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PERTURBED NEWTON-LIKE METHODS AND NONDIFFERENTIABLE OPERATOR EQUATIONS ON BANACH SPACES WITH A CONVERGENCE STRUCTURE

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Abstract. In this study, we use perturbed Newton-like methods to find solutions of nonlinear, nondifferentiable operator equations on Banach spaces with a convergence structure. This technique involves the introduction of a generalized norm as an operator from a linear space into a partially ordered Banach space. In this way the metric properties of the examined problem can be analyzed more precisely. Moreover, this approach allows us to derive from the same theorem, on the one hand, semi-local results of Kantorovich-type, and on the other hand, global results based on monotonicity considerations. Furthermore, we show that special cases of our results reduce to the corresponding ones already in the literature. Finally, our results are used to solve integral equations that cannot be solved with existing methods.

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I. INTRODUCTION

In this study, we are concerned with approximating a solution x^* of the nonlinear operator equation

$$F(x) + Q(x) = 0, \tag{1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in X, and Q is a nondifferentiable nonlinear operator with the same domain and values in X.

We introduce the perturbed Newton-like method

$$x_{n+1} = x_n + A(x_n)^*[-(F(x_n) + Q(x_n))] - z_n, \quad x_0 = 0 \quad (n \geq 0) \tag{2}$$

to approximate a solution x^* of equation (1). Here $A(x_n)^*$ $(n \geq 0)$ denotes a linear operator which is an approximation for $F'(x_n)^{-1}$ $(n \geq 0)$.

The importance of studying perturbed Newton-like methods comes from the fact that many commonly used variants of Newton's method can be considered procedures of this type. Indeed, approximation (2) characterizes any iterative process in which the corrections are taken as approximate solutions of the Newton equations. In particular for $A(x) = F'(x)$, $Q(x) = 0$ and $z_n = 0$ $(n \geq 0)$ we obtain the ordinary Newton-Kantorovich method [4].

The notion of a Banach space with a convergence structure was used in the elegant paper in [5] (see also [1], [2], [3], [8]) to solve equation (1), when $A(x) = F'(x)$, $Q(x) = 0$ for all $x \in D$ and $z_n = 0$ for all $n \geq 0$. However, there are many interesting real life applications already in the literature, where

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equation (1) contains a nondifferentiable term. See for example the applications at the end of this study. Moreover, since the Newton-like iterates can rarely be computed exactly, suitable choices of the points z_n will "correct" the sequence x_n and force it to converge to a solution x^* of equation (1). The case when $Q(x) = 0$ for all $x \in D$ and $F'(x_n)^* = F'(x_n)^{-1}$ $(n \geq 0)$ has already been considered but on a Banach space without generalized structure [3], [4], [7], [8], [9].

As in [6], we provide semi-local results of Kantorovich-type and global results based on monotonicity considerations from the same general theorem. Moreover, we show that our results can be reduced to the one obtained in [6], when $Q(x) = 0$ for all $x \in D$ and $z_n = 0$ $(n \geq 0)$, and furthermore to the ones obtained in [1], [2], [3], [5], [6], [7], [8] by further relaxing the requirements on X.

Finally, our results apply to solve a nonlinear integral equation involving a nondifferentiable term that cannot be solved with existing methods.

II. PRELIMINARIES

We will need the definitions:

Definition 1. The triple (X, V, E) is a Banach space with a convergence structure if

(C_1) $(X, \|\cdot\|)$ is a real Banach space;

(C_2) $(V, C, \|\ \|_v)$ is a real Banach space which is partially ordered by the closed convex cone C; the norm $\|\ \|_v$ is assumed to be monotone on C;

(C_3) E is a closed convex cone in $X \times V$ satisfying $\{0\} \times C \subseteq E \subseteq X \times C$;

(C_4) the operator $|\ |: D \to C$ is well defined:

$$|x| = \inf\{q \in C | (x, q) \in E\}$$

for

$$x \in D = \{x \in X | \exists q \in C : (x, q) \in E\};$$

and

(C_5) for all $x \in D$ $\quad \|x\| \leq \big\||x|\big\|_v$.

The set

$$U(a) = \{x \in X | (x, a) \in E\}$$

defines a sort of generalized neighborhood of zero.

Definition 2. An operator $L \in C^1(V_1 \to V)$ defined on an open subset V_1 of an ordered Banach space V is order convex on $[a, b] \subseteq V_1$ if

$$c, d \in [a, b],\ c \leq d \Rightarrow L'(d) - L'(c) \in L_+(V),$$

where for $m \geq 0$

$$L_+(V^m) = \{L \in L(V^m) | 0 \leq x_i \Rightarrow 0 \leq L(x_1, x_2, \ldots, x_m)\}$$

and $L(V^m)$ denotes the space of m-linear, symmetric, bounded operators on V.

Definition 3. The set of bounds for an operator $H \in L(X^m)$ is defined to be

$$B(H) = \{L \in L_+(V^m) | (x_i, q_i) \in E \Rightarrow (H(x_1, \ldots, x_m), L(q_1, \ldots, q_m)) \in E\}$$

Definition 4. Let $H \in L(X)$ and $y \in X$ be given, then

$$H^*(y) = z \Leftrightarrow z = T^\infty(0) = \lim_{n \to \infty} T^n(0),$$

$$T(x) = (I - H)(x) + y \Leftrightarrow z = \sum_{i=0}^{\infty} (I - H)^i y,$$

if this limit exists.

We will also need the Lemmas [2], [3], [5], [6]:

Lemma 1. Let $L \in L_+(V)$ and $a, q \in C$ be given such that:

$$L(q) + a \le q \text{ and } L^n(q) \to 0 \text{ as } n \to \infty. \tag{3}$$

Then the operator

$$(I - L)^* : [0, a] \to [0, a]$$

is well defined and continuous.

The following is a generalization of Banach's lemma [3], [5], [6].

Lemma 2. Let $H \in L(X)$, $L \in B(H)$, $y \in D$ and $q \in C$ be such that

$$L(q) + |y| \le q \text{ and } L^n(q) \to 0 \text{ as } n \to \infty.$$

Then the point $x = (I - H)^*(y)$ is well defined, $x \in S$ and

$$|x| \le (I - L)^* |y| \le q.$$

Moreover, the sequence

$$b_{n+1} = L(b_n) + |y|, \quad b_0 = 0$$

is well defined and

$$b_{n+1} \le q, \quad \lim_{n\to\infty} b_n = b = (I - L)^* |y| \le q.$$

III. CONVERGENCE ANALYSIS

Let K, L, M, M_1 be operators such that $L \in C^1(V_1 \to V)$, $K, M_1 \in L_+(V)$, $M \in C(V_1 \to V)$, and x_n be points in D. It is convenient for us to define the sequences $c_n, d_n, a_n, b_n \quad (n \ge 0)$ by

$$d_{n+1} = (L + K + M + M_1)(d_n) + L'(|x_n|)c_n, \quad d_0 = 0, \tag{4}$$

$$c_n = |x_{n+1} - x_n|, \tag{5}$$

$$a_n = (L + K + M + M_1)^n(a) \text{ for some } a \in C, \tag{6}$$

$$b_n = (L + K + M + M_1)^n(0), \tag{7}$$

and the point b by

$$b = (L + K + M + M_1)^\infty(0). \tag{8}$$

We can now state and prove the main result of this section.

Theorem 1. Let X be a Banach space with convergence structure (X, V, E) with $V = (V, C, \|\ \|_{\mathrm{v}})$, an operator $F \in C^1(D \to X)$ with $D \subseteq X$, an operator $Q \in C(D \to X)$, an operator $A(x) \in L(X)$, operators $L, L_1 \in C^1(V_1 \to V)$ with $V_1 \subseteq V$, operators $K, M_1(V_1 \to V)$ with $K, M_1 \in L_+(V)$, $M \in C(V_1 \to V)$, a point $a \in C$, and a null sequence $\{z_n\} \in D$ such that the following conditions are satisfied:

(C_6) the inclusions $U(a) \subseteq D$ and $[0, a] \subseteq V_1$ are true;

(C_7) L is order-convex on $[0, a]$ and satisfies

$$K + L'(|x| + |y|) - L'(|x|) \in B(A(x) - F'(x + y))$$

for all $x, y \in U(a)$ with $|x| + |y| \le a$;

(C_8) L_1 is order-convex on $[0, a]$ and satisfies

$$L_1'(|x|) - L_1'(0) \in B(A(0) - A(x))$$

for all $x, y \in U(a)$ with $|x| + |y| \le a$;

(C_9) the estimate $L_1' \le L'$ is true;

(C$_{10}$) M satisfies the conditions

$$0 \le (Q(x) - Q(x+y), M(|x| + |y|) - M(|x|) \in E$$

for all $x, y \in U(a)$ with $|x| + |y| \le a$,

$$M(w_1) - M(w_2) \le M(w_3) - M(w_4) \text{ and } M(w) \ge 0$$

for all $w, w_1, w_2, w_3, w_4 \in [0, a]$ with $w_1 \le w_3$ and $w_2 \le w_4$;

(C$_{11}$) M_1, x_n, z_n satisfy the inequality

$$0 \le (A(x_n)(z_n) - A(x_{n-1})(z_{n-1}), M_1(d_n - d_{n-1})) \in E \text{ for all } n \ge 1;$$

(C$_{12}$) $L_1'(0) \in B(I - A(0))$ and $\big(-(F(0) + Q(0) + A(0)(z_0)), L(0) + M(0) + M_1(0)\big) \in E$;

(C$_{13}$) $(L + K + M + M_1)(a) \le a$ with $0 \le L + K + M + M_1$;

and

(C$_{14}$) $(M + M_1 + L'(a) + K)^n a \to 0$ as $n \to \infty$.

Then

(i) The sequence $(x_n, d_n) \in (X \times V)^N$ is well defined, remains in E^N, is monotone and satisfies for all $n \ge 0$

$$d_n \le b$$

where d_n, b are given by (4) and (8) respectively and b is the smallest fixed point of $L + K + M + M_1$ in $[0, a]$.

(ii) Moreover, the iteration $\{x_n\}$ $n \ge 0$ generated by (2) converges to a solution $x^* \in U(b)$ of the equation $F(x) + Q(x) = 0$, which is unique in $U(a)$.

(iii) Furthermore, the following estimates are true for all $n \ge 0$:

$$b_n \le d_n \le b, \tag{9}$$

$$b_n \le a_n, \tag{10}$$

$$|x_{n+1} - x_n| \le d_{n+1} - d_n, \tag{11}$$

$$|x_n - x^*| \le b - d_n \tag{12}$$

and

$$|x_n - x^*| \le a_n - b_n \text{ for } M_1 = 0 \text{ and } z_n = 0 \quad (n \ge 0) \tag{13}$$

where d_n, a_n, b_n and b are given by (4), (6), (7) and (8) respectively.

Proof. We first note that b replacing a also satisfies the conditions of the theorem. Using conditions (C$_9$) and (C$_{10}$) we obtain

$$\begin{aligned}
|I - A(0)|(b) + |-(F(0) + Q(0) + A(0)(z_0)| &\le L_1'(0)(b) + L(0) + M(0) + M_1(0) \\
&\le L_1'(0)(b - 0) + L(0) + M(0) + M_1(0) \\
&\le (L_1'(0) + M + M_1)(b - 0) + L(0) + M(0) + M_1(0) \\
&\le L(b) - L(0) + M(b) - M(0) + M_1(b) - M_1(0) + L(0) + M(0) + M_1(0) \\
&\le (L + K + M + M_1)(b) = b.
\end{aligned}$$

Hence, by Lemma 2, x_1 is well defined and $(x_1, b) \in E$. We also get

$$x_1 = (I - A(0))(x_1) + \big[-\big(F(0) + Q(0) + A(0)(z_0)\big)\big]$$

$\Longrightarrow$

$$|x_1| \le L'(0)|x_1| + M(0) + L(0) + M_1(0) = d_1$$

and by the order convexity of L

$$
\begin{aligned}
d_1 &= L'(0)|x_1| + (M + M_1 + L)(0) \le L'(0)b + (M + M_1 + L)(0) \\
&\le \big(L'(0) + M + M_1\big)(b - 0) + (M + M_1 + L)(0) \\
&\le L(b) - L(0) + M(b) - M(0) + M_1(b) - M_1(0) + M(0) + M_1(0) + L(0) \\
&\le (M + M_1 + L + K)(b) = b.
\end{aligned}
$$

That is, we get $|x_1 - x_0| \le d_1 - d_0$ or

$$0 \le (x_0, d_0) \le (x_1, d_1).$$

We now assume that

$$0 \le (x_{k-1}, d_{k-1}) \le (x_k, d_k) \text{ for } k = 1, 2, \dots, n \text{ and } d_k \le b.$$

We need to find a bound for $I - A(x_n)$. We will show that $L_1'(|x_n|) \in B(I - A(x_n))$.

This fact follows from (C_7), (C_{12}) and the estimate

$$
\begin{aligned}
|I - A(x_n)| &\le |I - A(0)| + |A(0) - A(x_n)| \\
&\le L_1'(0) + L_1'(0 + |x_n|) - L_1'(0) = L_1'(|x_n|).
\end{aligned}
$$

Using (2) we obtain the approximation

$$
\begin{aligned}
-\big[F(x_n) + Q(x_n) + A(x_n)(z_n)\big] &= -F(x_n) - Q(x_n) - A(x_n)(z_n) \\
&\quad + A(x_{n-1})(x_n - x_{n-1}) + F(x_{n-1}) + Q(x_{n-1}) + A(x_{n-1})(z_{n-1}).
\end{aligned}
$$

Hence, by (C_7), (C_{10}), (C_{11}) and the induction hypotheses, we obtain in turn

$$
\begin{aligned}
&| - F(x_n) + F(x_{n-1}) + A(x_{n-1})(x_n - x_{n-1})| + |Q(x_{n-1}) - Q(x_n)| + |A(x_n)(z_n) - A(x_{n-1})(z_{n-1})| \\
&\le \int_0^1 [L'(|x_{n-1}| + tc_{n-1}) - L'(|x_{n-1}|) + K]\, c_{n-1}\, dt + M(|x_{n-1}| + |x_n - x_{n-1}|) - M(|x_{n-1}|) \\
&\qquad + M_1(d_n - d_{n-1}) \\
&\le L(|x_{n-1}| + c_{n-1}) - L(|x_{n-1}|) - L'(|x_{n-1}|)c_{n-1} + Kc_{n-1} + M(|x_{n-1}| + |x_n - x_{n-1}|) - M(|x_{n-1}|) \\
&\qquad + M_1(d_n - d_{n-1}) \\
&\le L(d_{n-1} + d_n - d_{n-1}) - L(d_{n-1}) - L'(|x_{n-1}|)c_{n-1} + K(d_n - d_{n-1}) \\
&\qquad + M(d_{n-1} + d_n - d_{n-1}) - M(d_{n-1}) + M_1(d_n - d_{n-1}) = (M + M_1 + L + K)(d_n) - d_n.
\end{aligned} \tag{14}
$$

We can now obtain that

$$
\begin{aligned}
&L_1'(|x_n|)(b - d_n) + | - (F(x_n) + Q(x_n) + A(x_n)(z_n))| + d_n \\
&\le L'(d_n)(b - d_n) + L(d_n) + M(d_n) + M_1(d_n) + K(d_n) - d_n + d_n \\
&\le (L'(d_n) + M + M_1 + K)(b - d_n) + (L + K + M + M_1)(d_n) \\
&\le K(b - d_n) + L(b) - L(d_n) + M(b) - M(d_n) + M_1(b) - M_1(d_n) + (L + K + M + M_1)(d_n) \\
&= (M + M_1 + L + K)(b) = b.
\end{aligned}
$$

That is, x_{n+1} is also well defined by Lemma 2 and $c_n \le b - d_n$. Hence d_{n+1} is well defined too and we obtain:

$$
\begin{aligned}
d_{n+1} &\le (M + M_1 + L + K)(d_n) + L'(d_n)(b - d_n) \le (M + M_1 + L + K)(d_n) \\
&\quad + (L'(d_n) + M + M_1)(b - d_n) \\
&\le (M + M_1 + L + K)(d_n) + L(b) - L(d_n) + M(b) - M(d_n) + M_1(b) - M_1(d_n) + K(b - d_n) \\
&= (M + M_1 + L + K)(b) = b.
\end{aligned}
$$

The monotonicity $(x_n, d_n) \le (x_{n+1}, d_{n+1})$ can be derived from

$$\begin{aligned} c_n + d_n &\le L'(|x_n|)c_n + |-(F(x_n) + Q(x_n) + A(x_n)(z_n))| + d_n \\ &\le L'(|x_n|)c_n + (M + M_1 + L + K)d_n = d_{n+1}. \end{aligned}$$

The induction has now been completed. We need to show that

$$b_n \le d_n \text{ for all } n \ge 1.$$

For $n = 1$ and from the definitions of b_n, d_n

$$b_1 = (L'(0) + K + M + M_1)^1(0) \le d_1.$$

Assume that

$$b_k \le d_k \text{ for } k = 1, 2, \dots, n.$$

Then, we obtain in turn

$$\begin{aligned} b_{n+1} = (L'(0) + K + M + M_1)^{n+1}(0) &= (L'(0) + K + M + M_1)(L'(0) + K + M + M_1)^n(0) \\ &\le (L'(0) + K + M + M_1)(d_n) \\ &\le L'(0)(d_n) + (M + K + M_1)(d_n) \\ &\le L(d_n) - L(0) + (M + K + M_1)(d_n) \\ &\le (L + K + M + M_1)d_n \le d_n \le d_{n+1}. \end{aligned}$$

Since $d_n \le b$, we have $b_n \le d_n \le b$. We also need to show that

$$0 \le a_n - b_n \le (L'(a) + K + M + M_1)^n(a) \qquad (n \ge 1). \tag{15}$$

For $n = 1$, we get

$$\begin{aligned} a_1 - b_1 &= (L + K + M + M_1)(a) - (L + K + M + M_1)(0) = (L(a) - L(0)) + (M(a) - M(0)) \\ &+ K(a - 0) + (M_1(a) - M_1(0)) = \int_0^1 [L'(ta) + K + M + M_1](a)\, dt \\ &\le \int_0^1 (L'(a) + K + M + M_1)(a)\, dt = (L'(a) + K + M + M_1)^1(a), \end{aligned}$$

which shows (15) for $n = 1$. Let us now assume that (15) is true for $k = 1, 2, \dots, n$, then we can obtain in turn

$$\begin{aligned} a_{n+1} - b_{n+1} &= (L + K + M + M_1)^{n+1}(a) - (L + K + M + M_1)^n(0) \\ &= (L + K + M + M_1)\big[(L + K + M + M_1)^n(a)\big] - (L + K + M + M_1) \\ &\quad \cdot \big[(L + K + M + M_1)^n(0)\big] \\ &= L\big[(L + K + M + M_1)^n(a)\big] - L\big[(L + K + M + M_1)^n(0)\big] \\ &+ (M + K + M_1)\big[(L + K + M + M_1)^n(a)\big] - (M + K + M_1)\big[(L + K + M + M_1)^n(0)\big] \\ &= \int_0^1 L'\big[(L + K + M + M_1)^n(0) + t\big((L + K + M + M_1)^n(a) - (L + K + M + M_1)^n(0)\big)\big] \\ &\quad \cdot \big[(L + K + M + M_1)^n(a) - (L + K + M + M_1)^n(0)\big]\, dt \\ &+ (M + K + M_1)\big[(L + K + M + M_1)^n(a) - (L + K + M + M_1)^n(0)\big] \\ &\le \int_0^1 L'\big((L + K + M + M_1)^n(a)\big)(L'(a) + K + M + M_1)^n(a)\, dt \\ &\quad + (M + K + M_1)\big((L'(a) + K + M + M_1)^n(a)\big) \\ &\le L'(a)(L'(a) + K + M + M_1)^n(a) + (M + K + M_1)(L'(a) + K + M + M_1)(a) \\ &= (L'(a) + K + M + M_1)(a). \end{aligned}$$

By condition (C_{14}) and the above, we deduce that the sequence $\{b_n\}$ $n \geq 0$ is Cauchy in a Banach space C, and as such, it converges to some $b = (L + K + M + M_1)^{\infty}(0)$. From $(L + K + M + M_1)(b) = (L + K + M + M_1)\big(\lim_{n\to\infty}(L + K + M + M_1)^n(0)\big) = \lim_{n\to\infty}(L + K + M + M_1)^{n+1}(0) = b$, we obtain

$$(L + K + M + M_1)(b) = b \leq a,$$

which makes b smaller than any solution of the inequality

$$(L + K + M + M_1)(p) \leq p. \qquad \text{(see also [2], [3], [5]).}$$

It also follows that the sequence $\{x_n\}$ $n \geq 0$ is Cauchy in X, and as such, it converges to some $x^* \in U(b)$. By letting $n \to \infty$ in (14) and using the hypotheses that $\lim_{n\to\infty} z_n = 0$, we deduce that x^* is a solution of the equation $F(x) + Q(x) = 0$.

To show uniqueness, let us assume that there exists another solution y^* of the equation $F(x)+Q(x) = 0$ in $U(a)$. Then, exactly as in [1], [2], [3], [5], [6], by considering the modified Newton-process

$$x_{n+1} = x_n - (F(x_n) + Q(x_n)),$$

we can show that this sequence converges, under the hypotheses of the theorem. Moreover, as above, we can easily show (see also [1], [2], [3], [5]) that

$$|y^* - x_n| \leq a_n - b_n \text{ for } M_1 = 0 \text{ and } z_n = 0 \ (n \geq 0),$$

from which follows that $x_n \to y^*$ as $n \to \infty$. Finally, estimates (12) and (13) are obtained by using standard majorization techniques [3], [4], [9].

That completes the proof of the theorem.

We will now introduce results on a posteriori estimates. It is convenient to define the operator

$$R_n(q) = (I - L'(|x_n|))^* S_n(q) + c_n$$

where

$$S_n(q) = (L + K + M + M_1)(|x_n| + q) - (L + K + M + M_1)(|x_n|) - L'(|x_n|)(q)$$

and the interval

$$I_n = [0, a - |x_n|].$$

It can easily be seen that the operators S_n are monotone on I_n. Moreover, the operators $R_n : [0, a - d_n] \to [0, a - d_n]$ are well defined and monotone. This fact follows from lemma 1 and the scheme

$$d_n + c_n \leq d_{n+1} \Rightarrow R(a) - d_{n+1} \leq a - d_n - c_n \Rightarrow S_n(a - d_n) + L'(|x_n|)(a - d_n - c_n) \leq a - d_n - c_n \quad (n \geq 0).$$

Then, exactly as in [6], we can show:

Proposition. The following implications are true:

(i) if $q \in I_n$ satisfy $R_n(q) \leq q$, then

$$c_n \leq R_n(q) = p \leq q$$

and

$$R_{n+1}(p - c_n) \leq p - c_n \text{ for all } n \geq 0;$$

(ii) under the hypotheses of Theorem 1, let $q_n \in I_n$ be a solution of $R_n(q) \leq q$, then

$$|x^* - x_m| \leq a_m \qquad (m \geq n)$$

where

$$a_n = q_n \text{ and } a_{m+1} = R_m(a_m) - c_m;$$

and

(iii) under the hypotheses of Theorem 1, any solution $q \in I_n$ of $R_n(q) \leq q$ is such that

$$|x^* - x_n| \leq R_n^\infty(0) \leq q.$$

Remark 1.

(a) The results obtained in Theorem 1 and the proposition reduce immediately to the corresponding ones in [6] (Theorem 5 and Lemmas 10–12) when $A(x) = F'(x_n)^{-1}$, $Q(x) = 0$ for all $x \in D$ and $z_n = 0$ for all $n \geq 0$ (M, M_1, K will be 0 then and $L = L_1$).

(b) As in [2], [3], [5], we can also show that if conditions (C_6)–(C_{12}) are satisfied and there exists $t \in (0,1)$ such that $(L + K + M + M_1)(a) \leq ta$, then there exists $a_1 \in [0, ta]$ satisfying conditions (C_6)–(C_{14}). The solution $x^* \in U(a_1)$ is unique in $U(a)$.

(c) From the approximation

$$A(x_n)(z_n) - A(x_{n-1})(z_{n-1}) = (A(x_n)(z_n) - z_n) + (-A(x_{n-1})(z_{n-1}) + z_{n-1}) + I(z_n - z_{n-1})$$

we observe that (C_{11}) will be true if

$$M_1 = 2L_0'(b) + I \text{ with } L_0 = L_1 \text{ or } L$$

and z_n are chosen so that

$$|z_n| + |z_{n-1}| + |z_n - z_{n-1}| \leq d_n - d_{n-1} \qquad (n \geq 1).$$

(d) Another choice for M_1, z_n can be $M_1 = |\varepsilon| I$ and $z_n = z_{n-1} + \varepsilon_n (x_n - x_{n-1})$ $(n \geq 1)$ and $|\varepsilon_n| \leq |\varepsilon|$ $(n \geq 0)$ provided that $F'(x) = I$ for all $x \in D$. It can then easily be seen that (C_{11}) is satisfied. The sequence z_n must still be chosen to be null. At the end of this paper, in Part V, we have given examples to cover this case. Several other choices are also possible.

(e) From the proof of the theorem, it can easily be seen that condition (C_9) can be replaced by the weaker condition (which is what we actually need)

$$L_1'(d_n)(b - d_n) \leq L(b) - L(d_n) \quad (n \geq 0).$$

This condition however is not easy to verify in general.

(f) It can easily be seen that the conclusion of Theorem 1 remains valid if in the hypotheses L_1 replaces L everywhere except in condition (C_7).

(g) Condition (C_7) can be replaced by the set of conditions

$$K + L_2'(|x| + |y|) - L_2'(|x|) \in B(F'(x) - F'(x + y))$$

and

$$L_3'(|x|) - L_3'(|0|) \in B(A(x) - F'(x))$$

for all $x, y \in [0, a]$ with $|x| + |y| \leq a$ as long as we can find L such that

$$L'(|x| + |y|) = L_3'(|x|) + L_2'(|x| + |y|) + K \text{ and } L'(|x|) = L_2'(|x|) + L_3'(|0|),$$

for all $x, y \in [0, a]$ with $|x| + |y| \leq a$. The operators $L, L_2, L_3 \in C^1(V_1 \to V)$ defined here must also be order convex.

(h) It can easily be seen from the proof of theorem 1 that the results obtained in theorem 1 remain valid if (C_{11}) is replaced by the weaker condition:

$$0 \leq (A(x_n)(z_n) - A(x_{n-1})(z_{n-1}), M_2|x_n - x_{n-1}|) \in E \quad (n \geq 1)$$

for some $M_2 \in L_+(V)$. This condition seems to be more helpful since it relates z_n with x_n and not an unknown sequence depending on M_1. Under this condition the term $M_1(b-b_n)$ appearing in the Proposition can be replaced by $M_2|x_n - x_{n-1}|$ $(n \geq 1)$. Other choices involving M_2 are possible.

(i) Define the residuals $r_n = -A(x_n)(z_n)$ $(n \geq 0)$ and set $\delta_n = x_{n+1} - x_n$ $(n \geq 0)$. Then from the approximation

$$r_n = [(I - A(x_n)) - I]\,(z_n)$$

we obtain

$$|r_n| \leq (L_1'(|x_n|) + I)\,|z_n|$$

which shows that $r_n \to 0$ as $n \to \infty$ if $z_n \to 0$ as $n \to \infty$. Consequently, the conclusions of theorem 1 remain true for the system

$$x_{n+1} - x_n = \delta_n, \quad A(x_n)\delta_n = -(F(x_n) + Q(x_n)) + r_n \quad (n \geq 0).$$

(j) (C_{11}) is equivalent to the condition that $(A(x_n)(z_n), M_1(d_n)) \in E^N$ $(n \geq 1)$ is an increasing sequence. Similar observations can be made for (C_{10}).

IV. THE MONOTONE CASE

Let $A \in L(X \to X)$ be a given operator. Define the operators $P, T(D \to X)$ by

$$P(x) = AT(x+u)$$
$$T(x) = G(x) + R(x), \quad P(x) = F(x) + Q(x)$$

and

$$F(x) = AG(x+u), \quad Q(x) = AR(x+u),$$

where G, R are as F, Q respectively. We deduce immediately that under the hypotheses of Theorem 1, the zero x^* of P is a zero of AT also, if $u = 0$.

We will now provide a monotonicity result to find a zero x^* of AT. The space X is assumed to be partially ordered and satisfies the conditions for V given in (C_1)–(C_5). Moreover, we set $X = V$, $D = C^2$ so that $|\ |$ turns out to be I.

We can now state and prove the main result of this section.

Theorem 2. Let V be a partially ordered Banach space satisfying conditions (C_1)–(C_5), Y be a Banach space, $G \in C^1(D \to Y)$, $R \in C(D \to Y)$ with $D \subseteq V$, $A, A(x) \in L(X \to V)$, $K, M_1(D \to V)$ with $K, M_1 \in L_+(V)$, $M \in C(V_1 \to V)$ and $u, v \in V$ such that:

(C_{15}) $[u,v] \subseteq D$;

(C_{16}) $2I - AG'(u) - A(v-u) + M + K + M_1 \in L_+(V)$;

(C_{17}) for all $w, w_1, w_2 \in [u,v] : w_1 \leq w_2 \Rightarrow AG'(u+w) \leq I$,

$$A(w_2) \leq A(w_1)$$
$$A(w_2) \leq AG'(w_1+u) + K$$
$$A(w_1) - A(w_2) + AG'(u+w_1) - AG'(u+w_2) \geq 0$$

and

$$-(Q(0) + A(0)(z_0)) \leq (M + K + M_1)(0);$$

(C_{18}) $AT(u) + A(u)(z_0) \leq 0$, $AT(v) + A(v)(z_0) \geq 0$ and $v - u \leq \int_0^{v-u} A(z)dz + AG(v) - (M+K+M_1)(v-u)$;

(C_{19}) condition (C_{10}) is satisfied;

(C_{20}) condition (C_{11}) is satisfied; and

(C_{21}) $(2I - AG'(v) - A(v-u) + M + K + M_1)^n (v-u) \to 0$ as $n \to \infty$.

Then the Newton sequence

$$y_0 = u, \quad y_{n+1} = y_n + (A(y_n))^*[-AT(y_n)] - z_n \qquad (n \geq 0)$$

is well defined for all $n \geq 0$, monotone and converges to a unique zero x^* of AT in $[u, v]$.

Proof. The proof follows immediately by setting $P(x) = AT(x+u)$, $L_1(e) = e - \int_0^e A(z)dz$, $L_2(e) = e - F(e)$, $L = L_1 + L_2$ and $a = v - u$.

Remark 2.

(a) By setting $R(x) = 0$ for all $x \in D$ and $z_n = 0 \quad (n \geq 0)$, we immediately deduce that our results reduce to the ones obtained in [6, thm 13].

(b) If M, R are linear operators, then condition (C_{19}) will certainly be satisfied if we set $M(e) = -Q(e)$.

(c) The operators L, L_1, L_2 can be chosen in general as $L_1(e) = z_1 + R_1(e) - \int_0^e A(z)dz$, $L_2 = z_2 + R_2(e) - F(e)$ and $L = L_1 + L_2$ with $z_1, z_2 \in [u, v]$ and $R_1, R_2 \in L(D \to Y)$. Conditions similar to the ones in Theorem 2 can then be easily produced for the above choices of L, L_1, L_2. For example, conditions (C_{16}) and (C_{21}) will become

$$R_1 + R_2 + K - AG'(u) - A(v-u) + M_1 + M \in L_+(V)$$

and

$$(R_1 + R_2 - AG'(u) - A(v-u) + M_1 + K + M)^n(v-u) \to 0 \text{ as } n \to \infty$$

respectively.

V. APPLICATIONS

We will complete this study with two examples that show how to choose L, M, M_1, z_n in practical applications. For simplicity, we assume that $A(x) = F'(x)$ for all $x \in D$.

Example 1. We discuss the case of a real Banach space with norm $\|\ \|$. Assume for simplicity that $F'(0) = I$ and there exists a monotone operator

$$f : [0, a] \to IR$$

such that

$$\|F''(x)\| \leq f(\|x\|) \text{ for all } x \in U(a)$$

and a continuous nondecreasing function g on $[0, r]$, $r \leq a$ such that

$$\|Q(x) - Q(y)\| \leq g(r)\|x - y\| \tag{16}$$

for all $x, y \in U(\frac{r}{2})$.

We showed in [3], (see also [9]) that (16) implies that

$$\|Q(x+\ell) - Q(x)\| \leq h(r + \|\ell\|) - h(r), \quad x \in U(a), \quad \|\ell\| \leq a - r, \tag{17}$$

where

$$h(r) = \int_0^r g(t)\, dt.$$

Conversely, it is not hard to see that we may assume, without loss of generality, that the function h and all functions $h(r+t) - h(r)$ are monotone in r. Hence, we may assume that $h(r)$ is convex and hence differentiable from the right. Then, as in [9], we show that (17) implies (16) and $g(r) = h'(r+0)$. Hence, we can set

$$L_1(q) = L(q) = \|F(0) + Q(0)\| + \int_0^q ds \int_0^s f(t)\, dt \tag{18}$$

and

$$M(q) = \int_0^q g(t)\, dt. \tag{19}$$

In Remark 1 (c) and (d) we have already provided some choices for M_1, z_n. Here, however, for simplicity let us choose $z_n = 0 \quad (n \geq 0)$ and $M_1 = 0$.

Then condition (C_{13}) will be true if

$$\frac{1}{2} f(a)a^2 - (1 - g(a))a + \|F(0) + Q(0)\| \leq 0. \tag{20}$$

If we set $Q = 0$ and $g = 0$, (20) is true if $\|F(0)\| f(a) \leq \frac{1}{2}$, which is a well known condition due to Kantorovich [4, Ch. 18]. If $Q \neq 0$, condition (20) is the same condition with the one found in [3], [9] for the Zincenko iteration.

In the example that follows, we show that our results can apply to solve nonlinear integral equations involving a nondifferentiable term, whereas the results obtained in [6] (or [5], [7], [8]) cannot apply.

Example 2. Let $X = V = C[0,1]$, and consider the integral equation

$$x(t) = \int_0^1 k(t, s, x(s))\, ds \text{ on } X, \tag{21}$$

where the kernel $k(t, s, x(s))$ with $(t, s) \in [0,1] \times [0,1]$ is a nondifferentiable operator on X. Consider (21) in the form (1) where $F, Q : X \to X$ are given by

$$F(x)(t) = Ix(t)$$

and

$$Q(x)(t) = -\int_0^1 k(t, s, x(s))\, ds.$$

The operator $|\ |$ is defined by considering the sup-norm. We assume that V is equipped with natural partial ordering, and there exists α, $a \in [0, +\infty)$, and a real function $\alpha(t, s)$ such that

$$\|k(t, s, x) - k(t, s, y)\| \leq \alpha(t, s)\|x - y\|$$

for all $t, s \in [0,1]$, $x, y \in U(\frac{a}{2})$, and

$$\alpha \geq \sup_{t \in [0,1]} \int_0^1 \alpha(t, s)\, ds.$$

Define the real functions h, f, g on $[0, a]$ by $h(r) = \alpha r$, $f(r) = 0$ and $g(r) = \alpha$ for all $r \in [0, a]$. By choosing L, M, M_1 as in (18) and (19) and Remark 1 (d) respectively, and by setting $K = 0$, we can easily see that the conditions (C_1)–(C_{12}) of Theorem 1 are satisfied. In particular, condition (C_{13}) becomes

$$(1 - \alpha - |\epsilon|)a - \|Q(0)\| \geq 0 \tag{22}$$

which is true in the following cases: if $0 \leq \alpha < 1 - |\epsilon|$, choose $a \geq \beta = \frac{\|Q(0)\|}{1-\alpha-|\epsilon|}$; if $\alpha = 1 - |\epsilon|$ and $Q(0) = 0$, choose $a \geq 0$; if $\alpha > 1 - |\epsilon|$ and $Q(0) = 0$, choose $a = 0$. If in (22) strict inequality is valid, then there exists a solution a^* of equation (22) satisfying condition (C_{14}). Note that if we choose $\alpha \in (0, 1 - |\epsilon|)$, $a \in (\beta, +\infty)$ and $\epsilon \in (-1, 1)$, condition (22) is valid as a strict inequality. Finally, we remark that the results obtained in [6] (or [5], [7], [8]) cannot apply here to solve equation (21), since Q is nondifferentiable on X and the z_n's are not necessarily zero.

CONCLUSION

In this study, we used perturbed Newton-like methods to find solutions of nonlinear, nondifferentiable operator equations on Banach spaces with a generalized structure. This technique involves the introduction of a generalized norm as an operator from a linear space into a partially ordered Banach space. This way the metric properties of the examined problem have been shown to be analyzed more precisely. Convergence results and error estimates have been improved compared with the real norm theory. Moreover, this approach also allowed us to derive, on the one hand, semi-local results of Kantorovich-type and, on the other hand, global results base on monotonicity considerations, from the same general theorem. All our convergence theorems were derived from metric estimates. Furthermore, we show that special cases of our results reduce to the corresponding ones already in the literature [1], [2], [5], [6], [7]. Finally, we used our results to solve applied problems that cannot be solved with existing methods.

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