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AN L_p-CRITERION OF AMENABILITY FOR A LOCALLY COMPACT GROUP

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Abstract. In this note we establish a criterion of amenability for a subgroup H of a second countable locally compact topological group G in terms of the left regular representation of H in $L_p(G)$.

1. Introduction

Throughout, we assume all topological groups separated.

Let G be a topological group and let $L_p(G)$ be the space of all complex-valued functions on G integrable to the power p over G with respect to a left-invariant Haar measure μ_G. The group G acts on $L_p(G)$ by the *left regular representation* $\lambda_G : G \to \mathcal{B}(L_p(G))$:

$$(\lambda_G(g)f)(x) = f(g^{-1}x), \quad g \in G,\ x \in G.$$

Here $\mathcal{B}(V)$ stands for the space of all bounded linear endomorphisms of a Banach space V.

A locally compact topological group is called *amenable* [5] if there exists a G-invariant mean on $L_\infty(G)$ or, equivalently, G possesses the *fixed point property*: for every continuous affine action on a separated locally convex space W and every convex compact set $Q \subset W$, there is a fixed point for G in Q.

Let V be a Banach G-module, i.e., a real or complex Banach space endowed with a continuous linear representation $\alpha : G \to \mathcal{B}(V)$. We say that V *almost has invariant vectors* if, for every compact subset $F \subset G$ and every $\varepsilon > 0$, there exists a unit vector $v \in V$ such that $\|\alpha(g)v - v\| \leq \varepsilon$ for all $g \in F$.

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Recently Bourdon, Martin, and Valette proved the following assertion ([3], Lemma 2).

Theorem 1. *Suppose that $p \in [1, \infty[$. Let X be a countable set on which a countable group H acts freely. The following are equivalent:*

(i) *The natural "permutation" representation λ_X of H on $L_p(X)$ almost has invariant vectors;*

(ii) *H is amenable.*

The main result of this note is the following generalization of Theorem 1:

Theorem 2. *Assume that $p \in [1, \infty[$. Let G be a second countable locally compact group and let H be a closed subgroup in G. The following are equivalent:*

(i) *The left regular representation of H on $L_p(G)$ almost has invariant vectors;*

(ii) *H is amenable.*

The reader is referred to [6] for an interesting investigation into unitary representations of amenable and non-amenable connected locally compact groups in terms of the reduced 1-cohomology.

2. Prerequisites

Before proving Theorem 2, we need to recall some basic facts and definitions in the theory of integration on locally compact groups.

Let G be a locally compact group and let H be a closed subgroup in G. Denote by μ_G and μ_H left-invariant Haar measures on G and H respectively and denote by π the projection $G \to G/H$.

Denote by Δ_K the modulus of a locally compact group K.

Given a function f and a class $u \in G/H$, take an arbitrary representative x in u and consider the function $\alpha : y \to f(xy)$ on H. If α is integrable over H, the left invariance of μ_H implies that $\int\limits_H f(xy)d\mu_H(y)$ is independent of the choice of x with $\pi(x) = u$.

It is well known that the homogeneous space G/H admits a *quasi-G-invariant* measure $\mu_{G/H}$ on H which is unique up to equivalence. Here the "quasi-G-invariance" means that all left translates of $\mu_{G/H}$ by the elements of G are equivalent to $\mu_{G/H}$. The measure $\mu_{G/H}$ can be described as follows (see [2], Chapter VII, 2.5 or [5]).

(a) There exists a positive continuous function $\rho > 0$ on G such that $\rho(xy) = \frac{\Delta_H(y)}{\Delta_G(y)}\rho(x)$ for all $x \in G$ and $y \in H$.

Put $\mu_{G/H} = (\rho\mu_G)/\mu_H$ (see [2], Definition 1 in Chapter VII, 2.2).

(b) If $f \in L_1(G, \rho\mu_G)$ then the set of $\overline{x} = \pi(x) \in G/H$ for which $y \mapsto f(xy)$ is not μ_H-integrable is $\mu_{G/H}$-negligible, the function $\overline{x} = \pi(x) \mapsto \int\limits_H f(xy)dy$ is $\mu_{G/H}$-integrable, and

$$\int\limits_G f(x)\rho(x)d\mu_G(x) = \int\limits_{G/H} d\mu_{G/H}(\overline{x}) \int\limits_H f(xy)d\mu_H(y).$$

(c) There exists a nonnegative continuous function h on G with $\int\limits_H h(xy)dy = 1$ for all $x \in G$ such that a function w on G/H is $\mu_{G/H}$-measurable ($\mu_{G/H}$-integrable) if and only if $h(w \circ \pi)$ is $\rho\mu_G$-measurable ($\rho\mu_G$-integrable). If $w \in L_1(G/H, \mu_{G/H})$ then

$$\int\limits_{G/H} w(u)d\mu_{G/H}(u) = \int\limits_{G} h(x)w(\pi(x))\rho(x)d\mu_G(x).$$

Note that a second countable locally compact space is Polish (*polonais*) (see [2]). As follows from Dixmier's lemma (see [4]), if G is a Polish group and H is a closed subgroup in G then there exists a Borel section $\sigma : G/H \to G$ (in particular, $\pi \circ \sigma = \mathrm{id}_{G/H}$). We will need the following technical assertion.

Lemma 1. *Suppose that G is a second countable locally compact group, H is a closed subgroup in G, $\sigma : G/H \to G$ is a Borel section, and $f \in L_1(G, \rho\mu_G)$. Then, in the above notations,*

$$\int\limits_{G} f(x)\rho(x)d\mu_G(x) = \int\limits_{H} d\mu_H(y) \int\limits_{G/H} f(\sigma(\overline{x})y)d\mu_{G/H}(\overline{x}).$$

Proof. By (b) and (c), we infer

$$\int\limits_{G} f(x)\rho(x)d\mu_G(x) = \int\limits_{G/H} d\mu_{G/H}(\overline{x}) \int\limits_{H} f(\sigma(\overline{x})y)d\mu_H(y) =$$

$$= \int\limits_{G} h(x)\rho(x)\Big(\int\limits_{H} f(xy)d\mu_H(y)\Big)d\mu_G(x) =$$

$$= \int\limits_{H} d\mu_H(y) \int\limits_{G} h(x)f(xy)\rho(x)d\mu_G(x) = \int\limits_{H} d\mu_H(y) \int\limits_{G/H} f(\sigma(\overline{x})y)d\mu_{G/H}(\overline{x}).$$

Here the third equality is guaranteed by the *scholie* in [1], p.96, since G is a countable union of compact sets and we may write the first two equalities also for $|f|$ and see that

$$\int\limits_{G} h(x)\rho(x)\Big(\int\limits_{H} |f(xy)|d\mu_H(y)\Big)d\mu_G(x)$$

exists. The lemma is proved.

3. Proof of Theorem 2

As in [3], we remark that (i) implies (ii) by the equivalence of amenability and the fulfillment of Reiter's condition (P_p) [5], p. 28:

(P_p) For every compact set F and every $\varepsilon > 0$, there exists a function $f \in L_p(H)$ with $f \geq 0$ and $\|f\|_{L_p(H)} = 1$ such that $\|\lambda_H(z)f - f\|_{L_p(H)} < \varepsilon$ for all $z \in F$.

Now, prove (ii)$\Rightarrow$(i). We suppose that $L_p(G)$ almost has invariant vectors for H and deduce from this that H meets Reiter's condition (P_1).

Take $\varepsilon > 0$ and a compact set $F \subset H$; choose $f \in L_p(G)$, $\|f\|_{L_p(G)} = 1$ with $\|\lambda_G(z)f - f\|_{L_p(G)} \leq \frac{\varepsilon}{2p}$ for all $z \in F$. We may assume that $\psi \geq 0$ by replacing f with $|f|$. If $z \in F$ then, putting $\varphi = f^p$, we have

$$\|\lambda_G(z)\varphi - \varphi\|_{L_1(G)} = \int\limits_{G} |f(z^{-1}x)^p - f(x)^p|d\mu_G(x)$$

$$\le p\int\limits_G |f(z^{-1}x)-f(x)||f(z^{-1}x)^{p-1}+f(x)^{p-1}|d\mu_G(x)$$

$$\le p\Big(\int\limits_G |f(z^{-1}x)-f(x)|^p d\mu_G(x)\Big)^{1/p}\Big(\int\limits_G (f(z^{-1}x)^{p-1}+f(x)^{p-1})^{p/(p-1)}d\mu_G(x)\Big)^{\frac{p-1}{p}}$$

$$\le p\|\lambda_G(z)f-f\|_{L_p(G)}\Big(2^{\frac{1}{p-1}}\int\limits_G (f(z^{-1}x)^p+f(x)^p)d\mu_G(x)\Big)^{\frac{p-1}{p}}=$$

$$=2p\|\lambda_G(z)f-f\|_{L_p(G)}\le\varepsilon.$$

This follows from the inequality $|a^p-b^p|\le p|a-b|(a^{p-1}+b^{p-1})$ $(a,b\ge 0)$; Hölder's inequality, the relation

$$(a+b)^{\frac{p}{p-1}}\le 2^{\frac{1}{p-1}}(a^{\frac{p}{p-1}}+b^{\frac{p}{p-1}}),\quad a,b\ge 0,$$

and the assumption that $\|f\|_{L_p(G)}=1$.

Now, let $\sigma: G/H\to G$ be a Borel section. Define a function Φ on H by setting

$$\Phi(y)=\int\limits_{G/H}\frac{\varphi(y\sigma(\overline{x}))}{\rho(\sigma(\overline{x}))}d\mu_{G/H}(\overline{x}),\quad y\in H.$$

Reckoning with Lemma 1, we obtain the following estimates:

$$\|\lambda_H(z)\Phi-\Phi||_{L_1(H)}=\int\limits_H\Big|\int\limits_{G/H}\frac{\varphi(z^{-1}y\sigma(\overline{x}))-\varphi(y\sigma(\overline{x}))}{\rho(\sigma(\overline{x}))}d\mu_{G/H}(\overline{x})\Big|d\mu_H(y)$$

$$\le\int\limits_H d\mu_H(y)\int\limits_{G/H}\Big|\frac{\varphi(z^{-1}y\sigma(\overline{x}))-\varphi(y\sigma(\overline{x}))}{\rho(\sigma(\overline{x}))}\Big|d\mu_{G/H}(\overline{x})=$$

$$=\int\limits_G\frac{|\varphi(z^{-1}x)-\varphi(x)|}{\rho(x)}\rho(x)d\mu_G(x)=\int\limits_G|\varphi(z^{-1}x)-\varphi(x)|d\mu_G(x)=$$

$$=\|\lambda_H(z)\varphi-\varphi\|_{L_1(G)}.$$

So, if $z\in F$ then $\|\lambda_G(z)\Phi-\Phi\|_{L_1(H)}\le\varepsilon$. Thus, H has property (P_1) and, hence, is amenable. Theorem 2 is proved.

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