitions:

Definition 1. The triple (X, V, E) is a Banach space with a convergence structure if

(C$_1$) $X, \|\cdot\|)$ is a real Banach space;

(C$_2$) $(V, C, \|\,\|_{\mathrm{v}})$ is a real Banach space which is partially ordered by the closed convex cone C; the norm $\|\,\|_{\mathrm{v}}$ is assumed to be monotone on C;

(C$_3$) E is a closed convex cone in $X \times V$ satisfying $\{0\} \times C \subseteq E \subseteq X \times C$;

(C$_4$) the operator $|\,| : D \to C$ is well defined:

$$|x| = \inf\{q \in C | (x, q) \in E\}$$

for

$$x \in D = \{x \in X | \exists q \in C : (x, q) \in E\};$$

and

(C$_5$) for all $x \in D$ $\|x\| \leq \||x|\|_{\mathrm{v}}$.

The set

$$U(a) = \{x \in X | (x, a) \in E\}$$

defines a sort of generalized neighborhood of zero.

Definition 2. An operator $L \in C^1(V_1 \to V)$ defined on an open subset V_1 of an ordered Banach space V is order convex on $[a, b] \subseteq V_1$ if

$$c, d \in [a, b],\ c \leq d \Rightarrow L'(d) - L'(c) \in L_+(V),$$

where for $m \geq 0$

$$L_+(V^m) = \{L \in L(V^m) | 0 \leq x_i \Rightarrow 0 \leq L(x_1, x_2, \ldots, x_m)\}$$

and $L(V^m)$ denotes the space of m-linear, symmetric, bounded operators on V.

Definition 3. The set of bounds for an operator $H \in L(X^m)$ is defined to be

$$B(H) = \{L \in L_+(V^m) | (x_i, q_i) \in E \Rightarrow (H(x_1, \ldots, x_m), L(q_1, \ldots, q_m)) \in E\}.$$

Definition 4. A partially ordered topological space V is called regular if every order bounded increasing sequence has a limit.

We will need the following proposition due to Kantorovich [3], [5].

Proposition. Let V be a regular partially ordered topological space (POTL-space) and let x, y be two points of V such that $x \leq y$. If $H : [x, y] \to V$ is a continuous isotone operator having the property that $x \leq Hx$ and $y \geq Hy$, then there exists a point $z \in [x, y]$ such that $z = Hz$.

III. CONVERGENCE ANALYSIS

We will need the following basic result:

Lemma. Let V be a regular partially ordered topological space, an operator $L \in C^1(V_1 \to V)$ with $[0, a] \subseteq V_1 \subseteq V$ for some $a \in V$, an operator $M \in C(V_1 \to V)$, operators $R < I$, $B > I$, T, $K \in L_+(V)$, a point $c \in V$ with $c > 0$ and a point $p \in [0, a]$.

Assume:

(a) The equation

$$\begin{aligned} g(q) = {} & BT\big[L(p + q) - L(p) - L'(p)q + M(p + q) - M(p) + K(p + q) - K(p)\big] \\ & - (I - R)q + c = 0 \end{aligned} \tag{4}$$

has solutions in the interval $[0, a]$ and denote by q^* the least of them.

(b) Let $G \in L_+(V)$ be given and $R_+ \in L_+(V)$, $c_+, p_+ \in V$ be such that the following conditions are satisfied:

$$R_+ < \min\{(G-2I)BTL'(p) + BTGL'(p_+) + G(R-I) + I, I\} = \alpha \tag{5}$$

$$\begin{aligned} 0 < c_+ \leq\ & BTG(L(p_+) + M(p_+) + K(p_+)) + GBT(L(p) + M(p) + K(p)) \\ & -2BT(L(p) + M(p) + K(p)) + (R_+ + G - I - BTGL'(p_+))c = \beta \end{aligned} \tag{6}$$

and

$$0 \leq p_+ \leq p + c, \tag{7}$$

where α and β are functions of the operators and points involved.

(c) The following estimate is true

$$M(p) \leq M(p+q) \tag{8}$$

for all $p, q \in [0, a]$.

Then the equation

$$\begin{aligned} g_+(q) = & BGT[L(p_+ + q) - L(p_+) - L'(p_+)q + M(p_+ + q) - M(p_+) + K(p_+ + q) - K(p_+)] \\ & - (I - R_+)q + c_+ = 0 \end{aligned} \tag{9}$$

has nonnegative solutions and the least of them, denoted by q_+^* lies in the interval $[c_+, q^* - c]$.

Proof. Using the hypothses $g(q^*) = 0$ and $R < I$ we deduce from (4) that $c \leq q^*$. We will show that

$$g_+(q^* - c) \leq 0. \tag{10}$$

From equation (9) and using (6), we obtain in turn

$$\begin{aligned} g_+(q^* - c) \leq\ & BGT[L(p + c + q^* - c) - L(p_+) - L'(p_+)(q^* - c) \\ & + M(p + c + q^* - c) - M(p_+) + K(p + c + q^* - c) - K(p_+)] - (I - R_+)(q^* - c) + c_+ \\ & = g(q^*) - BT[L(p + q^*) - L(p) - L'(p)q^* + M(p + q^*) - M(p) + K(p + q^*) - K(p)] \\ & + BTG[L(p + q^*) - L(p_+) - L'(p_+)(q^* - c) + M(p + q^*) - M(p_+) + K(p + q^*) - K(p_+)] \\ & + (I - R)q^* - c - (I - R_+)(q^* - c) + c_+ \\ & = [BTL'(p) - BTGL'(p_+) + (I - R) - (I - R_+)]q^* \\ & + (G - I)\{(I - R)q^* - c - BT[L(p) + M(p) + K(p) + L'(p)q^*]\} \\ & + BT[L(p) + M(p) + K(p) - GL(p_+) - GM(p_+)] \\ & + BTGL'(p_+)c + (I - R_+)c + c_+ - c \\ & = [(2I - G)BTL'(p) - BTGL'(p_+) + R_+ + G(I - R) - I]q^* \\ & + 2BT(L(p) + M(p) + K(p)) - BTG(L(p_+) + M(p_+) + K(p_+)) \\ & - GBT(L(p) + M(p) + K(p)) + (BTGL'(p_+) - G + I - R_+)c + c_+ \leq 0 \end{aligned}$$

since (5) and (6) are satisfied.

Moreover from (9) for $q = c_+$, we obtain

$$g_+(c_+) \geq 0. \tag{11}$$

By inequalities (10), (11), the fact that g is continuous and isotone on $[c_+, q^* - c]$, and since V is a regular partially ordered topological space, from the proposition, we deduce that there exists a point q_+^* with

$$g_+(q_+^*) = 0 \tag{12}$$

and

$$c_+ \leq q_+^* \leq q^* - c. \tag{13}$$

We can assume that q_+^* denotes the least of the solutions of equation (9).

That completes the proof of the lemma.

The following result is a consequence of the above lemma.

Theorem 1. Let $\{c_n\} \in V$, $\{T_n\}$, $\{R_n\}$, $\{G_n\} \in L_+(V)$ $(n \geq 0)$ be sequences and V as in the above lemma. Assume:

(a) There exists a sequence $\{p_n\} \in [0, a] \subseteq V_1 \subseteq V$ for some $a \in V$ with $p_0 = 0$ and

$$p_{n+1} \leq \sum_{j=0,1,\ldots,n} c_j \text{ for } n \geq 0. \tag{14}$$

(b) $R_0 < I$ and the function

$$\begin{aligned} g_0(q) &= BT_0\big[L(p_0+q) - L(p_0) - L'(p_0)q + M(p_0+q) - M(p_0) + K(p_0+q) - K(p_0)\big] \\ &\quad - (I - R_0)q + c_0 = 0 \end{aligned} \tag{15}$$

has roots on $[0, a]$, where B, L, M, K are as in the above lemma. Denote by q_0^* the least of them.

(c) The following conditions are satisfied for all $n \geq 0$

$$R_{n+1} < \alpha_{n+1}, \tag{16}$$

$$0 < c_{n+1} \leq \beta_{n+1} \tag{17}$$

and

$$0 \leq p_{n+1} \leq p_n + c_n. \tag{18}$$

(d) The linear operators T_n are boundedly invertible for all $n \geq 0$, and set $G_n = T_{n+1}T_n^{-1}$ $(n \geq 0)$.

(e) Condition (8) is satisfied.

Then, the equation

$$\begin{aligned} g_n(q) &= BG_nT_n\big[L(p_n+q) - L(p_n) - L'(p_n)q + M(p_n+q) - M(p_n) + K(p_n+q) - K(p_n)\big] \\ &\quad - (I - R_n)q + c_n = 0 \end{aligned} \tag{19}$$

has solutions in $[0, a]$ for every $n \geq 0$ and denoting by q_n^* the least of them, we have

$$\sum_{j=n,\ldots,\infty} c_j \leq q_n^* \quad (n \geq 0). \tag{20}$$

Proof. Let us assume that for some nonnegative integer n, $I - R_n > 0$, $g_n(q)$ has roots on $[0, a]$ and denote by q_n^* the least of them. We use induction on n. We also observe that this is true by hypothesis (b) for $n = 0$. Using (14), (16), (17), (18), the lemma, and setting $c = c_n$, $c_+ = c_{n+1}$, $R = R_n$, $R_+ = R_{n+1}$, $G = G_n$ we deduce that q_{n+1}^* exists, and

$$c_{n+1} \leq q_{n+1}^* \leq q_n^* - c_n. \tag{21}$$

The induction is now complete and (20) follows immediately from (21).

That completes the proof of the theorem. From now on we assume that X is a Banach space with a convergence structure in the sense of [4].

The following result is an immediate consequence of Theorem 1.

Theorem 2. Assume:

(a) the hypotheses of Theorem 1 are satisfied;

(b) there exists a sequence $\{x_n\}$ $(n \geq 0)$ in a Banach space X with a convergence structure such that $|x_{n+1} - x_n| \leq c_n$. Then

(i) the sequence $\{x_n\}$ $(n \geq 0)$ converges to some point x^*;

(ii) moreover the following error bounds are true

$$|x^* - x_n| \leq q_n^* \tag{22}$$

and

$$|x^* - x_{n+1}| \leq q_n^* - c_n \text{ for all } n \geq 0. \tag{23}$$

We can introduce the main result:

Theorem 3. Let X be a Banach space with convergence structure (X, V, E) with $V = (V, C, \|\,\|_v)$, an operator $F \in C^1(D \to X)$ with $D \subseteq X$, an operator $Q \in C(D \to X)$, an operator $A(x) \in C(X \to D)$, an operator $L \in C^1(V_1 \to V)$ with $V_1 \subseteq V$, an operator $M \in C(V_1 \to V)$, an operator $K \in L_+(V)$, and a point $a \in C$ such that the following conditions are satisfied:

(a) the inclusions $U(a) \subseteq D$ and $[0, a] \subseteq V_1$ are true;

(b) L is order-convex on $[0, a]$ and satisfies

$$K + L'(|x| + |y|) - L'(|x|) \in B(A(x) - F'(x + y)) \tag{24}$$

for all $x, y \in U(a)$ with $|x| + |y| \leq a$;

(c) M satisfies the condition

$$0 \leq (Q(x) - Q(x + y), M(|x| + |y|) - M(|x|) \in E, \quad M(0) = 0 \tag{25}$$

for all $x, y \in U(a)$ with $|x| + |y| \leq a$;

(d) for the sequences $\{c_n\}$, $\{T_n\}$, $\{R_n\}$, $\{G_n\}$, $\{p_n\}$ $(n \geq 0)$ the hypotheses (14), (b), (16), (18) and (d) of Theorem 1 are satisfied;

(e) the following conditions are also satisfied

$$|\delta_n| \leq c_n \leq T_n| - (F(x_n) + Q(x_n))| \leq \gamma_n \leq \beta_n \quad (n \geq 1), \tag{26}$$

$$| - r_n| \leq T_n^{-1} R_n c_n \tag{27}$$

where

$$\gamma_n = T_n\big[L(p_n + c_n) - L(p_n) - L'(p_n)c_n + M(p_n + c_n) - M(p_n) + K(p_n + c_n) - K(p_n)\big] + R_n c_n \quad (n \geq 1). \tag{28}$$

Then

(i) the sequence $\{x_n\}$ $(n \geq 0)$ generated by

$$x_{n+1} = x_n + \delta_n \text{ with } x_0 = 0$$

remains in $\overline{U}(x_0, t_0^*)$ and converges to a solution x^* of equation $F(x) = 0$;

(ii) moreover, the error estimates (22) and (23) are true where q_n^* is the least root in $[0, a]$ of the function $g_n(q)$ defined in (19), with $p_n = \|x_n - x_0\|$ $(n \geq 0)$.

Proof. Let us assume that $x_n, x_{n+1} \in U(x_0, q_0^*)$, where the existence of q_0^* is guaranteed from hypotheses (d). We note that $|\delta_0| \leq c_0$. Using the approximation

$$-(F(x_{n+1}) + Q(x_{n+1})) = (F(x_n) - F(x_{n+1}) + A(x_n)(x_{n+1} - x_n)) + (Q(x_n) - Q(x_{n+1})) - r_n, \tag{29}$$

(24), (25), (27) and setting $p_n = |x_n - x_0|$, we obtain in turn

$$\begin{aligned} &| - (F(x_{n+1}) + Q(x_{n+1}))| \leq |F(x_n) - F(x_{n+1}) + A(x_n)(x_{n+1} - x_n)| + |Q(x_n) - Q(x_{n+1})| + | - r_n| \\ &\leq \int_0^1 \big[L'(p_n + t|x_{n+1} - x_n|) - L'(p_n) + K\big]|x_{n+1} - x_n|\,dt + M(p_n + |x_{n+1} - x_n|) - M(p_n) + | - r_n| \\ &\leq L(p_n + c_n) - L(p_n) - L'(p_n)c_n + Kc_n + M(p_n + c_n) - M(p_n) + T_n^{-1} R_n c_n. \end{aligned}$$

Hence, by (26) we get

$$c_{n+1} \leq T_{n+1}| - (F(x_{n+1}) + Q(x_{n+1}))| \leq \gamma_n \leq \beta_n \quad (n \geq 1),$$

which shows (17).

It can easily be seen that by using induction on n, the hypotheses of Theorem 2 are satisfied. Hence, by (24) and (27) the iteration $\{x_n\}$ $(n \geq 0)$ remains in $U(x_0, q_0^*)$ and converges to x^* so that (22) and (23) are satisfied. Moreover, from the estimate

$$| - (F(x_n) + Q(x_n))| \leq |A(x_n) - F'(x_n)|c_n + |F'(x_n)|c_n + | - r_n|,$$

(24), (27), the continuity of F, F', A, T_n, R_n and $c_n \to 0$ as $n \to \infty$, we deduce that

$$F(x^*) + Q(x^*) = 0.$$

That completes the proof of the theorem.

IV. APPLICATIONS

We will complete this study with an example that shows how to choose L, M, in practical applications. For simplicity, we assume that $A(x) = F'(x)$ for all $x \in D$.

Example. We discuss the case of a real Banach space with norm $\|\ \|$. Assume that $F'(0) = I$ and there exists a monotone operator

$$f : [0, a] \to IR$$

such that

$$\|F''(x)\| \leq f(\|x\|) \text{ for all } x \in U(a) \tag{30}$$

and a continuous nondecreasing function ℓ on $[0, r]$, $r \leq a$ such that

$$\|Q(x) - Q(y)\| \leq \ell(r)\|x - y\| \tag{31}$$

for all $x, y \in U(\frac{r}{2})$.

We showed in [3], (see also [9]) that (18) implies that

$$\|Q(x + e) - Q(x)\| \leq h(r + \|e\|) - h(r), \quad x \in U(a), \quad \|e\| \leq a - r, \tag{32}$$

where

$$h(r) = \int_0^r \ell(t)\, dt. \tag{33}$$

Conversely, it is not hard to see that we may assume, without loss of generality, that the function h and all functions $h(r + t) - h(r)$ are monotone in r. Hence, we may assume that $h(r)$ is convex and hence differentiable from the right. Then, as in [9], we show that (32) implies (31) and $\ell(r) = h'(r + 0)$. Hence, we can set

$$L_1(q) = L(q) = \|F(0) + Q(0)\| + \int_0^q ds \int_0^s f(t)\, dt \tag{34}$$

and

$$M(q) = \int_0^q \ell(t)\, dt. \tag{35}$$

Define the functions f_1, f_2, f_3 on $[0, a]$ by

$$f_1(q) = L(q) - q, \; f_2(q) = f_1(q) + h(q) \text{ and } f_3(q) = \int_0^q f(t)\, dt.$$

Choose $B = T_0 = 1$, $R_0 = K = 0$ and $p_0 = 0$. Then by (15) we get

$$g_0(q) = f_2(q).$$

It can easily be seen that with the above choices of L and M conditions (24) and (25) are satisfied.

Suppose that the function g_0 has a unique zero q_0^* in $[0, a]$ and $g_0(a) \leq 0$. It is then known [3], [9] that equation (1) admits a solution x^* in $U(q_0^*)$, this solution is unique in $U(a)$, and the iteration $\{x_n\}$ $(n \geq 0)$ given by (2) is well defined, remains in $U(q_0^*)$ for all $n \geq 0$ and converges to x^*. By applying the Banach lemma on invertible operators we can show that $A(x_n)$ is invertible and $\|A(x_n)^{-1}\| \leq -f_1'(\|x_n\|)^{-1} = T_n$ $(n \geq 0)$.

Moreover, suppose that (30) is satisfied. Furthermore, assume that instead of conditions (16) and (17), the weaker condition (10) is satisfied. Using (30) and the approximation

$$r_n = \big[(F'(x_n) - F'(x_0)) + F'(x_0)\big]\delta_n + F(x_n) + Q(x_n)$$

we obtain

$$\|r_n\| \leq \big(f_3(\|x_n\|) + 1\big)c_n + T_n^{-1}\gamma_n \quad (n \geq 1).$$

The above estimate provides us with a possible (but not the only) choice for the R_n's. Indeed, (27) will be true if the R_n's $(n \geq 0)$ can be chosen so that

$$(f_3(\|x_n\|) + 1)c_n + T_n^{-1}\gamma_n \leq T_n^{-1}R_n c_n \quad (n \geq 1).$$

Using the above choices and Theorem 3, it finally also follows that estimates (22) and (23) are satisfied for all $n \geq 0$.

REFERENCES

[1] Argyros, I.K. The secant method in generalized Banach spaces, Appl. Math. Comp. 39, (1990), 111–121.

[2] ______________. Newton-like methods in generalized Banach spaces, Functiones et Approximatio Commentarii Mathematici, XXII, (1993), 13–20.

[3] Argyros, I.K. and Szidarovszky, F. The Theory and Applications of iteration methods, C.R.C. Press, Inc. Boca Raton Florida, U.S.A. 1993.

[4] Argyros, I.K. Perturbed Newton-like methods and nondifferentiable operator equations on Banach spaces with a convergence structure, Southwest Journal of Pure and Applied Mathematics (SWJPAM) Vol. 1, No. 1 (1995), 1–12.

[5] Kantorovich, L.V. and Akilov, G.P. Functional Analysis in normed spaces, Academic Press, New York, 1978.

[6] Meyer, P.W. Newton's method in generalized Banach spaces, Numer. Funct. Anal. Optimiz. 9 (3 and 4), (1987), 249–259.

[7] ______________. A unifying theorem for Newton's method, Numer. Funct. Anal. Optimiz. 13 (5 and 6), (1992), 463–473.

[8] Moret, I. A Kantorovich-type theorem for inexact Newton methods, Numer. Funct. Anal. and Optimiz. 10 (3 and 4), (1989), 351–365.

[9] Zabrejko, P.P. and Nguen, D.F. The majorant method in the theory of Newton-Kantorovich approximations and the Ptak error estimates, Numer. Funct. Anal. Optimiz. 9 (5 and 6), (1987), 671–684.

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Vol. No. 01, 1995, pp. 37-45.
Submitted: June 8, 1995. Published: October 26, 1995.

A PROBABILISTIC APPROACH TO THE ESTIMATION OF THE DISTRIBUTION OF THE TARGET LOCATION POINT

M.A. Tabatabai

Abstract. A probabilistic approach is used to estimate the probability distribution of the target location point. The results can be used in missile-testing and target acquisition fire support model simulation. The target may be a mobile multiple launcher rocket, an antiaircraft gun or a ship in mid-ocean. In each case, there is a time lag between the detection of the target and the arrival of the projectile.

(1991) A.M.S. (MOS) Subject Classification Codes. 60, 62.

Key Words and Phrases. Probability density function, Gamma function.

I. Introduction

The target location problem frequently appears in missile-testing and artillery field programs. In this article, interest focuses on the probability distribution of the target location point for a multiple launcher rocket and a Howitzer. Our main concern is to find the probability density function of a mobile MLR location which is deliberately moving so as to confound prediction of its position. Let us now describe the situation by calling the two parties Red and Blue. The mission of the Red multiple launcher rocket is to strike by firing and move as quickly as possible in a specific direction with an average speed R. The reaction of the Blue commander is to predict the target location and launch a round of artillery at an anticipated target location.

Let the initial target location point (the point at which the Red initially fires) be represented by a bivariate normal random vector (X, Y), where X denotes the horizontal coordinate, Y denotes the vertical coordinate and X and Y are assumed to be independent random variables with a mean vector

$$(\mu_X, \mu_Y) = (0, 0)$$

and a standard deviation vector

$$(\sigma_X, \sigma_Y) = (100, 100)$$

where the mean and standard deviation for both X and Y are in meters.

Let T denote the Blue artillery reaction time. This random variable is the total response time from the Red multiple launch rocket firing to the first rounds of the Blue artillery landing on target. The distribution of the random variable T will be derived in Section 2. The Red displacement time using daylight values ranges from 1.5 minutes to 2 minutes.

Department of Mathematical Sciences, Cameron University, Lawton, OK 73505
E-mail Address: mohammad@tiger.cameron.edu

II. Distribution of Reaction Time

A sample of 103 response times from the Red firing to the first rounds of the Blue artillery landing on target is included in the analysis. The data was provided by the Fire Support Center at Cameron University. The goal of this section is to find the distribution of Blue reaction time.

Before making a specificaiton of the distribution of response time, it is a good practice to look at the data in a graphical form in order to see if they could have come from one of a known family of distributions.

One of the simplest ways of looking at the data is to make a stem-and-leaf plot. Figure 1 displays the stem-and-leaf plot for Blue reaction time (in minutes). In this display we immediately see the asymmetry of the main part of the data. The distribution is positively skewed and the figure indicates the presence of some extreme outliers. In Figure 2, we have the Boxplot. It provides a visual impression of several important aspects of the underlying distribution of the Blue reaction time. From a Boxplot we can pick out the following features of a data set:

Location
Skewness
Spread
Tail length
Outlying data points

Stem-and-leaf of Blue Re   N=103

Leaf Unit=0.10

2	1246777789999
3	00001112222233333444555566666666666777777888999
4	0011112223344444446668899
5	00014
6	2
7	24578
8	
9	79
10	49
11	44
12	1
13	
14	
15	
16	7

Figure 1. Stem-and-leaf Plot of Blue Reaction Time

The mean reaction time for this data set is 4.554 minutes with a standard deviation of 2.396 minutes. The data ranges from a low of 2.160 minutes to a high of 16.770 minutes. The first, second and third quartiles are 3.300, 3.740, and 4.630 respectively and the trimmed mean is 4.238 minutes.

```
            -----
    ----I +    I ------     * * *         0   0 0   0     0                      0
            -----
---------+----------+----------+----------+----------+----------
        3 . 0       6 . 0       9 . 0      1 2 . 0     1 5 . 0
```

FIGURE 2. Boxplot of Blue Reaction Time

Let T be the Blue reaction time and M be the Red movement time. More specifically, M is the difference between Blue reaction time and Red displacement time. If we use 2 minutes for the Red displacement time, then we can write the random variable M as $M = T - 2$. Figure 3 displays the stem-and-leaf graph for Red movement time.

0	1246777789999
1	0000111222223333344455556666666666677777788899 9
2	001111222334444446668899
3	00014
4	2
5	24578
6	
7	79
8	49
9	44
10	1
11	
12	
13	
14	7

FIGURE 3. Stem-and-leaf Plot for Red Movement Time

To have a more reliable estimate for the parameters of underlying distrubtion of random variable M, we examine the Boxplot for the Blue reaction time. Figure 4 shows the Boxplot for M.

```
          -----                                       0
     ---I   +  I ------     * * *         0 0 0   0 0     0                      0
          -----
--+----------+----------+----------+----------+-----------+---
 0 . 0      3 . 0       6 . 0       9 . 0      1 2 . 0     1 5 . 0
```

FIGURE 4. Boxplot of Red Movement Time

The small hollow circles in the Boxplot indicate the presence of eight extreme outliers. There is a widespread awareness of the dangers posed by the occurrence of extreme outliers, which may be a result of keypunch errors, recording or transmission errors, misplaced decimal points, exceptional phenomena such as a lack of adequate training in firing rockets, or members of a different population slipping into the sample. Outliers occur frequently in real data. When the data contain outliers, one approach is to remove the outliers and then calculate your sample statistics. A second approach is to use robust and resistant estimators to estimate the population parameters. Robust estimators limit the influence of extreme observations and are insensitive to outliers. For more details about robust statistical techniques, see Hoaglin, Mosteller, and Tukey (1983); Roussaauw and Leroy (1987); Tabatabai and Argyros (1993). In this article we first remove the extreme outliers and then calculate some statistics.

Table 1 shows a summary of some relevant estimates of the parameters of Red movement time.

N	MEAN	MEADIAN	TRMEAN	STDEV	SEMEAN
95	1.960	1.690	1.838	1.124	0.115

MIN	MAX	Q1	Q3
0.160	5.850	1.270	2.410

TABLE 1. Summary of Red Movement Time

Let us now turn our attention to the theory underlying estimation of the density function of M. Let the range be partitioned into k class intervals $I_j,\ j = 1, 2, \ldots, k$ and $I_j = (a_{j-1}, a_j),\ j = 1, 2, \ldots, k$ where $a_0 < a_1 < \cdots < a_k$. Here we assume that $a_j - a_{j-1} = 1$ minute for $j = 1, 2, \ldots, k$. Define

$$\hat{f}_M(m) = \begin{cases} f_j/[n(a_j - a_{j-1})] & \text{for } a_{j-1} < m \leq a_j, \quad j = 1, 2, \ldots, k \\ 0, & \text{elsewhere} \end{cases}$$

where f_j is the frequency of the class interval I_j and $n = \sum_{j=1}^{k} f_j$ is the total number of observations. The function $\hat{f}_M$ defined on $(-\infty, \infty)$ is the relative frequency histogram for M. We note that

$$\hat{f}_M(m) \geq 0 \text{ for all } m \in (-\infty, \infty), \tag{1}$$

$$\int_{-\infty}^{\infty} \hat{f}_M(m)\, dm = \int_{a_0}^{a_k} \hat{f}_M(m)\, dm = \sum_{j=1}^{k} \int_{a_{j-1}}^{a_j} \frac{f_j}{n(a_j - a_{j-1})}\, dm = \sum_{j=1}^{k} \frac{f_j}{n} = 1. \tag{2}$$

Hence $\hat{f}_M$ defines a density function and can be used as an estimate of the underlying density function f_M. In particular, an estimate of the probability of an event A is given by

$$\hat{P}(A) = \int_A \hat{f}_M(m) dm.$$

Moreover, $\hat{P}(A)$ is an unbiased and consistent estimate of $P(A)$.

For our data on Red movement time, the estimate of probability distribution of M is given by

m	$\hat{f}_M(m)$	m	$\hat{f}_M(m)$
.5	13/103	8.5	2/103
1.5	47/103	9.5	2/103
2.5	24/103	10.5	1/103
3.5	5/103	11.5	0/103
4.5	1/103	12.5	0/103
5.5	5/103	13.5	0/103
6.5	0/103	14.5	1/103
7.5	2/103		

where $m_j = (a_{j-1} + a_j)/2$ is the class mark of class interval I_j. The mean of $\hat{f}_M(m)$ is given by

$$\hat{\mu}_M = \int_{a_0}^{a_k} m\hat{f}_M(m)\,dm = \sum_{j=1}^{k} \int_{a_{j-1}}^{a_j} \frac{mf_j}{n(a_j - a_{j-1})}\,dm = \sum_{j=1}^{k} \frac{f_j}{n}\left(\frac{a_j + a_{j-1}}{2}\right) = \sum_{j=1}^{k} \frac{f_j m_j}{n}.$$

So far we have answered th following questions: What does the underlying probability density function f_M look like? How do we estimate f_M? The next question that one needs to answer is: Does f_M look like a known or specified distribution? The probability density function of M is Gamma distribution with parameters α and β. We have

$$f_M(m) = \begin{cases} \dfrac{m^{\alpha-1}\exp(-m/\beta)}{\Gamma(\alpha)\beta^\alpha}, & m > 0 \\ 0, & m \le 0 \end{cases}$$

where $\Gamma(\alpha) = \int\limits_0^\infty x^{\alpha-1}e^{-x}\,dx$.

The mean $\mu_M = \alpha\beta$ and standard deviation $\sigma_M = \beta\sqrt{\alpha}$. To estimate the parameters α and β, we use the method of moments technique. Hence, the moment estimators $\hat{\alpha}$ and $\hat{\beta}$ are

$$\hat{\alpha} = \frac{(\text{sample mean})^2}{\text{sample variance}} \text{ and } \hat{\beta} = \frac{\text{sample mean}}{\hat{\alpha}}.$$

If we use Table 1, then the moment estimates are

$$\hat{\alpha} = \frac{(1.960)^2}{(1.124)^2} = 3.0419 \text{ and } \hat{\beta} = \frac{1.960}{3.0419} = 0.6443.$$

Now, we use the chi-square goodness-of-fit test to test the following hypothesis?

$$H_0 : f_M(m) = \frac{m^{3.0419-1}\exp(-m/0.6443)}{\Gamma(3.0419)(0.6443)^{3.0419}}$$

vs

$$H_1 : f_M(m) \neq \frac{m^{3.0419-1}\exp(-m/0.6443)}{\Gamma(3.0419)(0.6443)^{3.0419}}.$$

The chi-square test statistic is 4.9153. The associated degrees of freedom is 3 and the $P-$ value is 0.1781. Therefore, we have to conclude that the Gamma distribution with parameters $\alpha = 3.0419$ and $\beta = 0.6443$ does provide an excellent fit for the data. Similar conclusions were reached when we used the Kolmogorov-Smirnov test.

The maximum likelihood estimates of α and β are solutions of the following equations

$$\ln \alpha - \left(\frac{d}{d\alpha} \Gamma(\alpha) \right) / \Gamma(\alpha) = \ln \overline{m} - \frac{1}{n} \sum_{j=1}^{n} \ln m_j \text{ and } \beta = \frac{\overline{m}}{\alpha}$$

where $\overline{m}$ is the sample mean.

These type of equations are not easily solvable and it is necessary to resort to numerical methods.

Let $F_T(t)$ denote the distribution function of Blue reaction time, then we have

$$\begin{aligned} F_T(t) &= P(T \leq t) = P(M + 2 \leq t) \\ &= P(M \leq t-2) = \int_{-\infty}^{t-2} f_M(m)\, dm \\ &= \int_{-\infty}^{t-2} \frac{m^{\alpha-1} \exp(-m/\beta)}{\Gamma(\alpha)\beta^{\alpha}}\, dm. \end{aligned}$$

Then, the probability density function $f_T(t)$ of Blue reaction time is given by

$$f_T(t) = \frac{d}{dt} F_T(t) = \begin{cases} \dfrac{(t-2)^{\alpha-1} \exp(-(t-2)/\beta)}{\Gamma(\alpha)\beta^{\alpha}}, & t > 2 \\ 0, & \text{elsewhere} \end{cases}$$

where the values of α and β can be estimated by 3.0419 and 0.6443 respectively. Then mean for Blue reaction time is estimated by $\hat{\mu}_T = 3.960$ and the standard deviation is estimated by $\hat{\sigma}_T = 1.124$.

III. Distribution of Target Location Point

Let the random vector (D_1, D_2) be the target location point and be defined by

$$\begin{aligned} D_1 &= X + RM \cos\theta \\ D_2 &= Y + RM \sin\theta \end{aligned}$$

where R is the average speed of Red, M is the Red movement time, θ is the Red angle of movement and X, Y are independent normal random variables with $\mu_X = \mu_Y = 0$ meters and $\sigma_X = \sigma_Y = 100$ meters.

Our goal is to find the joint probability density function of (D_1, D_2). We begin this section by first finding the probability distribution of RM.

Let $W = RM$ and $F_W(w)$ denote the distribution function of the random variable W. Then

$$\begin{aligned} F_W(w) &= P(W \leq w) = P(RM \leq w) = P(M \leq \frac{1}{R} w) \\ &= \int_{-\infty}^{\frac{1}{R}w} f_M(m)\, dm = \int_{-\infty}^{\frac{1}{R}w} \frac{m^{\alpha-1}\exp(-m/\beta)}{\Gamma(\alpha)\beta^{\alpha}}\, dm. \end{aligned}$$

Then the probability density function $f_W(w)$ is given by

$$f_W(w) = \frac{d}{dW} F_W(w) = \begin{cases} \dfrac{W^{\alpha-1}\exp(-w/(R\beta))}{\Gamma(\alpha)(R\beta)^{\alpha}}, & W > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Thus, we can write

$$\begin{aligned} D_1 &= X + W\cos\theta \\ D_2 &= Y + W\sin\theta \end{aligned}$$

where the random variable θ is uniformly distributed on the interval $[0, 2\pi)$ with the probability density function $g(\theta)$ defined by

$$g(\theta) = \begin{cases} \dfrac{1}{2\pi} & 0 \le \theta < 2\pi \\ 0, & \text{elsewhere} \end{cases}$$

and the random variables X, Y, θ, and W are independent reandom variables with joint probability density function given by

$$\begin{aligned} f(x, y, \theta, w) &= f_X(x)\, f_Y(y)\, g(\theta)\, f_W(w) \\ &= \begin{cases} \dfrac{w^{\alpha-1}\exp\left[-w/(R\beta) - \frac{1}{2}(x^2/100^2 + y^2/100^2)\right]}{40000\pi^2\Gamma(\alpha)(R\beta)^{\alpha}}, & \begin{array}{c} -\infty < y < \infty \\ -\infty < x < \infty \\ 0 \le \theta < 2\pi \\ 0 < w < \infty \end{array} \\ 0, & \text{elsewhere} \end{cases} \end{aligned}$$

Let

$$\begin{aligned} D_1 &= g_1(X, Y, \theta, W) = X + W\cos\theta, \\ D_2 &= g_2(X, Y, \theta, W) = Y + W\sin\theta, \\ D_3 &= g_3(X, Y, \theta, W) = \theta, \\ D_4 &= g_4(X, Y, \theta, W) = W, \end{aligned}$$

$$\begin{aligned} A &= \{(x, y, \theta, w) : -\infty < x < \infty,\ -\infty < y < \infty,\ 0 \le \theta < 2\pi,\ 0 < w < \infty\}, \\ B &= \{(D_1, D_2, D_3, D_4) : -\infty < D_1 < \infty,\ -\infty < D_2 < \infty,\ 0 \le D_3 < 2\pi,\ 0 < D_4 < \infty\}. \end{aligned}$$

Define the transformation $g : A \xrightarrow[\text{onto}]{1-1} B$ as

$$\begin{aligned} g &= (g_1(x, y, \theta, w),\ g_2(x, y, \theta, w),\ g_3(x, y, \theta, w),\ g_4(x, y, \theta, w)) \\ &= (D_1, D_2, D_3, D_4) \end{aligned}$$

then the inverse transformation is $g^{-1} : B \xrightarrow[\text{onto}]{1-1} A$ which can be written as

$$\begin{aligned} g^{-1} &= (g_1^{-1}(D_1, D_2, D_3, D_4),\ g_2^{-1}(D_1, D_2, D_3, D_4),\ g_3^{-1}(D_1, D_2, D_3, D_4),\ g_4^{-1}(D_1, D_2, D_3, D_4)) \\ &= (x, y, \theta, w) \end{aligned}$$

where

$$\begin{aligned} X &= D_1 - D_4\cos D_3 \\ Y &= D_2 - D_4\sin D_3 \\ \theta &= D_3 \\ W &= D_4. \end{aligned}$$

Let

$$J\left(\frac{x,y,\theta,w}{D_1,D_2,D_3,D_4}\right)=\begin{vmatrix}\frac{\partial x}{\partial D_1} & \cdot & \cdot & \cdot & \frac{\partial x}{\partial D_4}\\ \frac{\partial y}{\partial D_1} & \cdot & \cdot & \cdot & \frac{\partial y}{\partial D_4}\\ \frac{\partial \theta}{\partial D_1} & \cdot & \cdot & \cdot & \frac{\partial \theta}{\partial D_4}\\ \frac{\partial w}{\partial D_1} & \cdot & \cdot & \cdot & \frac{\partial w}{\partial D_4}\end{vmatrix}$$

be the Jacobian of inverse transformation g^{-1}. Then

$$J\left(\frac{x,y,\theta,w}{D_1,D_2,D_3,D_4}\right)=\begin{vmatrix}1 & 0 & D_4\sin D_3 & -\cos D_3\\ 0 & 1 & -D_4\cos D_3 & -\sin D_3\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{vmatrix}=1.$$

The joint probability density function of random variables D_1, D_2, D_3, D_4 is denoted by $h(D_1,D_2,D_3,D_4)$ and is given by

$$h(D_1,D_2,D_3,D_4)=|J|\cdot f\left(g_1^{-1},g_2^{-1},g_3^{-1},g_4^{-1}\right)$$
$$=\begin{cases}\dfrac{D_4^{\alpha-1}\exp\left\{-D_4/(R\beta)-\left[D_1^2+D_2^2+D_4^2-2D_4(D_1\cos D_3+D_2\sin D_3)\right]/20000\right\}}{40000\pi^2\Gamma(\alpha)(R\beta)^\alpha}, & (D_1,D_2,D_3,D_4)\in B\\ 0, & \text{elsewhere.}\end{cases}$$

Finally, the distribution of the target location point is denoted by $h(D_1,D_2)$ and is given by

$$h(D_1,D_2)$$
$$=\begin{cases}\dfrac{\exp\left\{-\left(D_1^2+D_2^2\right)/20000\right\}}{40000\pi^2(R\beta)^\alpha\Gamma(\alpha)}\displaystyle\int_0^{2\pi}\int_0^{\infty}\exp\left\{-\left[D_4^2-2D_4(D_1\cos D_3+D_2\sin D_3)+\frac{20000}{R\beta}D_4\right]/20000\right\}dD_4dD_3, & \\ & -\infty<D_1<\infty\\ & -\infty<D_1<\infty\\ 0, & \text{elsewhere.}\end{cases}$$

IV. Some Special Cases

Case 1. Suppose the initial point (X,Y) is predetermined and is equal to $X=a$, and $Y=b$. Then

$$D_1=a+W\cos\theta$$
$$D_2=b+W\sin\theta$$

Define the transformation $S:A_1\xrightarrow[\text{onto}]{1-1}B_1$ as

$$S=(S_1(w,\theta),S_2(w,\theta))=(D_1,D_2)$$

where

$$A_1=\{(w,\theta):0<w<\infty,\ 0\le\theta<2\pi\}$$

and

$$B_1 = \{(D_1, D_2) : -\infty < D_1 < \infty,\ -\infty < D_2 < \infty\}.$$

Then the inverse transformation $S^{-1} : B_1 \to A_1$ is a one-to-one and onto function defined by

$$S^{-1} = (S_1^{-1}(D_1, D_2), S_2^{-1}(D_1, D_2)) = (W, \theta)$$

where

$$W = \left[(D_1 - a)^2 + (D_2 - b)^2\right]^{\frac{1}{2}}$$

and

$$\theta = \arctan\left(\frac{D_2 - b}{D_1 - a}\right).$$

The Jacobian of inverse transformation is

$$J\left(\frac{W, \theta}{D_1, D_2}\right) = \begin{vmatrix} \frac{\partial W}{\partial D_1} & \frac{\partial W}{\partial D_2} \\ \frac{\partial \theta}{\partial D_1} & \frac{\partial \theta}{\partial D_2} \end{vmatrix} = \frac{1}{\left[(D_1 - a)^2 + (D_2 - b)^2\right]^{\frac{1}{2}}}.$$

The joint probability density function of the target location point conditioned upon knowledge of $(X, Y) = (a, b)$ is denoted by $h_1(D_1, D_2)$ and is given by

$$h_1(D_1, D_2) = \left| J\left(\frac{W, \theta}{D_1, D_2}\right)\right| \cdot f_W(S_1^{-1}) \cdot g(S_2^{-1})$$
$$= \begin{cases} \dfrac{\left[(D_1 - a)^2 + (D_2 - b)^2\right]^{\frac{\alpha}{2} - 1}}{2\pi (R\beta)^\alpha \Gamma(\alpha)} \cdot \exp\left\{-\left[(D_1 - a)^2 + (D_2 - b)^2\right]^{\frac{1}{2}} / (R\beta)\right\}, & \text{if } (D_1, D_2) \in B_1 \\ 0, & \text{elsewhere.} \end{cases}$$

Case 2. If $\theta = 0$, then the joint probability density function of D_1 and D_2 is denoted by $h_0(D_1, D_2)$ and is equal to

$$h_0(D_1, D_2)$$
$$= \begin{cases} \dfrac{\exp\left\{-[D_1^2 + D_2^2]\right\}}{20000\pi (R\beta)^\alpha \Gamma(\alpha)} \displaystyle\int_0^\infty D_4^{\alpha - 1} \exp\left\{-\left[D_4^2 - 2D_1 D_4 + \frac{20000}{R\beta} D_4\right] / 20000\right\} dD_4, & \begin{matrix} -\infty < D_1 < \infty \\ -\infty < D_2 < \infty \end{matrix} \\ 0, & \text{elsewhere.} \end{cases}$$

In general, if $\theta = C$, then the joint probability density function of D_1 and D_2 is denoted by $h_C(D_1, D_2)$ and is given by

$$h_C(D_1, D_2)$$
$$= \begin{cases} \dfrac{\exp\left\{-[D_1^2 + D_2^2]\right\}}{20000\pi (R\beta)^\alpha \Gamma(\alpha)} \displaystyle\int_0^\infty D_4^{\alpha - 1} \exp\left\{-\left[D_4^2 - 2D_4(D_1 \cos C + D_2 \sin C) + \frac{20000}{R\beta} D_4\right] / 20000\right\} dD_4, & \\ & -\infty < D_1 < \infty \\ & -\infty < D_2 < \infty \\ 0, & \text{elsewhere.} \end{cases}$$

References

[1] Hoaglin, D.C., Mosteller, F., Tukey, J., *Understanding Robust and Exploratory Data Analysis*, John Wiley & Sons, Inc. (1983).

[2] Rousseeuw, P.J., and Leroy, A.M., *Robust Regression and Outlier Detection*, John Wiley & Sons, Inc. (1987).

[3] Tabatabai, M.A., and Argyros, I.K., *Robust Estimation and Testing for General Nonlinear Regression Models*, Applied Math and Comp **58/1** (1993), 85–101.

Information for Authors cont.

Copyright. Authors will be asked, upon acceptance of an article, to transfer copyright of the article to the publisher. This will ensure the widest possible dissemination of information under copyright laws. Preprints are available via anonymous ftp at ftp://rattler.cameron.edu/SWJPAM-preprints. Authorization to copy is granted provided that the SWJPAM copyright symbol is not removed.

Tables. Tables must be numbered consecutively with roman numerals and have titles. They should be prepared in such a manner that no break is necessary.

Figures. All graphs and diagrams should be referred to as 'figure' (not abbreviated) in the text. A detailed caption should be provided for each figure, and figures should be numbered consecutively.

Footnotes. The use of footnotes should be minimized.

Notification. The author will be notified of the posting of his/her work and the time of release for SWJPAM.

References. References should be listed at the end of each paper. They should be numbered consecutively using arabic numbers between square brackets in the text. In the list of references, the numbers should appear without square brackets. A journal reference should comprise of name(s) and initial(s) of author(s), title of article, abbreviated name of journal (following the style of 'Abbreviations of Names of Serials reviewed in Mathematical Reviews'), volume number, year of publication (within parentheses), and full page numbers, e.g.

1. Argyros, I.K. and Szidarovszky, F.: On the Monotone Convergence of Algorithmic Models, Appl. Math. Com. 48, (1992), 167–176.

A book reference should comprise of name(s) and initial(s) of author(s), title of book, publisher, town, year of publication, e.g.

11. Ortega, J.M. and Rheinboldt, W.C.: Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1970

An edited volume reference should comprise of name(s) and initial(s) of author(s), title of paper, name(s) of editor(s), title of book, publisher, town, year, inclusive page numbers, e.g.

12. Prenter, P.: On Polynomial Operators and Equations, L.B. Rall (ed), Computational Solution of nonlinear operator equations, Johy Wiley Publisher, New York, 1971, pp. 361–398.

Indexing and Abstracting services. Articles published in SWJPAM are selectively indexed or abstracted in: Mathematical Reviews of A.M.S., Zentralblatt für Mathematik, European Mathematical Society (EMIS).

SOUTHWEST JOURNAL
OF
PURE AND APPLIED MATHEMATICS
VOLUME 1, JULY–DECEMBER 1995

CONTENTS

ISSN 1083-0464