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ALGEBRAS OF FUNCTIONS ON MAPPINGS

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Abstract. Topological algebras of functions on mappings are defined and investigated. It is proved that each algebra satisfying certain conditions (which are necessary and sufficient) is topologically (and isometrically with respect to its semi-norms) isomorphic to a subalgebra of an algebra of functions on some mapping. It is interesting to note that a completely regular space does not define a topology of its algebra of continuous functions uniquely if this space contains an infinite compact subspace, while a mapping do this, of course, amongst topologies of the definite kind. It is possible to define a new conception (connected with mappings) of a completeness of algebras and to prove some usual properties of complete algebras.

Contents

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§ 1. Introduction

1.1. We shall denote by $\mathbb{C}$ the field and the (complex) algebra of complex numbers, by $\mathbb{R}$ the field and the (real) algebra of real numbers, by $\mathbb{N} = \{1, 2, 3, \dots\}$ the set of natural numbers with the usual algebraic and topological structures.

All desirable definitions concerning algebras may be found in [7] or [8].

All topological definitions which are absent here may be found in [1] or [5].

The term "mapping" will mean "continuous map". Unlike [5] we shall use the term "compact" instead of "quasi-compact". Besides that we shall assume that a homomorphism of algebras preserves the unit if it exists in the domain of the homomorphism, and the involution if it exists in both these algebras.

If $f\colon X \to Y$ is a map and $A \subseteq X$ then the symbol $f|_A$ stands for the restriction of the map f to the set A; the symbol $[A]_X$ stands for the closure of the set A in the topological space X.

We formulate the following definitions to avoid misunderstandings ($f\colon X \to Y$ is a map of topological spaces; no axioms of separability are assumed).

1.2. **Definition.** The map f is called *closed* if for each closed subset $F \subseteq X$ the set $fF = \{fx : x \in F\}$ is closed in the space Y.

1.3. **Definition.** The map f is called *compact* if for each point $y \in Y$ the set $f^{-1}y = \{x \in X : fx = y\}$ is a compact subspace of the space X.

1.4. **Definition.** The map f is called *perfect* if it is continuous, closed and compact.

1.5. Let X be a topological space, $C(X)$ be an algebra of all continuous complex functions on the space X, $C^*(X) \subseteq C(X)$ be a subalgebra consisting of all bounded functions.

The algebra $C^*(X)$ has the natural topology generated by the norm $\|g\| = \sup\{|gx| : x \in X\}$ for $g \in C^*(X)$. Of course, this algebra has many other topologies, but they are not so interesting, except, maybe, the topology of the pointwise convergence.

The algebra $C(X)$ has many different topologies. Usually topologies under consideration are minimal or maximal in some sense (for example, the compact-open topology or the topology of the pointwise convergence) because these topologies are more natural than others.

It is well known that each commutative complex topological algebra satisfying certain conditions is isomorphic to a subalgebra of an algebra $C(X)$ for some completely regular space X (see [2], [10], [8], [7]). The space X does not define a topological structure of this algebra. If we want to get a topological isomorphism, we must define some additional structure on the space X (see [10]; unfortunately, Theorem 5 of this paper is not correct).

1.6. We can consider mappings as a generalization of topological spaces (a non-empty space X corresponds to the constant mapping $X \to \{*\}$ onto the singleton). In the paper [9] there are many definitions of properties of mappings which are analogous to properties of spaces. In particular, a compactification of a mapping (see, for example, [11]) is an analog of compactification of a space. What is an analog for a mapping of the normed algebra $C^*(X)$ for a space X?

Let us note that an answer depends on considered class of mappings. In general case this analog is a sheaf of topological algebras. In the simplest case of a globally completely regular mapping $f\colon X \to Y$ (see Definition 2.18) satisfying the condition $[fX]_Y = Y$ it is the topological algebra $C(f)$ defined in the section 2. The topology of the algebra $C(f)$ is naturally defined by the mapping f analogously to the algebra $C^*(X)$.

1.7. In the second section we investigate topological and categorial properties of mappings and their algebras of functions.

The third section is devoted to maximal ideals of the algebra $C(f)$.

In the fifth section we prove that each commutative complex algebra with the unit and with Hausdorff topology generated by a family of semi-norms, satisfying the conditions 1)–7) of Theorem 2.7 (see 5.1), is topologically isomorphic to a subalgebra of the algebra $C(f)$ for some mapping $f\colon X \to Y$.

1.8. In the sixth section we prove that the completeness of the algebra $C(f)$ with respect to the natural uniformity is a rare property.

Proposition 2.15 and Corollary 2.25 lead to a supposition that a topology on a set Y of semi-norms of an algebra C plays a part of a "uniformity", and a "completeness" of the algebra C with respect to this "uniformity" means that the embedding of the algebra C into the algebra $C(\tilde{f})$, constructed in the items 5.22 and 5.32, is closed. Let us note that there exists the smallest topology on the set Y which is compatible with the algebra C; this topology is described in the item 5.14.

In the section we prove an internal characterization of this "completeness" and prove some usual properties of complete algebras: the existence of inverse elements, of functions of elements, the closedness of maximal ideals and so on. The class of "complete" algebras is very wide as it follows from Example 1.9.

1.9. **Example.** Let $\mathfrak{M}$ be a completely regular space and let $\{\mathfrak{M}_y : y \in Y\}$ be its covering by compact sets. Let $C(\mathfrak{M})$ be the algebra of all continuous complex functions defined on the space $\mathfrak{M}$ with the topology generated by the family of semi-norms $\{n_y : y \in Y\}$, where $n_y g = \sup\{|gM| : M \in \mathfrak{M}_y\}$ for all $g \in C(\mathfrak{M})$ and $y \in Y$. It follows from Corollary 5.28 and Definition 6.13 that the algebra $C(\mathfrak{M})$ is "complete".

§ 2. An algebra of functions on a mapping

2.1. We shall consider algebras of complex functions, but all results of this section will be valid for algebras of real functions too (in this case involution is identity: $g^* = g$).

Let us fix a mapping $f\colon X \to Y$ of topological spaces. We shall assume that $[fX]_Y = Y$, but this condition is not essential.

A. Algebras of functions

2.2. **Definition.** A function $g\colon X \to \mathbb{C}$ will be called *f-bounded* if for each point $y \in Y$ there exists a neighborhood $Uy \subseteq Y$ such that the function $g|_{f^{-1}Uy}$ is bounded.

2.3. Let $B(f)$ be the set of all f-bounded functions, $C(f)$ be the set of all continuous f-bounded functions, $C(X)$ be the set of all continuous functions, $C^*(X)$

be the set of all continuous bounded functions with the standard norm $\|g\| = \sup\{|gx| : x \in X\}$ for $g \in C^*(X)$.

Of course, $C^*(X) \subseteq C(f) = C(X) \cap B(f)$. All these sets have the structure of complex algebras with the involution (the complex conjugation) and with the unit g_e, and the algebra $C^*(X)$ is complete.

2.4. **Assertion.** *If the mapping f is closed and the set $f^{-1}y$ is pseudocompact for each point $y \in Y$ then $C(f) = C(X)$.*

Proof. Let us note that $fX = Y$ since f is closed and fX is dense in Y. Let $g \in C(X)$, $y \in Y$. The space $f^{-1}y$ is pseudocompact, therefore the function g is bounded on $f^{-1}y$, that is, there exists a number $M \in \mathbb{R}$ such that $|gx| \leqslant M$ for all $x \in f^{-1}y$.

Since the function g is continuous, for every point $x \in f^{-1}y$ there exists a neighborhood $Ux \subseteq X$ such that $|gx' - gx| < 1$ for all $x' \in Ux$. Then the set $U = \bigcup\{Ux : x \in f^{-1}y\}$ is a neighborhood of the set $f^{-1}y$. The set $Uy = f^{\#}U = \{y' \in Y : f^{-1}y' \subseteq U\}$ is a neighborhood of the point y since the mapping f is closed.

Due to the definition of the set Ux we have $|gx| < M + 1$ for every $x \in f^{-1}Uy$, therefore the function g is f-bounded and $g \in C(f)$. □

2.5. **Assertion.** *If the space Y is countably compact then $C(f) = C^*(X)$.*

Proof. Let $g\colon X \to \mathbb{C}$ be an unbounded function. Let us prove that g does not belong to $B(f)$.

Since the function g is not bounded, there is a set $A = \{x_n : n \in \mathbb{N}\}$ such that $|gx_n| > n$ for each $n \in \mathbb{N}$. Let $M = fA$.

If the set M is finite then there is a point $y \in M$ such that the set $A \cap f^{-1}y$ is infinite.

If the set M is infinite then it has a strict limit point $y \in Y$ since the space Y is countably compact ([5], Theorem 3.10.3).

In both cases the function g is unbounded on $f^{-1}Uy$ for any neighborhood $Uy \subseteq Y$ of the point y. □

B. **Semi-norms**

2.6. For each point $y \in Y$ let us define a function $n_y\colon B(f) \to \mathbb{R}$ by the equality

$$n_y g = \inf\{\sup\{|gx| : x \in f^{-1}Uy\} : Uy \subseteq Y \text{ is a neighborhood of the point } y\}$$

for all $g \in B(f)$.

2.7. **Theorem.** *For every point $y \in Y$ the function n_y has the following properties (g, g_1, $g_2 \in B(f)$, $c \in \mathbb{C}$; g_e is the unit function, that is, $g_e x = 1$ for all $x \in X$; the symbol "$*$" denotes the involution):*

1) $n_y g \geqslant 0$;
2) $n_y(c \cdot g) = |c| \cdot n_y g$;
3) $n_y(g_1 + g_2) \leqslant n_y g_1 + n_y g_2$;
4) $n_y(g_1 \cdot g_2) \leqslant n_y g_1 \cdot n_y g_2$;
5) $n_y g_e = 1$;
6) $n_y(g \cdot g^*) = n_y g \cdot n_y g^*$;
7) $n_y g^* = n_y g$.

Proof is standard. □

2.8. In particular, the properties 1)–5) mean that for every $y \in Y$ the function n_y is a semi-norm on the algebra $B(f)$ (and on $C(f)$), while the properties 6) and 7) mean that this semi-norm is completely regular ([8], §16.1). Of course, if a function $g \in C(f)$ satisfies the condition $n_y g = 0$ for all $y \in Y$ then $gx = 0$ for all $x \in X$.

2.9. **Proposition.** *If the mapping f is closed then*

$$n_y g = \sup\{|gx| : x \in f^{-1}y\}$$

for all $y \in Y$ and $g \in C(f)$.

Proof. Let $y \in Y$, $g \in C(f)$ and $\sup\{|gx| : x \in f^{-1}y\} = A$. Obviously, $A \leqslant n_y g$.

Let us take an arbitrary number $\varepsilon > 0$. Since the function g is continuous, for each point $x \in f^{-1}y$ there exists a neighborhood $Ux \subseteq X$ such that $|gx' - gx| < \varepsilon$ for all $x' \in Ux$. Then the set $U = \bigcup\{Ux : x \in f^{-1}y\}$ is a neighborhood of the set $f^{-1}y$. Since the mapping f is closed, the set $Uy = f^{\#}U$ is a neighborhood of the point y.

Due to the construction of the set Uy we have $|gx| < A + \varepsilon$ for all points $x \in \in f^{-1}Uy$, therefore the inequality $n_y g \leqslant A + \varepsilon$ is valid.

Thus, we get the inequality $A \leqslant n_y g \leqslant A + \varepsilon$ for all $\varepsilon > 0$. Hence, $n_y g = A$. $\square$

2.10. **Proposition.** *Let $g \in B(f)$, $y_0 \in Y$. Then*

$$n_{y_0} g = \inf\{\sup\{n_y g : y \in Uy_0\} : Uy_0 \subseteq Y \textit{ is a neighborhood of the point } y_0\}.$$

Proof follows from the definition 2.6 of the semi-norms n_y, $y \in Y$. $\square$

2.11. **Corollary.** *For each function $g \in B(f)$ and each number $\varepsilon > 0$ the set $U_{g,\varepsilon} = \{y \in Y : n_y g < \varepsilon\}$ is open, that is, the function $\tilde{g} \colon Y \to \mathbb{R}$, defined by the formula $\tilde{g}y = n_y g$ for all $y \in Y$, is upper semicontinuous* ([5], 1.7.14; compare with 1.7.16).

2.12. **Corollary.** *If a function $g \in B(f)$, a point $y_0 \in Y$ and a number $\varepsilon > 0$ are given then there exists a neighborhood $Uy_0 \subseteq Y$ such that $n_y g < n_{y_0} g + \varepsilon$ for all $y \in Uy_0$.*

Proof. The open set $U_{g,t}$ for $t = n_{y_0} g + \varepsilon$ is a desirable neighborhood. $\square$

2.13. **Assertion.** *If a set $Z \subseteq Y$ is countably compact and $g \in B(f)$ then the function $\tilde{g}$ defined in 2.11 is bounded on Z, and for every number $\varepsilon > 0$ there exists a neighborhood $UZ \subseteq Y$ such that $n_y g < M + \varepsilon$ for all $y \in UZ$, where $M = \sup\{n_y g : y \in Z\}$.*

Proof. Let $g \in B(f)$ and $\varepsilon > 0$. Since the space Z is countably compact, $Z \subseteq \subseteq \bigcup\{U_{g,n} : n \in \mathbb{N}\} = Y$ and $U_{g,n} \subseteq U_{g,n+1}$ for all $n \in \mathbb{N}$, there exists a number $n_0 \in \mathbb{N}$ such that $Z \subseteq U_{g,n_0}$. Therefore $M = \sup\{n_y g : y \in Z\} \leqslant n_0$ is finite, and the set $UZ = U_{g,M+\varepsilon}$ is a required neighborhood. $\square$

C. Topologies on the algebras $B(f)$ and $C(f)$

2.14. Let us equip the algebras $B(f)$ and $C(f)$ with the topologies generated by the family of semi-norms $\{n_y : y \in Y\}$. Bases of these topologies consists of sets

$$V_B(g_0, \varepsilon, M) = \{g \in B(f) : n_y(g - g_0) < \varepsilon \text{ for all } y \in M\},\ g_0 \in B(f),$$

and

$$V_C(g_0, \varepsilon, M) = \{g \in C(f) : n_y(g - g_0) < \varepsilon \text{ for all } y \in M\},\ g_0 \in C(f),$$

where $M \subseteq Y$ is a finite set and $\varepsilon > 0$.

In the standard way we can prove that all algebraic operations of $B(f)$ and $C(f)$ are continuous relatively these topologies.

2.15. **Proposition.** *The algebra $C(f)$ is a closed subalgebra of $B(f)$.*

Proof. It is necessary to prove the closedness of $C(f)$ in $B(f)$ only.

Let $g_0 \in [C(f)]_{B(f)}$. Let us prove that the function g_0 is continuous at a point $x_0 \in X$ which is chosen arbitrarily. Let us denote $M = \{fx_0\}$, and let $\varepsilon > 0$ be arbitrary. Due to the definition of the topology of $B(f)$ there is a function $g \in \in V_B\left(g_0, \frac{\varepsilon}{3}, M\right) \cap C(f)$. Since the function g is continuous, there is a neighborhood

$U_0x_0 \subseteq X$ such that $|gx - gx_0| < \frac{\varepsilon}{3}$ for all $x \in U_0x_0$. By Corollary 2.12 there is a neighborhood $Ufx_0 \subseteq Y$ such that $n_y(g - g_0) < \frac{\varepsilon}{3}$ for all $y \in Ufx_0$. Hence for each point $x \in Ux_0 = U_0x_0 \cap f^{-1}Ufx_0$ we get

$$|g_0x - g_0x_0| = |g_0x - gx + gx - gx_0 + gx_0 - g_0x_0| \leqslant |g_0x - gx| + |gx - gx_0| + \\ + |gx_0 - g_0x_0| < n_{fx}(g - g_0) + \frac{\varepsilon}{3} + n_{fx_0}(g - g_0) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

therefore the function g_0 is continuous. □

2.16. **Proposition.** *The identity embedding* $C^*(X) \xrightarrow{\subseteq} C(f)$ *is continuous and* $C^*(X)$ *is dense in* $C(f)$.

Proof. The continuity of this embedding follows from the inequality $\|g\| \geqslant n_y g$ for all $g \in C^*(X)$ and $y \in Y$ (in general this embedding is not homeomorphic).

Let us take an arbitrary function $g \in C(f)$ and prove that $g \in [C^*(X)]_{C(f)}$. To this end, let us define a function $g_t \in C^*(X)$ for each number $t \geqslant 0$ by putting for $x \in X$

$$g_t x = \begin{cases} gx \text{ if } |gx| \leqslant t, \\ \frac{t}{|gx|} \cdot gx \text{ if } |gx| > t. \end{cases}$$

Let us denote $A_r = \{g_t : t \geqslant 0\}$ and prove that $g \in [A_g]_{C(f)} \subseteq [C^*(X)]_{C(f)}$. Let an arbitrary neighborhood $V_C(g, \varepsilon, M)$ of the function g be given ($M \subseteq Y$ is a finite set, $\varepsilon > 0$). Let $t_0 = \max\{n_y g : y \in M\}$. Then for all $t \geqslant t_0$ and $y \in M$ we have $n_y(g_t - g) = 0$, therefore $g_t \in V_C(g, \varepsilon, M)$ and $A_g \cap V_C(g, \varepsilon, M) \neq \varnothing$. Hence, the required statement is valid. □

2.17. **Proposition.** a) *If the space* Y *has a finite topology then the algebra* $C(f)$ *is topologically isomorphic to the algebra* $C^*(X)$.

b) *If the mapping* f *is perfect and every compact subset of the space* Y *has a finite topology then the algebra* $C(f)$ *is topologically isomorphic to the algebra* $C(X)$ *with the compact-open topology* ([5], §3.4).

c) *If the mapping* f *is closed and for each point* $y \in Y$ *the set* $f^{-1}y$ *has a finite topology then the algebra* $C(f)$ *is topologically isomorphic to the algebra* $C(X)$ *with the topology of the pointwise convergence* ([5], §2.6).

Proof. a) Due to Assertion 2.5 we have $C(f) = C^*(X)$ since the space Y is compact. For each point $y \in Y$ let us define the set

$$\bar{y} = \bigcap\{U \setminus V : U, V \subseteq Y \text{ are open subsets, } y \in U \setminus V\}.$$

It follows from the definition 2.6 that if $y_1, y_2 \in Y$ and $y_1 \in \bar{y}_2$ then $n_{y_1}g = n_{y_2}g$ for all functions $g \in C(f)$. Let $M \subseteq Y$ be a set satisfying the condition $|M \cap \bar{y}| = 1$ for all $y \in Y$. Obviously, the set M is finite. It is easily seen that for each $g \in C(f) = C^*(X)$ the equality $\|g\| = \max\{n_y g : y \in M\}$ is satisfied, therefore the topologies of the algebras $C(f)$ and $C^*(X)$ are identical.

b) Due to Assertion 2.4 we have $C(f) = C(X)$. Since each compact subset $B \subseteq Y$ has a finite topology, for such B there exists a finite set $M_B \subseteq B$ such that $B \subseteq \bigcup\{\bar{y} : y \in M_B\}$ (see the proof of the statement a)). Hence all sets of the form

$$V(g_0, \varepsilon, \bar{y}, F) = \{g \in C(X) : gF \subseteq U(g_0F, \varepsilon)\},$$

where $y \in Y$, $F \subseteq f^{-1}\bar{y}$ is a compact subset, $g_0 \in C(X)$, $\varepsilon > 0$ and

$$U(g_0F, \varepsilon) = \{c \in \mathbb{C} : \text{ there exists a point } x \in F \text{ such that } |c - g_0x| < \varepsilon\}$$

constitute a subbase for the compact-open topology.

Comparing the definitions we see that $V_c(g_0, \varepsilon, \{y\}) \subseteq V(g_0, \varepsilon, \bar{y}, F)$ for all $g_0 \in C(f) = C(X)$, $\varepsilon > 0$, $y \in Y$ and compact subsets $F \subseteq f^{-1}\bar{y}$ since $n_{y'}g = n_y g$ for all $y' \in \bar{y}$ and $g \in C(f)$, therefore the identity map $C(f) \to C(X)$ is continuous since the sets $V_C(g_0, \varepsilon, \{y\})$, $g_0 \in C(f)$, $\varepsilon > 0$, $y \in Y$, constitute a subbase for the topology of $C(f)$.

Let $y \in Y$, $g_0 \in C(f)$, $\varepsilon > 0$ and $\Phi = g_0 f^{-1} y$. The set Φ is compact since the set $f^{-1}y$ is compact and the function g_0 is continuous. Let us consider a covering of the set Φ by open disks $U\left(c_0, \frac{\varepsilon}{4}\right) = \left\{c \in \mathbb{C} : |c - c_0| < \frac{\varepsilon}{4}\right\}$, $c_0 \in \Phi$. This covering contains a finite subcovering $\left\{U\left(c_k, \frac{\varepsilon}{4}\right) : k = 1, 2, \ldots, m\right\}$. For $k = 1, 2, \ldots, m$ let $F_k = \left[g_0^{-1} U\left(c_k, \frac{\varepsilon}{4}\right)\right]_X \cap f^{-1}y$; the set F_k is compact and $f^{-1}y = \bigcup\{F_k : k = = 1, 2, \ldots, m\}$.

It is easily seen that for all $c', c'' \in U\left(g_0 F_k, \frac{\varepsilon}{4}\right)$, $k = 1, 2, \ldots, m$, we have $|c' - - c''| < \varepsilon$, therefore by Proposition 2.9 $\bigcap\left\{V\left(g_0, \frac{\varepsilon}{4}, \bar{y}, F_k\right) : k = 1, 2, \ldots, m\right\} \subseteq \subseteq V_C(g_0, \varepsilon, \{y\})$. Hence the identity map $C(X) \to C(f)$ is continuous too; that is why this mapping is a homeomorphism.

c) Due to Assertion 2.4 we have $C(f) = C(X)$. It is easily seen that the topology of the pointwise convergence is generated by the family of semi-norms $\{n_x : x \in X\}$ where $n_x g = |gx|$ for all $x \in X$ and $g \in C(X)$.

For each point $x \in X$ let us define the set

$$\bar{x} = \bigcap\{f^{-1}fx \cap U \setminus V : U, V \subseteq X \text{ are open subsets, } x \in U \setminus V\}.$$

For each $y \in Y$ let $M_y \subseteq f^{-1}y$ be a set satisfying the condition $|M_y \cap \bar{x}| = 1$ for all $x \in f^{-1}y$. Obviously, the set M_y is finite for every $y \in Y$.

Let us note that each function $g \in C(f) = C(X)$ is constant on any set $\bar{x}$, $x \in X$. Hence, for all $g \in C(X)$, $y \in Y$, $x \in M_y$ and $x' \in \bar{x}$ the equality $n_{x'} g = n_x g$ holds.

Due to Proposition 2.9 the equality $n_y g = \max\{n_x g : x \in M_y\}$ holds for all $y \in Y$ and $g \in C(f)$. Therefore the finite family of semi-norms $\{n_x : x \in M_y\}$ and the semi-norm n_y, $y \in Y$, generate the same topologies. □

D. **A conversion to globally completely regular mappings**

2.18. **Definition.** A mapping $f\colon X \to Y$ of topological spaces will be called *globally completely regular* if it satisfies the next conditions:

a) for an arbitrary point $x \in X$ and for every point $x' \in f^{-1}fx \setminus \{x\}$ there exists a neighborhood $Ofx \subseteq Y$ and a continuous function $g\colon X \to [0, 1]$ such that $gx = 0$ and $gx' = 1$;
b) for an arbitrary point $x \in X$ and for every neighborhood $Ux \subseteq X$ there exists a neighborhood $Ofx \subseteq Y$ and a continuous function $g\colon X \to [0, 1]$ such that $gx' = 1$ for all $x' \in f^{-1}Ofx \setminus Ux$.

2.19. Of course, each globally completely regular mapping is Tychonoff (see [9]; compare with §7) and separable (see [11], Definition 3).

2.20. **Theorem.** *There exist a space X_f, a perfect globally completely regular mapping $p_f\colon X_f \xrightarrow{\text{onto}} Y$ and a mapping $q_f\colon X \to X_r$, $[q_f X]_{X_f} = X_f$, satisfying the following conditions:*

1) *$p_f q_f = f$;*
2) *a map $\varphi\colon C(p_f) \to C(f)$ defined by the equality $\varphi g = g q_f$ for all $g \in C(p_f)$ is a topological isomorphism onto $C(f)$ which preserves all semi-norms n_y, $y \in Y$;*
3) *if $p'\colon X' \to Y$ and $q'\colon X \to X'$ are mappings satisfying the conditions* 1) *and* 2) *then there exists and is unique a mapping $h\colon X' \to X_f$ such that $q_f = hq'$ and $p' = p_f h$; if p' is perfect then h is perfect too and, hence, $hX' = X_f$;*
4) *if a space X'_f and mappings $p'_f\colon X'_f \to Y$ and $q'_f\colon X \to X'_f$ satisfy the conditions* 1), 2) *and* 3) (*except the uniqueness and the perfectness of the mapping h*) *then the mapping $h'\colon X_f \to X'_f$, which exists by the virtue of*

the condition 3)*, is a homeomorphic embedding and the set* $h'X_f$ *is a retract of the space* X'_f*.*

The space X_f *and the mappings* p_f *and* q_f *satisfying the conditions* 1)–4) *are unique. If the mapping* f *is perfect and globally completely regular then the mapping* q_f *is a homeomorphism onto* X_f*.*

Proof. Let $C(f) = \{g_\alpha : \alpha \in \mathfrak{A}\}$, and let $Z_\alpha = \mathbb{C} \cup \{\infty\}$ be a one-point compactification of the space of complex numbers (a complex sphere) for all $\alpha \in \mathfrak{A}$.

For each $\alpha \in \mathfrak{A}$ let us set $Y_\alpha = Y \times Z_\alpha$, and let $p_\alpha\colon Y_\alpha \xrightarrow{\text{onto}} Y$ and $f_\alpha\colon X \to Y_\alpha$ be mappings defined by the formulas $p_\alpha(y,z) = y$ for all $(y,z) \in Y \times Z_\alpha$ and $f_\alpha x = (fx, g_\alpha x)$ for all $x \in X$ (see [5], §2.3). Let us set $X_\alpha = [f_\alpha X]_{Y_\alpha}$. It is useful to note that the closed sets X_α and $Y \times \{\infty\}$ are disjoint since the function g_α is f-bounded.

Let us put $Y_\mathfrak{A} = Y \times \prod\{Z_\alpha : \alpha \in \mathfrak{A}\}$ and define mappings $\pi\colon Y_\mathfrak{A} \xrightarrow{\text{onto}} Y$, $p\colon Y_\mathfrak{A} \xrightarrow{\text{onto}} \prod\{Z_\alpha : \alpha \in \mathfrak{A}\}$, $\pi_\alpha\colon Y_\mathfrak{A} \xrightarrow{\text{onto}} Y_\alpha$, $\alpha \in \mathfrak{A}$, and $f_\mathfrak{A}\colon X \to Y_\mathfrak{A}$ by the equalities $\pi\{y, z_\beta : \beta \in \mathfrak{A}\} = y$, $p\{y, z_\beta : \beta \in \mathfrak{A}\} = \{z_\beta : \beta \in \mathfrak{A}\}$, $\pi_\alpha\{y, z_\beta : \beta \in \mathfrak{A}\} = (y, z_\alpha)$ for all $\alpha \in \mathfrak{A}$ and $\{y, z_\beta : \beta \in \mathfrak{A}\} \in Y_\mathfrak{A}$, $f_\mathfrak{A} x = \{fx, g_\beta x : \beta \in \mathfrak{A}\}$ for $x \in X$ (see [5], §2.3). If the mapping f is globally completely regular then the mapping $f_\mathfrak{A}$ is a homeomorphic embedding.

Let $X_f = [f_\mathfrak{A} X]_{Y_\mathfrak{A}}$, $p_f = \pi|_{X_f}$, and let $q_f\colon X \to X_f$ be the mapping which coincides with $f_\mathfrak{A}$. The mapping p_f is perfect since π is perfect due to the compactness of $\prod\{Z_\alpha : \alpha \in \mathfrak{A}\}$. Moreover, the mappings π and p_f are separable since the space $\prod\{Z_\alpha : \alpha \in \mathfrak{A}\}$ is Hausdorff. Therefore if the mapping f is perfect then the mapping q_f is perfect due to Lemma 8 of the paper [11].

Let us prove that the mappings π and p_f are globally completely regular. It is sufficient to prove this statement for the mapping π only. Let $x = (y_0, z_0) \in Y_\mathfrak{A} = Y \times \prod\{Z_\alpha : \alpha \in \mathfrak{A}\}$ be a point ($y_0 \in Y$, $z_0 \in \prod\{Z_\alpha : \alpha \in \mathfrak{A}\}$) and $Ux \subseteq Y_\mathfrak{A}$ be its neighborhood (in the case a) let $Ux = Y_\mathfrak{A} \setminus p^{-1}px'$ for $x' \in \pi^{-1}y_0 \setminus \{x\}$). By the definition of the product topology there are neighborhoods $Uy_0 \subseteq Y$ and $Uz_0 \subseteq \prod\{Z_\alpha : \alpha \in \mathfrak{A}\}$ such that $x \in Uy_0 \times Uz_0 = \pi^{-1}Uy_0 \cap p^{-1}Uz_0 \subseteq Ux$. Since the space $\prod\{Z_\alpha : \alpha \in \mathfrak{A}\}$ is completely regular ([5], Theorem 2.3.11), there is a continuous function $g_0\colon \prod\{Z_\alpha : \alpha \in \mathfrak{A}\} \to [0,1]$ such that $g_0 z_0 = 0$ and $g_0 z = 1$ for all $z \in \prod\{Z_\alpha : \alpha \in \mathfrak{A}\} \setminus Uz_0$. Then the neighborhood $O\pi x = Uy_0$ and the function $g = g_0 p$ have the required properties.

Let us note also that for each $\alpha \in \mathfrak{A}$ the equality $\pi_\alpha f_\mathfrak{A} = f_\alpha$ is valid, therefore $\pi_\alpha X_f \subseteq X_\alpha$ (indeed $\pi_\alpha X_f = X_\alpha$ since the mapping π_α is perfect by Lemma 8 of the paper [11]) and, hence, $z_\alpha \neq \infty$ for all $\{y, z_\beta : \beta \in \mathfrak{A}\} \in X_f$; consequently, we can define a continuous function $\bar{g}_\alpha\colon X_f \to \mathbb{C}$ by putting $\bar{g}_\alpha\{y, z_\beta : \beta \in \mathfrak{A}\} = z_\alpha$ for all $\{y, z_\beta : \beta \in \mathfrak{A}\} \in X_f$. Obviously, for each $\alpha \in \mathfrak{A}$ the equality $g_\alpha = \bar{g}_\alpha q_f$ is valid.

If $g \in C(p_f)$ then $\varphi g = g q_f \in C(f)$, hence, there exists $\alpha \in \mathfrak{A}$ such that $\varphi g = g_\alpha$. The functions g and $\bar{g}_\alpha$ satisfy the same equality which defines them uniquely on the dense set $q_f X \subseteq X_f$. Since the space $\mathbb{C}$ is Hausdorff we have $g = \bar{g}_\alpha$, hence the map φ is an (algebraic) isomorphism. The preservation of all semi-norms n_y, $y \in Y$, is the consequence of the fact that $q_f X$ is dense X_f. Therefore φ is a homeomorphism also.

Let mappings $p'\colon X' \to Y$ and $q'\colon X \to X'$ satisfy the conditions 1) and 2). We shall construct a mapping $h\colon X' \to X_f$ satisfying the conditions $q_f = hq'$ and $p' = p_f h$.

For every $\alpha \in \mathfrak{A}$ let $\tilde{g}_\alpha \in C(p')$ be the right (which exists and is unique by virtue of the condition 2)) element which satisfies the equality $g_\alpha = \tilde{g}_\alpha q'$. Let us define a mapping $h'\colon X' \to Y_\mathfrak{A}$ by the equality $h'x = \{p'x, \tilde{g}_\alpha x : \alpha \in \mathfrak{A}\}$ for all $x \in X'$. Let us note that $q_f X \subseteq h'X'$ since $f = p'q'$ and $g_\alpha = \tilde{g}_\alpha q'$ for all $\alpha \in \mathfrak{A}$. It is obvious that the equalities $f_{\mathfrak{A}} = h'q'$ and $p' = \pi h'$ are true.

It is necessary to prove that $h'X' \subseteq X_f$. Let us suppose that there is a point $x_0 \in h'X' \setminus X_f$. Since the set X_f is closed and the mapping π is globally completely regular, there exist a neighborhood $U\pi x_0 \subseteq Y$ and a continuous function $g\colon Y_{\mathfrak{A}} \to$ $\to [0,1]$ such that $gx_0 = 1$ and $gx = 0$ for all $x \in X_f \cap \pi^{-1}U\pi x_0 = p_f^{-1}U\pi x_0$. Let us define the continuous functions $g_0 = gf_{\mathfrak{A}} \in C(f)$ and $g_0' = gh' \in C(p')$. It is obvious that these functions satisfy the condition $g_0 = g_0'q'$, therefore due to the condition 2) the equality $n_{y_0}g_0 = n_{y_0}g_0'$ must be valid, but we have $n_{y_0}g_0 = 0$ and $n_{y_0}g_0' = 1$. This contradiction proves that $h'X' \subseteq X_f$, and we have the mapping $h\colon X' \to X_f$ coinciding with h'. The conditions $q_f = hq'$ and $p' = p_f h$ are satisfied.

Let us prove the uniqueness of the mapping h. Assume that there is a mapping $\tilde{h}\colon X' \to X_f$ such that $q_f = \tilde{h}q'$, $p' = p_f\tilde{h}$ and $\tilde{h} \neq h$. Then there exists a point $x_0 \in X'$ such that $\tilde{h}x_0 \neq hx_0$. Then we have $p_f\tilde{h}x_0 = p_f hx_0$ since $p_f\tilde{h} = p_f h = p'$. Therefore there is $\alpha_0 \in \mathfrak{A}$ such that $\bar{g}_{\alpha_0}\tilde{h}x_0 \neq \bar{g}_{\alpha_0}hx_0$. But in this case $\tilde{g}_1 = \bar{g}_{\alpha_0}\tilde{h}$ and $\tilde{g}_2 = \bar{g}_{\alpha_0}h$ are two different continuous functions belonging to $C(p')$ such that $g_{\alpha_0} = \bar{g}_{\alpha_0}q_f = \tilde{g}_1q' = \tilde{g}_2q' \in C(f)$. This contradicts the condition 2), therefore $h' = h$.

If the mapping p' is perfect then h is perfect due to Lemma 8 of the paper [11].

If $X' = X_f$, $p' = p_f$ and $q' = q_f$ then the mapping h must be the identity since it is unique.

If a space X_f' and mappings p_f' and q_f' satisfy the conditions 1)–3) then there exist mappings $h\colon X_f' \to X_f$ and $h'\colon X_f \to X_f'$ such that $q_f = hq_f'$, $p_f' = p_f h$, $q_f' =$ $= h'q_f$, $p_f = p_f'h'$. Due to the preceding observation the mapping hh' is identity, hence, the mapping h' is a homeomorphic embedding and the mapping $h'h$ is a retraction.

If the space X_f' and the mappings p_f' and q_f' satisfy the conditions 1)–4) then both these mappings h and h' must be homeomorphic embeddings and "onto", hence, they are mutually inverse homeomorphisms. □

2.21. **Theorem.** *Let mappings $f_1\colon X_1 \to Y_1$, $f_2\colon X_2 \to Y_2$, $h_1\colon X_1 \to X_2$ and $h_2\colon Y_1 \to Y_2$ such that $[f_1X_1]_{Y_1} = Y_1$, $[f_2X_2]_{Y_2} = Y_2$ and $h_2f_1 = f_2h_1$ be given. Then there exists a unique mapping $h\colon X_{f_1} \to X_{f_2}$ such that $hq_{f_1} = q_{f_2}h_1$ and $h_2p_{f_1} = p_{f_2}h$. Moreover,*

1) *if h_2 is separable then h is separable too;*
2) *if h_2 is perfect then h is perfect also;*
3) *if h_2 is perfect and $[h_1X_1]_{X_2} = X_2$ then $hX_{f_1} = X_{f_2}$.*

Proof is analogous to the proof of Theorem 2.20 with some simplifications. □

2.22. *Remark.* Due to Theorems 2.20 and 2.21 we are in a position to replace the given mapping $f\colon X \to Y$ by the perfect globally completely regular mapping $p_f\colon X_f \xrightarrow{\text{onto}} Y$, when it is useful for our purposes.

E. **Homomorphisms of algebras**

2.23. **Theorem.** *Let f_1, f_2, h_1 and h_2 be the same mappings as in Theorem 2.21. Then the map $\varphi\colon C(f_2) \to C(f_1)$ defined by the equality $\varphi g = gh_1$ for all $g \in C(f_2)$ is a continuous homomorphis satisfying the condition $n_y g \geqslant \sup\{n_{y'}(\varphi g) : y' \in$ $\in h_2^{-1}y\}$ for all $y \in h_2Y_1$ and $g \in C(f_2)$. Moreover,*

1) *if $[h_1X_1]_{X_2} = X_2$ then φ is a continuous isomorphism onto a subalgebra of $C(f_1)$;*
2) *if the mapping $h_1\colon X_1 \xrightarrow{\text{onto}} h_1X_1$ is quotient (in particular, if h_1 is closed or open), the set h_1X_1 is clopen in X_2 and one of the conditions*

a) *the mapping* $f_2|_{h_1X_1}$ *is closed and the set* $f_2^{-1}y \cap h_1X_1$ *is pseudocompact for each* $y \in f_2h_1X_1$ *or*

b) *the mapping* h_2 *is closed and* $h_2^{-1}y$ *is countably compact for each* $y \in \in h_2Y_1$

is valid, then φ *is a homomorphism onto a closed subalgebra of* $C(f_1)$*;*

3) *if the mapping* h_2 *is perfect,* h_2Y_1 *is clopen subset of* Y_2 *and* $[h_1X_1]_{X_2} = = f_2^{-1}h_2Y_1$ *then* φ *is a homomorphism onto a closed subalgebra of* $C(f_1)$ *too, and* $n_yg = \sup\{n_{y'}(\varphi g) : y' \in h_2^{-1}y\}$ *for all* $y \in h_2Y_1$ *and* $g \in C(f_2)$*;*

4) *if the mapping* h_2 *is closed, the set* $h_2^{-1}y$ *has a finite topology for each* $y \in \in Y_2$ *and* $[h_1X_1]_{X_2} = X_2$*, then* φ *is a topological isomorphism onto a closed subalgebra of* $C(f_1)$*, and* $n_yg = \max\{n_{y'}(\varphi g) : y' \in h_2^{-1}y\}$ *for all* $y \in h_2Y_1$ *and* $g \in C(f_2)$*.*

Proof. The homomorphism φ is continuous since for all $y \in Y_1$ and $g \in C(f_2)$ the inequality $n_y(\varphi g) = n_y(gh_1) \leqslant n_{h_2y}g$ is fulfilled. From the last inequlity it follows that $\sup\{n_{y'}(\varphi g) : y' \in h_2^{-1}y\} \leqslant n_yg$ for all $y \in h_2Y_1$ and $g \in C(f_2)$.

1). This statement is obvious.

2). Let $g_0 \in [\varphi C(f_2)]_{C(f_1)}$. Let us take arbitrary points $x_1, x_2 \in X_1$ such that $h_1x_1 = h_1x_2$ and prove that $g_0x_1 = g_0x_2$. Of course, for all $g \in \varphi C(f_2)$ the equality $gx_1 = gx_2$ is true.

Let us take $\varepsilon > 0$ and consider the neighborhood $V = V_C\left(g_0, \frac{\varepsilon}{2}, f_1\{x_1, x_2\}\right) \subseteq \subseteq C(f_1)$. By the choice of g_0 there is a function $g \in V \cap \varphi C(f_2)$. Then we have $gx_1 = gx_2$ and

$$|g_0x_1 - g_0x_2| = |g_0x_1 - gx_1 + gx_2 - g_0x_2| \leqslant |g_0x_1 - gx_1| + |gx_2 - g_0x_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since the number $\varepsilon > 0$ is arbitrary it means that $g_0x_1 = g_0x_2$.

Thus we can define a function $g\colon X_2 \to \mathbb{C}$ by putting

$$gx = \begin{cases} g_0x' \text{ if } x \in h_1X_1 \text{ and } x = h_1x' \text{ for some } x' \in X_1, \\ 0 \text{ if } x \in X_2 \setminus h_1X_1. \end{cases}$$

This function is continuous since the mapping h_1 is quotient and the set h_1X_1 is clopen. To prove that the function g is f_2-bounded we can use Assertion 2.4 in the case a), or Assertion 2.13 and the closedness of h_2 in the case b). The equality $\varphi g = g_0$ is obvious.

3). Due to Theorems 2.20 and 2.21 we can suppose that the mappings f_1 and f_2 are perfect and f_2 is separable (otherwise we can replace the mappings f_1, f_2 and h_1 by the mappings p_{f_1}, p_{f_2} and h from Theorems 2.20 and 2.21). Then the mapping h_1 is perfect too since h_2 is perfect ([3], Chapter I, §10, Proposition 5, and [11], Lemma 8). Therefore all conditions of the statement 2) are fulfilled and, hence, the set $\varphi C(f_2)$ is closed in $C(f_1)$.

Let $y_0 \in h_2Y_1$, $g \in C(f_2)$ and $A = \sup\{n_y(\varphi g) : y \in h_2^{-1}y_0\}$. Let us prove that $n_{y_0}g = A$. It is sufficient to prove that $n_{y_0}g \leqslant A$. Let us take an arbitrary number $\varepsilon > 0$ and set $Uh_2^{-1}y_0 = U_{g,A+\varepsilon}$ (see 2.11). Then we get the inequality $\sup\{|gh_1x| : x \in f_1^{-1}Uh_2^{-1}y_0\} \leqslant A + \varepsilon$ on account of the definition 2.6. The set $Uy_0 = h_2^{\#}Uh_2^{-1}y_0 \cap h_2Y_1$ is a neighborhood of the point y_0 since the mapping h_2 is closed and the set h_2Y_1 is clopen in Y_2. Then the inequality

$$n_{y_0}g \leqslant \sup\{|gx| : x \in f_2^{-1}Uy_0\} \leqslant \sup\{|gh_1x| : x \in f_1^{-1}Uh_2^{-1}y_0\} \leqslant A + \varepsilon$$

holds since $f_2^{-1}Uy_0 \subseteq [h_1f_1^{-1}Uh_2^{-1}y_0]_{X_2}$. As the number $\varepsilon > 0$ is arbitrary, we get $n_{y_0}g \leqslant A$, and the poof is concluded.

4). It follows from the statements 1) and 3) that φ is a continuous isomorphism onto a closed subalgebra of $C(f_1)$ and the required equality is fulfilled (with "sup" instead of "max"). The rest can be proved by analogy with the proof of the statement c) in Proposition 2.17. □

2.24. *Remark.* Let the mappings f_1, f_2, h_1 and h_2 be the same. Let a point $y_0 \in Y_2$ satisfy the following conditions:

1) there exists a neighborhood $U_0 y_0 \subseteq Y_2$ such that $f_2^{-1} U_0 y_0 \subseteq [h_1 X_1]_{X_2}$,
2) for each neighborhood $U h_2^{-1} y_0 \subseteq Y_1$ of the set $h_2^{-1} y_0$ there exists a neighborhood $U y_0 \subseteq Y_2$ such that $h_2^{-1} U y_0 \subseteq U h_2^{-1} y_0$ (that is, the mapping h_2 is closed at the point y_0).

Then the same reasonings as in the proof of the statement 3) of Theorem 2.23 proves the equality $n_{y_0} g = \sup\{n_y(gh_1) : y \in h_2^{-1} y_0\}$ for all functions $g \in C(f_2)$.

2.25. **Corollary.** *Let $f_1 \colon X_1 \to Y$, $f_2 \colon X_2 \to Y$ and $h \colon X_1 \to X_2$ be mappings such that $f_1 = f_2 h$ and $[f_2 X_2]_Y = Y$.*[1] *Then the map $\varphi \colon C(f_2) \to C(f_1)$ defined by the equality $\varphi g = gh$ for all $g \in C(f_2)$ is a continuous homomorphism onto a closed subalgebra of $C(f_1)$. Moreover, if $[hX_1]_{X_2} = X_2$ then φ is a topological isomorphism onto a closed subalgebra of $C(f_1)$ preserving all semi-norms n_y, $y \in$ $\in Y$.*

2.26. **Theorem.** *Let mappings $f_1 \colon X_1 \to Y_1$, $f_2 \colon X_2 \xrightarrow{\text{onto}} Y_2$ and $h_2 \colon Y_1 \to Y_2$ be given, moreover, let the mapping f_2 be perfect and globally completely regular and $[f_1 X_1]_{Y_1} = Y_1$. Besides that, let $\varphi \colon C(f_2) \to C(f_1)$ be a homomorphism such that the inequality $n_y(\varphi g) \leqslant n_{h_2 y} g$ is valid for all $y \in Y_1$ and $g \in C(f_2)$. Then there exists and is unique a mapping $h_1 \colon X_1 \to X_2$ such that $f_2 h_1 = h_2 f_1$ and $g h_1 = \varphi g$ for all $g \in C(f_2)$. Moreover,*

1) *if both mappings f_1 and h_2 are perfect or separable then the mapping h_1 is, respectively, perfect or separable too;*
2) *if for some point $y_0 \in h_2 Y_1$ the equality $n_{y_0} g = \sup\{n_y(\varphi g) : y \in h_2^{-1} y_0\}$ is fulfilled for all functions $g \in C(f_2)$ then $f_2^{-1} y_0 \subseteq [h_1 X_1]_{X_2}$.*

Proof is almost identical to the proof of Theorem 2.20. □

2.27. **Corollary.** *Let $f_1 \colon X_1 \to Y$ be a mapping,*[2] *$f_2 \colon X_2 \xrightarrow{\text{onto}} Y$ be a perfect completely regular mapping and $\varphi \colon C(f_2) \to C(f_1)$ be a homomorphism such that the inequality $n_y(\varphi g) \leqslant n_y g$ holds for all $y \in [f_1 X_1]_Y$ and $g \in C(f_2)$. Then there exists and is unique a mapping $h \colon X_1 \to X_2$ such that $f_2 h = f_1$ and $gh = \varphi g$ for all $g \in C(f_2)$; hence, $\varphi C(f_2)$ is a closed subalgebra of $C(f_1)$. Moreover,*

1) *if the mapping f_1 is perfect or separable then the mapping h is, respectively, perfect or separable too;*
2) *if for some point $y \in [f_1 X_1]_Y$ the equality $n_y g = n_y(\varphi g)$ is fulfilled for all functions $g \in C(f_2)$ then $f_2^{-1} y \subseteq [hX_1]_{X_2}$.*

§ 3. Mappings and maximal ideals

3.1. Let us fix a perfect globally completely regular mapping $f \colon X \xrightarrow{\text{onto}} Y$. If a given mapping is not perfect or globally completely regular then we can use Theorems 2.20 and 2.21.

We shall consider algebras of complex functions but all results of §3 will be valid for algebras of real functions too.

A. Closed ideals and quotient algebras

3.2. For each point $y \in Y$ let $I_y = \{g \in C(f) : n_y g = 0\}$; of course, this is a closed ideal of the algebra $C(f)$.

[1] We assume that semi-norm n_y, $y \in Y$, is defined on the algebra $C(f_1)$ iff $y \in [f_1 X_1]_Y$.

[2] See the preceding foot-note.

For an arbitrary finite set $A \subseteq Y$ let $I_A = \bigcap\{I_y : y \in A\}$ and let $n_A g = \max\{n_y g : y \in A\}$ for all $g \in C(f)$; of course, n_A is a seminorm satisfying the conditions 1)–7) of Theorem 2.7, and $I_A = \{g \in C(f) : n_A g = 0\}$.

Therefore we can define a homomorphism $p_A\colon C(f) \xrightarrow{\text{onto}} C_A = C(f)/I_A$ onto a quotient algebra for any finite set $A \subseteq Y$ and a norm on the algebra C_A by the formula $\|\bar{g}\| = n_A g$ for all $\bar{g} \in C_A$, where $g \in C(f)$ is a function such that $p_A g = \bar{g}$; of course, the homomorphism p_A is continuous.

3.3. **Lemma.** *Let $A \subseteq Y$ be a finite subset. Then*

a) *for each element $\bar{g} \in C_A$ there exists a function $g \in C^*(X) \subseteq C(f)$ such that the equalities $p_A g = \bar{g}$ and $\|g\| = \|\bar{g}\|$ are satisfied;*

b) *the algebra C_A is complete.*

Proof. a) Let $\bar{g} \in C_A$ and $g \in C(f)$ be some elements satisfying the condition $p_A g = \bar{g}$. Let $t = \|\bar{g}\| = n_A g$ and let $g_t \in C^*(x)$ be the function defined in the proof of Proposition 2.16. Then $\|g_t\| = t = \|\bar{g}\|$ and $n_A(g_t - g) = 0$, that is, $g_t - g \in I_A$ and, hence, $p_A g_t = p_A g = \bar{g}$.

b) Let $\{\bar{g}_n : n \in \mathbb{N}\}$ be a Cauchy sequence of elements of C_A. Without loss of generality we can suppose that $\|\bar{g}_{n+1} - \bar{g}_n\| \leqslant 2^{-n}$ for all $n \in \mathbb{N}$. Due to the statement a), for each $n \in \mathbb{N}$ there exists a function $g_n \in C^*(X) \subseteq C(f)$ such that $p_A g_n = \bar{g}_{n+1} - \bar{g}_n$ and $\|g_n\| = \|\bar{g}_{n+1} - \bar{g}_n\| \leqslant 2^{-n}$. Since the algebra $C^*(X)$ is complete, the series $\sum\limits_{n=1}^{\infty} g_n$ converges to some function $g_0 \in C^*(X)$. By Proposition 2.16 this convergence takes place in $C(f)$ too. Due to the continuity of the homomorphism p_A, we have

$$\bar{g}_1 + p_A g_0 = \bar{g}_1 + p_A \sum_{n=1}^{\infty} g_n = \bar{g}_1 + \sum_{n=1}^{\infty} p_A g_n = \bar{g}_1 + \sum_{n=1}^{\infty} (\bar{g}_{n+1} - \bar{g}_n),$$

that is, the last series converges; since its n-th partial sum equals $\bar{g}_n$ for all $n \in \mathbb{N}$, the given sequence converges to $\bar{g}_1 + p_A g_0$. □

3.4. **Corollary.** *Every maximal ideal of the algebra $C(f)$ containing an ideal I_A for some finite $A \subseteq Y$ is closed.*

Proof. Let $M \subset C(f)$ be a maximal ideal, $A \subseteq Y$ be a finite set such that $I_A \subseteq M$. Then $\bar{M} = p_A M$ is a maximal ideal (otherwise M would be contained in a larger ideal) which is closed by the statement II 4) of §9.4 of the book [8]. Therefore $M = p_A^{-1}\bar{M}$ (since M is maximal) is closed. □

B. Closed maximal ideals of $C(f)$ and points of X

3.5. **Lemma.** a) *For each point $x \in X$ the set $M_x = \{g \in C(f) : gx = 0\}$ is a closed maximal ideal of the algebra $C(f)$ and $I_{fx} \subseteq M_x$.*

b) *If points $x_1, x_2 \in X$ satisfy the conditions $fx_1 = fx_2$ and $x_1 \neq x_2$ then $M_{x_1} \neq M_{x_2}$.*

c) *If $M \subset C(f)$ is a closed maximal ideal and $y \in Y$ is a point such that $I_y \subseteq M$ then there exists a point $x \in f^{-1}y$ such that $M_x = M$* (see also 4.4).

d) *If points $x_1, x_2 \in X$ satisfy the condition $M_{x_1} = M_{x_2}$ then $gx_1 = gx_2$ for all functions $g \in C(f)$.*

Proof. a) Let $x \in X$ be an arbitrary point. Then we can define a homomorphism $p_x\colon C(f) \xrightarrow{\text{onto}} \mathbb{C}$ by the formula $p_x g = gx$ for all $g \in C(f)$. The homomorphism p_x is continuous, therefore its kernel $M_x = p_x^{-1}0$ is a closed ideal. This ideal is maximal since the algebra $\mathbb{C}$ has no ideals except $\{0\}$.

b) Let $x_1, x_2 \in X$ be points such that $fx_1 = fx_2$ and $x_1 \neq x_2$. Since the mapping f is globally completely regular, there exists a function $g \in C(f)$ such that $gx_1 = 0$ and $gx_2 = 1$. Then $g \in M_{x_1}$ and $g \notin M_{x_2}$, that is, $M_{x_1} \neq M_{x_2}$.

c) Let M be a closed maximal ideal of the algebra $C(f)$ and $y \in Y$ be a point such that $I_y \subseteq M$. Let us suppose that for each point $x \in f^{-1}y$ there exists a function $g_x \in M$ such that $g_x x \neq 0$; the sets $Ux = \{x' \in X : g_x x' \neq 0\}$, $x \in \in f^{-1}y$, constitute an open covering of the compact space $f^{-1}y$. Let us choose a finite subcovering $\{Ux_k : k = 1, 2, \ldots, n\}$ and put $g_0 = \sum\limits_{k=1}^{n} g_{x_k} \cdot g^*_{x_k}$. Then the (real) function g_0 belongs to M and $g_0 x = \sum\limits_{k=1}^{n} g_{x_k} x \cdot g^*_{x_k} = \sum\limits_{k=1}^{n} |g_{x_k} x|^2 > 0$ for all $x \in f^{-1}y$. Let $t = \inf\{g_0 x : x \in f^{-1}y\}$. Then $t > 0$ since $f^{-1}y$ is compact ([5], Corollary 3.2.9), and we can define a function $g\colon X \to \mathbb{R} \subset \mathbb{C}$ by putting for each $x \in X$

$$gx = \begin{cases} g_0 x \text{ if } g_0 x \geqslant t, \\ t \text{ if } g_0 x < t. \end{cases}$$

Due to Proposition 2.9 we have $n_y(g - g_0) = 0$, therefore $g - g_0 \in I_y \subseteq M$ and $g \in M$. But for the element g of the algebra $C(f)$ there is an inverse element since $gx \neq 0$ for all $x \in X$, and we have got the contradiction: elements of ideals have not inverse elements ([8], §7.4, the statement I). Hence there is a point $x \in f^{-1}y$ such that $gx = 0$ for all $g \in M$ and, consequently, $M = M_x$ since M is a maximal ideal.

d) If $x_1, x_2 \in X$ are points such that $M_{x_1} = M_{x_2}$ then the equality $p_{x_1} = p_{x_2}$ is fulfilled since these homomorphisms have the identical kernels. Hence, $gx_1 = = p_{x_1} g = p_{x_2} g = gx_2$. □

C. The space of closed maximal ideals and the space X

3.6. Let $\mathfrak{M}_f$ be the set of all closed maximal ideals of the algebra $C(f)$. For each $y \in Y$ let $\mathfrak{M}_y = \{M \in \mathfrak{M}_f : I_y \subseteq M\}$. It follows from Lemma 4.4 that $\mathfrak{M}_f = \bigcup\{\mathfrak{M}_y : y \in Y\}$.

Let us define a map $\pi_f\colon X \xrightarrow{\text{onto}} \mathfrak{M}_f$ by the equality $\pi_f x = M_x$ for all $x \in X$ (the equality $\pi_f X = \mathfrak{M}_f$ follows from Lemma 3.5 c) and the preceding equality).

Due to Lemma 3.5 d) for every function $g \in C(f)$ we can define the function $\bar{g}\colon \mathfrak{M}_f \to \mathbb{C}$ by the equality $\bar{g} = g\pi_f^{-1}$ since $\pi_f X = \mathfrak{M}_f$. Let us equip the set $\mathfrak{M}_f$ with the smallest topology in which all functions $\bar{g}$, $g \in C(f)$, are continuous. Then we get the homomorphism $\varphi_f\colon C(f) \to C(\mathfrak{M}_f)$ defined by the formula $\varphi_f g = \bar{g}$ for all $g \in C(f)$.

For every point $y \in Y$ and an arbitrary function $g \in C(f)$ let us set $n_y g = = \sup\{|gM| : M \in \mathfrak{M}_y\}$. As it will be proved in Theorem 3.7, the function n_y is a semi-norm on the algebra $C(\mathfrak{M}_f)$ satisfying the conditions 1)–7) of Theorem 2.7 for each point $y \in Y$. Therefore we can equip the algebra $C(\mathfrak{M}_f)$ with the topology generated by the family of semi-norms $\{n_y : y \in Y\}$ (see 2.14).

3.7. **Theorem.** a) *The map π_f is continuous.*

b) *The mapping $\pi_y = \pi_f|_{f^{-1}y}\colon f^{-1}y \xrightarrow{\text{onto}} \mathfrak{M}_y \subseteq \mathfrak{M}_f$ is a homeomorphism for each point $y \in Y$; hence, the space $\mathfrak{M}_y$ is compact.*

c) *For each point $y \in Y$ the function $n_y\colon C(\mathfrak{M}_f) \to \mathbb{R}$ is a semi-norm satisfying the conditions* 1)–7) *of Theorem 2.7.*

d) *The map $\varphi_f\colon C(f) \xrightarrow{\text{onto}} C(\mathfrak{M}_f)$ is a topological isomorphism preserving all semi-norms n_y, $y \in Y$.*

Proof. a) By the definition of the topology of the space $\mathfrak{M}_f$ all sets of the form $V(g, U) = \bar{g}^{-1}U$, where $g \in C(f)$, $\bar{g} = \varphi_f g$ and $U \subseteq \mathbb{C}$ is an open set, constitute a subbase for this topology; obviously, $\pi_f^{-1}V(g, U) = g^{-1}U$ is an open subset of the space X, therefore the map π_f is continuous.

b) This statement follows from the preceding one, Lemma 3.5 b), c) and Theorem 3.1.13 of the book [5].

c). It follows from the preceding statement that $n_y g$ has a finite value for all $y \in Y$ and $g \in C(\mathfrak{M}_f)$. The rest can be proved in a standard way.

d) The equality $\varphi_f C(f) = C(\mathfrak{M}_f)$ follows from the statement a), since we can define the inverse map $C(\mathfrak{M}_f) \to C(f)$; the equality $n_y(\varphi_f g) = n_y g$ for every $y \in$ $\in Y$ and $g \in C(f)$ follows from the statement b) and Proposition 2.9; the rest is obvious. □

3.8. *Remark.* It follows from the results of this section that a perfect globally completely regular mapping $f\colon X \xrightarrow{\text{onto}} Y$ can be restored (up to a homeomorphism) if its algebra $C(f)$ and the space Y (and the corresponding semi-norms n_y, $y \in Y$) are known: we can set

$$X = \{(y, M) : y \in Y,\ M \subset C(f) \text{ is a maximal ideal such that } I_y \subseteq M\},$$

define the maps $f\colon X \xrightarrow{\text{onto}} Y$ and $\pi_f\colon X \xrightarrow{\text{onto}} \mathfrak{M}_f$ by the equalities $f(y, M) = y$ and $\pi_f(y, M) = M$ for all $(y, M) \in X$, and equip X with the smallest topology in which the maps f and π_f are continuous (the space $\mathfrak{M}_f$ in this case is the set of all closed maximal ideals of the algebra $C(f)$ equipped with the topology described in the book [8], §11.3; see also the item 5.3 below).

§ 4. Some auxiliary lemmas

4.1. **Lemma.** *Let C be a normed (complex or real) commutative algebra with the unit e_C and with a norm satisfying the conditions 1)–7) of Theorem 2.7, and let $\tilde{C}$ be its completion. Then*

a) *for every closed maximal ideal $M \subset C$ the set $\tilde{M} = [M]_{\tilde{C}}$ is a maximal ideal of the algebra $\tilde{C}$;*

b) *for every maximal ideal $\tilde{M} \subset \tilde{C}$ the set $M = \tilde{M} \cap C$ is a closed maximal ideal of the algebra C;*

c) *the maps described in a) and b) above are inverse to each other, and the equality $gM = g\tilde{M}$ is valid for all elements $g \in C$ and closed maximal ideals $M \subset C$* (see 5.3).

Proof. a) The set $\tilde{M}$ is an ideal of the algebra $\tilde{C}$ due to the continuity of the multiplication. The quotient algebra C/M is isomorphic to the algebra $\mathbb{C}$ (or $\mathbb{R}$) since the ideal M is maximal. Hence the quotient algebra $\tilde{C}/\tilde{M}$ contains C/M as dense (since C is dense in $\tilde{C}$) and closed (since C/M is complete) subset. Therefore $\tilde{C}/\tilde{M}$ coincides with the algebra C/M, that is, it is isomorphic to $\mathbb{C}$ (or $\mathbb{R}$), and the ideal $\tilde{M}$ is maximal.

The statements b) and c) are obvious. □

4.2. In the following lemmas let C be a (complex or real) commutative algebra with the unit e_C equipped with a Hausdorff topology generated by the family of semi-norms $\{n_y : y \in Y\}$ satisfying the conditions 1)–7) of Theorem 2.7.

4.3. **Lemma.** *Let $I \subset C$ be a closed ideal and let $p\colon C \xrightarrow{\text{onto}} C/I$ be a homomorphism onto the quotient algebra. Then*

a) *for every closed maximal ideal $M \subset C$ satisfying the condition $I \subseteq M$ the set $\bar{M} = pM$ is a closed maximal ideal of the algebra C/I;*

b) *for every closed maximal ideal $\bar{M} \subset C/I$ the set $M = p^{-1}\bar{M}$ is a closed maximal ideal of the algebra C such that $I \subseteq M$;*

c) *the maps described in a) and b) above are inverse to each other, and the equality $(pg)\bar{M} = gM$ is valid for all elements $g \in C$ and closed maximal ideals $M \subset C$ satisfying the condition $I \subseteq M$* (see 5.3).

Proof is obvious. □

4.4. **Lemma.** *For each closed maximal ideal $M \subset C$ there is a point $y \in Y$ such that $I_y \subseteq M$, where $I_y = \{g \in C : n_y g = 0\}$* (see 5.2).

Proof. Let us note that if $M \subset C$ is a maximal ideal and $I \subset C$ is an ideal such that $I \setminus M \neq \varnothing$, then there are elements $g_M \in M$ and $g_I \in I$ such that $g_M + g_I = e_C$. Indeed, if $g_0 \in I$ is an element such that $g_0 \notin M$ then there are elements $g_1 \in M$ and $g \in C$ such that $g \cdot (g_1 + g_0) = e_C$ since the ideal M is maximal. Then we can take $g_M = g \cdot g_1$ and $g_I = g \cdot g_0$.

Let $M \subset C$ be a closed maximal ideal. By the definition of the topology of the algebra C there exist a number $\varepsilon > 0$ and a finite set $A \subseteq Y$ such that $V(e_C, \varepsilon, A) \cap M = \varnothing$, where

$$V(e_C, \varepsilon, A) = \{g \in C : n_y(g - e_C) < \varepsilon \text{ for all } y \in A\}.$$

Let us prove that

$$I_A = \bigcap\{I_y : y \in A\} = \{g \in C : n_y g = 0 \text{ for all } y \in A\} \subseteq M.$$

If, on the contrary, there is some element $g \in I_A \setminus M$, then there are $g_M \in M$ and $g_{I_A} \in I_A$ such that $g_M + g_{I_A} = e_C$. Hence, we have $p_A g_M = p_A(g_M + g_{I_A}) = p_A e_C = e_A$, where $p_A \colon C \xrightarrow{\text{onto}} C/I_A$ is a homomorphism onto the quotient algebra and e_A is the unit of the algebra C/I_A. But it is obvious that $p_A^{-1} e_A \subseteq V(e_C, \varepsilon, A)$, therefore $g_M \in V(e_C, \varepsilon, A) \cap M$, that is, $V(e_C, \varepsilon, A) \cap M \neq \varnothing$. But this is impossible by the choice of the neighborhood $V(e_C, \varepsilon, A)$. Hence, $I_A \subseteq M$.

Let us suppose that $I_y \setminus M \neq \varnothing$ for all $y \in A$. Analogously, for each $y \in A$ there are elements $g_y \in M$ and $g_{I_y} \in I_y$ such that $g_y + g_{I_y} = e_C$. Then we have $\prod\{g_y + g_{I_y} : y \in A\} = e_C$. Removing the brackets we get

$$\prod\{g_y : y \in A\} + \sum\{g_{I_y} \cdot \prod\{g_z : z \in A \setminus \{y\}\} : y \in A\} + \ldots + \prod\{g_{I_y} : y \in A\} = e_C.$$

In this sum all terms, with the exception of the last one, belongs to M, and $\prod\{g_{I_y} : y \in A\} \in \bigcap\{I_y : y \in A\} = I_A$. Hence, we have got an equality of the type $g_M + g_{I_A} = e_C$, where $g_M \in M$ and $g_{I_A} \in I_A$. But this is impossible since $g_{I_A} \in I_A \subseteq M$ and $e_C \notin M$. Therefore there is an element $y \in A$ such that $I_y \subseteq M$. □

4.5. **Lemma.** *For each closed ideal $J \subset C$ there is a closed maximal ideal $M \subset C$ such that $J \subseteq M$.*

Proof. Since the ideal J is closed, there exist a number $\varepsilon > 0$ and a finite set $A \subseteq Y$ such that $V(e_C, \varepsilon, A) \cap J = \varnothing$. Let us set $I_A = \bigcap\{I_y : y \in A\}$. Then the set $J'_A = \{g_1 + g_2 : g_1 \in J, g_2 \in I_A\}$ is an ideal such that $I_A \cup J \subseteq J'_A$ and $V(e_C, \varepsilon, A) \cap J'_A = \varnothing$. By the statement VII, §8.1 of the book [8], the set $J_A = [J'_A]_C$ is a closed ideal, $I_A \cup J \subseteq J_A$ and $V(e_C, \varepsilon, A) \cap J_A = \varnothing$.

Let us define the semi-norm n_A by the equality $n_A g = \max\{n_y g : y \in A\}$ for all $g \in C$. It is easily verified that all conditions 1)–7) of Theorem 2.7 are valid. We can consider the quotient algebra $C_A = C/I_A$ and the corresponding homomorphism $p_A \colon C \xrightarrow{\text{onto}} C_A$. The algebra C_A is normed with the norm defined by the equality $\|\bar{g}\| = n_A g$ for $\bar{g} \in C_A$ where $g \in C$ is any element such that $p_A g = \bar{g}$ (it is easily seen that $n_A g = n_A g'$ if $p_A g = p_A g'$ since $n_A(g - g') = 0$). Let $\tilde{C}_A$ be a completion of the algebra C_A.

The set $\tilde{j} = [p_A J_A]_{\tilde{C}_A}$ is a closed ideal of the algebra $\tilde{C}_A$. There is a maximal ideal $\tilde{M} \subset \tilde{C}_A$ such that $\tilde{j} \subseteq \tilde{M}$ (see [8], §7.4, the statement IV). The ideal $\tilde{M}$ is closed by the statement II 4), §9.4 of the book [8]. By Lemmas 4.1 and 4.3 the set $M = p^{-1}(\tilde{M} \cap C_A)$ is a closed maximal ideal, and $J \subseteq M$. □

§ 5. Realizations of abstract algebras

5.1. Further on let C be a commutative complex algebra with the unit e_C (except the item 5.22) and the zero 0_C, equipped with a Hausdorff topology generated (see 2.14) by a family of semi-norms $\{n_y : y \in Y\}$ satisfying the conditions 1)–7) of Theorem 2.7 (if the algebra C has not an involution then we put $g^* = g$ for all $g \in C$ in the conditions 6) and 7); if the algebra C has not a unit[3] then we can add the unit e using the method of the item III, §16.1 of the book [8][4]). It is not necessary to suppose that $n_{y_1} \neq n_{y_2}$ for $y_1, y_2 \in Y$ such that $y_1 \neq y_2$.

Let $\{n_y : y \in Y_C\}$ be the family of all continuous semi-norms on the algebra C satisfying the conditions 1)–7) of Theorem 2.7. We shall assume that $n_{y_1} \neq n_{y_2}$ for all $y_1, y_2 \in Y_C$ such that $y_1 \neq y_2$. Of course, we can use this family instead of the preceding one.

A. The space of closed maximal ideals

5.2. Let $\mathfrak{M}$ be the set of all closed maximal ideals of the algebra C.

Let $y \in Y$ (or $y \in Y_C$) be an arbitrary point. Then the set $I_y = \{g \in C : n_y g = 0\}$ is a closed ideal of the algebra C. Let us set $\mathfrak{M}_y = \{M \in \mathfrak{M} : I_y \subseteq M\}$. It follows from Lemma 4.4 that $\mathfrak{M} = \bigcup\{\mathfrak{M}_y : y \in Y\} = \bigcup\{\mathfrak{M}_y : y \in Y_C\}$.

5.3. Every element $g \in C$ can be considered as a function $g\colon \mathfrak{M} \to \mathbb{C}$: for any $M \in \mathfrak{M}$ let $p_M\colon C \xrightarrow{\text{onto}} C/M = \mathbb{C}$ be the homomorphism onto the quotient algebra; then we put $gM = p_M g$ (see [8], §11.2).

Let us equip the set $\mathfrak{M}$ with the smallest topology in which all functions $g \in C$ are continuous (see [8], §11.3). The space $\mathfrak{M}$ with this topology is completely regular.

5.4. **Lemma.** *Let $y \in Y$ (or Y_C) be an arbitrary point. Then*

a) *the subspace $\mathfrak{M}_y \subseteq \mathfrak{M}$ is compact;*

b) *for each element $g \in C$ the equality $n_y g = \sup\{|gM| : M \in \mathfrak{M}_y\}$ is satisfied.*

Proof. Let $y \in Y$ and let $p_y\colon C \xrightarrow{\text{onto}} C_y = C/I_y$ be the homomorphism onto the quotient algebra equipped with the norm which is defined by the equality $\|\bar{g}\| = n_y g$ for every $\bar{g} \in C_y$, where $g \in C$ is an element such that $p_y g = \bar{g}$. Let $\tilde{C}_y$ be the completion of the algebra C_y. Since the conditions 1)–7) of Theorem 2.7 are preserved under the completion, as in the proof of Theorem 1, §16.2 of the book [8], we get $\|\tilde{g}^2\| = \|\tilde{g}\|^2$ and $\sup\{|\tilde{g}\tilde{M}| : \tilde{M} \in \tilde{\mathfrak{M}}_y\} = \|\tilde{g}\|$ for all functions $\tilde{g} \in \tilde{C}_y$, where $\tilde{\mathfrak{M}}_y$ is the space of all maximal ideals of the algebra $\tilde{C}_y$. The space $\tilde{\mathfrak{M}}_y$ is compact due to Theorem 2 of §11.3 of the book [8].

Using Lemmas 4.1 and 4.3 we get the one-to-one map $\varphi_y\colon \tilde{\mathfrak{M}}_y \xrightarrow{\text{onto}} \mathfrak{M}_y$ defined by the formula $\varphi_y \tilde{M} = p_y^{-1}(\tilde{M} \cap C_y)$ for all $\tilde{M} \in \tilde{\mathfrak{M}}_y$. This map satisfies the condition $g(\varphi_y \tilde{M}) = \bar{g}\tilde{M}$ for all $\tilde{M} \in \tilde{\mathfrak{M}}_y$ and $g \in C$, where $\bar{g} = p_y g$. It follows from the comparison of the topologies that the map φ_y is continuous, and hence it is a homeomorphism since the space $\tilde{\mathfrak{M}}_y$ is compact ([5], Theorem 3.1.13). Finally,

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[3]It is possible that there is an element $g_0 \in C$ such that $g_0 \cdot g = g$ for all $g \in C$, but $n_y g_0 = 0$ for some $y \in Y$ (of course, $n_y g = 0$ for all $g \in C$ in this case). We shall call this element by the quasi-unit since it does not satisfy the condition 5) of Theorem 2.7.

[4]If the algebra C has not a unit then we consider the algebra $C' = \{\lambda \cdot e + g' : \lambda \in \mathbb{C}, g' \in C\}$, where e is the unit, and for each $y \in Y$, $\lambda \in \mathbb{C}$ and $g' \in C$ we define

$$n_y(\lambda\cdot e+g') = \begin{cases} \sup\{n_y(\lambda \cdot g + g' \cdot g) : g \in C, n_y g = 1\} & \text{if there exists } g \in C \text{ such that } n_y g > 0, \\ |\lambda| & \text{if } n_y g = 0 \text{ for all } g \in C. \end{cases}$$

we have

$$n_y g = \|\bar{g}\| = \sup\{|\bar{g}\tilde{M}| : \tilde{M} \in \tilde{\mathfrak{M}}_y\} = \sup\{|gM| : M \in \mathfrak{M}_y\}$$

for any $g \in C$, where $\bar{g} = p_y g$. □

B. **Preorders on the sets of semi-norms**

5.5. Let us equip the set Y with the preorder[5] by putting $y_1 \leqslant y_2$ for $y_1, y_2 \in Y$ if $n_{y_1} g \leqslant n_{y_2} g$ for all $g \in C$.

Analogously the preorder is defined for the set Y_C; in this case this preorder is a partial order.

Let us define the map $j_Y : Y \to Y_C$ by the equality $j_Y y = y'$ for $y \in Y$ where $y' \in Y_C$ is an element (unique for any $y \in Y$) such that $n_{y'} = n_y$. Obviously, this map preserves the preorder, that is, if $y_1, y_2 \in Y$ and $y_1 \leqslant y_2$ then $j_Y y_1 \leqslant j_Y y_2$.

5.6. **Assertion.** *For all $y_1, y_2 \in Y$ (or Y_C) the following conditions are equivalent:*

1) $y_1 \leqslant y_2$;
2) $I_{y_1} \supseteq I_{y_2}$;
3) $\mathfrak{M}_{y_1} \subseteq \mathfrak{M}_{y_2}$.

Proof. The implications 1) $\Longrightarrow$ 2) and 2) $\Longrightarrow$ 3) are obvious, the implication 3) $\Longrightarrow$ 1) follows from Lemma 5.4b). □

5.7. **Assertion.** a) *For every finite set $A \subseteq Y_C$ there exists* $\max A \in Y_C$.

b) *For each maximal ideal $M \in \mathfrak{M}$ there exists and is unique an element $y_M \in \in Y_C$ such that $n_{y_M} g = |gM|$ for all $g \in C$; moreover, the element y_M is minimal in the set Y_C.*

c) *A maximal element of the set Y_C exists iff the algebra C is normed.*

Proof. a) See the item 3.2.

b) It follows from the statement I, §11.2 of the book [8], and Assertion 5.6.

c) This is obvious, since an element $y \in Y_C$ is maximal iff the semi-norm n_y is a norm on the algebra C. □

5.8. **Lemma.** *For each point $y_0 \in Y_C$ there exists a finite subset $A \subseteq Y$ such that for all $g \in C$ the inequality $n_{y_0} g \leqslant n_A g$ is satisfied, that is, $y_0 \leqslant \max j_Y A$* (see the item 3.2).

Proof. The set $V(0_C, 1, \{y_0\}) = \{g \in C : n_{y_0} g < 1\}$ is open since the semi-norm n_{y_0} is continuous. Therefore there are a finite set $A \subseteq Y$ and a number $\varepsilon > 0$ such that $V(0_C, \varepsilon, A) = \{g \in C : n_y g < \varepsilon \text{ for all } y \in A\} \subseteq V(0_C, 1, \{y_0\})$.

Let us assume that $I_A = \bigcap\{I_y : y \in A\}$ is not contained in I_{y_0}, that is, there exists an element $g_0 \in I_A \setminus I_{y_0}$. Then we have $n_A g_0 = \max\{n_y g_0 : y \in A\} = 0$ and $n_{y_0} g_0 = c \neq 0$. But then for $g = \frac{1}{c} \cdot g_0$ we get $g \in V(0_C, \varepsilon, A)$ and $g \notin V(0_C, 1, y_0)$ since $n_{y_0} g = 1$, and this contradicts the choice of the set $V(0_C, \varepsilon, A)$. Therefore $I_A \subseteq I_{y_0}$. Since the semi-norm n_A has all properties 1)–7) of Theorem 2.7, there is $y \in Y_C$ such that $n_A = n_y$ and $I_A = I_y$ (that is, $y = \max j_Y A$). Using Assertion 5.6 we get the desirable inequality. □

5.9. **Definition.** A family of continuous semi-norms on the algebra C satisfying the conditions 1)–7) of Theorem 2.7 will be called *sufficient* if it defines the given topology of the algebra C in the sense of 2.14.

5.10. Of course, the families $\{n_y : y \in Y\}$ and $\{n_y : y \in Y_C\}$ are sufficient.

5.11. **Corollary.** *A family $\{n_y : y \in Y'\}$ of continuous semi-norms on the algebra C satisfying the conditions 1)–7) of Theorem 2.7 is sufficient iff for each $y \in Y_C$ there is a finite subset $A \subseteq Y'$ such that $y \leqslant \max j_{Y'} A$.*

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[5]A preorder is a transitive reflexive relation.

C. **An involution**

5.12. **Lemma.** *If the algebra C has an involution then it is symmetric, that is, $g^*M = \overline{gM}$ for all $g \in C$ and $M \in \mathfrak{M}$*[6] (see [8], §14).

Proof. Let $g \in C$, $M \in \mathfrak{M}$ and let $y \in Y$ be an element such that $M \in \mathfrak{M}_y$. Let $\tilde{C}_y$ be defined as in the proof of Lemma 5.4 and let $\tilde{M} = [p_y M]_{\tilde{C}_y}$ and $\bar{g} = p_y g$. Then we have the equalities $g^*M = \bar{g}^*\tilde{M}$ and $gM = \bar{g}\tilde{M}$.

The algebra $\tilde{C}_y$ satisfies the conditions of Theorem 1 of §16.2 of the book [8] and, hence, it is symmetric, that is, $\bar{g}^*\tilde{M} = \overline{\bar{g}\tilde{M}}$. Therefore the algebra C is symmetric too, since $g^*M = \bar{g}^*\tilde{M} = \overline{\bar{g}\tilde{M}} = \overline{gM}$. □

5.13. Let $C(\mathfrak{M})$ be the algebra of all continuous functions $g\colon \mathfrak{M} \to \mathbb{C}$.

For each $g \in C(\mathfrak{M})$ and $y \in Y$ let $n_y g = \sup\{|gM| : M \in \mathfrak{M}_y\}$. Since the set $\mathfrak{M}_y$ is compact for every $y \in Y$, the function n_y is a semi-norm on the algebra $C(\mathfrak{M})$ satisfying the conditions 1)–7) of Theorem 2.7. Moreover, if $g \in C(\mathfrak{M})$ is a function such that $n_y g = 0$ for all $y \in Y$, then $gM = 0$ for all $M \in \mathfrak{M}$, since $\mathfrak{M} = \bigcup\{\mathfrak{M}_y : y \in Y\}$.

Let us equip the algebra $C(\mathfrak{M})$ with the topology generated by the family of semi-norms $\{n_y : y \in Y\}$.

Due to Corollary 5.11 this topology is independent of the choice of a sufficient family of semi-norms. Moreover, the algebra $C(\mathfrak{M})$ defines the same preorders on the sets Y and Y_C as C does (see Assertion 5.6; this is true for any subalgebra $\tilde{C} \subseteq C(\mathfrak{M})$ such that $C \subseteq \tilde{C}$).

As it was shown in the item 5.3, we can regard the algebra C as a subalgebra of $C(\mathfrak{M})$. Since this embedding preserves all semi-norms n_y, $y \in Y$ (see Lemma 5.4b)), it is a topological isomorphism onto a subalgebra.

The algebra $C(\mathfrak{M})$ has the involution defined by the equality $g^*M = \overline{gM}$ for all $g \in C(\mathfrak{M})$ and $M \in \mathfrak{M}$. By Lemma 5.12, if the algebra C has an involution, then this involution agrees with the involution of the algebra $C(\mathfrak{M})$.

D. **Topologies on sets of semi-norms**

5.14. Let us fix some subalgebra $\tilde{C} \subseteq C(\mathfrak{M})$ with an involution such that $C \subseteq \tilde{C}$. Of course, we can take $\tilde{C} = C(\mathfrak{M})$; if the algebra C has an involution, then we can take $\tilde{C} = C$.

For each function $g \in \tilde{C}$ let $U_g = \{y \in Y : n_y g < 1\}$ (compare with Corollary 2.11). Let us equip the set Y with a topology by taking as a subbase of this topology the family $\{U_g : g \in \tilde{C}\}$.

Analogously, a topology is defined for the set Y_C.

It follows from Corollaries 2.11 and 5.20 that the defined topology on the set Y (or Y_C) is the smallest one which is compatible with the algebra C, but we can use any larger topology (see 5.32). Nevertheless we shall use the smallest topology until the item 5.32.

5.15. **Lemma.** a) *The family $\{U_g : g \in \tilde{C}\}$ is a base for the topology of the space Y.*

b) *The space Y_C is a T_0-space.*

c) *The map $j_Y\colon Y \to Y_C$ (see 5.5) is continuous. The mapping j_Y is a homeomorphism onto $j_Y Y \subseteq Y_C$ iff $n_{y_1} \neq n_{y_2}$ for all $y_1, y_2 \in Y$ such that $y_1 \neq y_2$.*

d) *For each $y_0 \in Y$ the equality*

$$\{y \in Y : y \leqslant y_0\} = \bigcap\{Uy_0 : Uy_0 \subseteq Y \text{ is a neighborhood of the point } y_0\}$$

holds.

[6] The long line means the complex conjugation.

Proof. a) Let elements $g_1, g_2 \in \tilde{C}$ and $y_0 \in Y$ satisfy the condition $y_0 \in U_{g_1} \cap U_{g_2}$. In order to prove our statement we have to find a function $g \in \tilde{C}$ such that $y_0 \in \in U_g \subseteq U_{g_1} \cap U_{g_2}$.

By the equality $U_g = U_{g \cdot g^*}$ for all $g \in \tilde{C}$, we can assume that the functions g_1 and g_2 are real and non-negative. Let $c_1 = n_{y_0} g_1$ and $c_2 = n_{y_0} g_2$. Then $0 \leqslant c_1 < 1$ and $0 \leqslant c_2 < 1$. Therefore there exist numbers $m_1, m_2 \in \mathbb{N}$ such that $c_1^{m_1} < \frac{1}{2}$ and $c_2^{m_2} < \frac{1}{2}$. Hence the function $g = g_1^{m_1} + g_2^{m_2} \in \tilde{C}$ satisfies the conditions, which follows from the definition of seminorms n_y, $y \in Y$ (see 5.13):

1) $n_{y_0} g \leqslant n_{y_0}(g_1^{m_1}) + n_{y_0}(g_2^{m_2}) = c_1^{m_1} + c_2^{m_2} < \frac{1}{2} + \frac{1}{2} = 1$,
2) if $n_y g_1 \geqslant 1$ or $n_y g_2 \geqslant 1$ for some $y \in Y$ then $n_y g \geqslant 1$.

b) Let $y_1, y_2 \in Y_C$ and $y_1 \neq y_2$. Since $n_{y_1} \neq n_{y_2}$ by the definition of the set Y_C, there exists a function $g_0 \in C \subseteq \tilde{C}$ such that $n_{y_1} g_0 \neq n_{y_2} g_0$. Let us suppose, for example, that $n_{y_1} g_0 < n_{y_2} g_0$. Let $c = \frac{1}{2} \cdot (n_{y_1} g_0 + n_{y_2} g_0)$ and $g = \frac{1}{c} \cdot g_0$. Then $y_1 \in U_g$ and $y_2 \notin U_g$.

c) This statement is obvious.

d) It is sufficient to note that the following statements hold:

1) if elements $y, y_0 \in Y$ and $g \in \tilde{C}$ satisfy the conditions $y_0 \in U_g$ and $y \leqslant y_0$ then $y \in U_g$;
2) if elements $y, y_0 \in Y$ do not satisfy the condition $y \leqslant y_0$ then there exists a function $g_0 \in C \subseteq \tilde{C}$ such that $n_y g_0 > n_{y_0} g_0$; if $c = \frac{1}{2} \cdot (n_y g_0 + n_{y_0} g_0)$ and $g = \frac{1}{c} \cdot g_0$ then $y \notin U_g$ and $y_0 \in U_g$.

Therefore the required equality is fulfilled. □

E. **A construction of mappings**

5.16. Let us set
$$X = \{(y, M) \in Y \times \mathfrak{M} : M \in \mathfrak{M}_y\}$$
and define maps $f \colon X \xrightarrow{\text{onto}} Y$ and $\pi \colon X \xrightarrow{\text{onto}} \mathfrak{M}$ by the equalities $f(y, M) = y$ and $\pi(y, M) = M$ for all $(y, M) \in X$. For each function $g \in C(\mathfrak{M})$ let us define a function $\bar{g} \colon X \to \mathbb{C}$ by the equality $\bar{g}(y, M) = gM$ for all $(y, M) \in X$, that is, $\bar{g} = g\pi$.

Let us equip the set X with the smallest topology in which all maps f and $\bar{g}$, $g \in C(\mathfrak{M})$, are continuous. Obviously, this topology coincides with the topology of the subspace of the product $Y \times \mathfrak{M}$, and the map π is continuous.

Thus, we have the homomorphism $\varphi \colon C(\mathfrak{M}) \to C(f)$ defined by the formula $\varphi g = g\pi$ for all $g \in C(\mathfrak{M})$.

We shall write X_C, f_C, π_C and φ_C instead of X, f, π and φ if $Y = Y_C$ in this construction.

The algebra $\tilde{C}$ is defined in the item 5.14.

5.17. **Lemma.** *For each point $x \in X$ and each closed set $F \subset X$ such that $x \notin F$ there exist a neighborhood $Ufx \subseteq Y$ and a function $g \in \tilde{C}$ such that the following conditions are satisfied:*

1) *gM is real and $gM \geqslant -\frac{1}{2}$ for all $M \in \mathfrak{M}$;*
2) *$\bar{g}x = g\pi x = -\frac{1}{2}$;*
3) *$\bar{g}x' \geqslant 1$ for all $x' \in F \cap f^{-1}Ufx$.*

Proof. By the definition of the topology of the space X (see 5.3 and 5.16) there are functions $g_1, g_2, \ldots, g_n \in C$ and neighborhoods $Ufx \subseteq Y$ and $U\bar{g}_1 x, U\bar{g}_2 x, \ldots, U\bar{g}_n x \subseteq \mathbb{C}$ such that $x \in f^{-1}Ufx \cap \bigcap\{\bar{g}_k^{-1} U\bar{g}_k x : k = 1, 2, \ldots, n\} \subseteq X \setminus F$.

For $k = 1, 2, \ldots, n$ let $c_k = \inf\{|c - \bar{g}_k x|^2 : c \in \mathbb{C} \setminus U\bar{g}_k x\}$. Then $c_k > 0$ for $k = 1, 2, \ldots, n$, and we can define the function

$$g = \sum_{k=1}^{n} \frac{3}{2c_k} \cdot (g_k - \bar{g}_k x \cdot e_C) \cdot (g_k - \bar{g}_k x \cdot e_C)^* - \frac{1}{2} \cdot e_C.$$

The function $g \in \tilde{C}$ and the neighborhood $Ufx \subseteq Y$ have the required properties 1), 2) and 3). □

5.18. **Lemma.** *For each point $y \in Y$, each compact set $\Phi \subseteq f^{-1}y$ and each closed set $F \subseteq X$ such that $F \cap \Phi = \varnothing$ there are a neighborhood $Uy \subseteq Y$ and a function $g \in \tilde{C}$ such that the following conditions hold:*

1) *gM is real and $gM \geqslant -\frac{1}{2}$ for all $M \in \mathfrak{M}$;*
2) *$-\frac{1}{2} \leqslant \bar{g}x = g\pi x < 0$ for all $x \in \Phi$;*
3) *$\bar{g}x \geqslant 1$ for all $x \in F \cap f^{-1}Uy$.*

Proof. For each point $x \in \Phi$ there exist a neighborhood $U_x y \subseteq Y$ and a function $g_x \in \tilde{C}$ satisfying the conditions 1)–3) of Lemma 5.17. The sets $Ux = \{x' \in \in X : \bar{g}_x x' < 0\}$, $x \in \Phi$, $\bar{g}_x = g_x\pi$, form an open covering of the compact set Φ. Let $\{Ux_1, Ux_2, \ldots, Ux_n\}$ be its finite subcovering and $Uy = \bigcap\{U_{x_k}y : k = 1, 2, \ldots, n\}$.

If $n = 1$ then our Lemma is proved. Let us suppose that $n > 1$ and show that the number n can be decreased.

Let $A = \sup\{\max\{0, \bar{g}_{x_1}x, \bar{g}_{x_2}x\} : x \in Ux_1 \cup Ux_2\}$. The function $g = 2 \cdot g_{x_1} \cdot g_{x_2}$ satisfies the following conditions ($\bar{g} = g\pi$):

α) $\bar{g}x \geqslant 2$ for all $x \in F \cap f^{-1}Uy$;
β) $-A \leqslant \bar{g}x \leqslant \frac{1}{2}$ for all $x \in Ux_1 \cup Ux_2$;
γ) $\bar{g}x \geqslant -A$ for all $x \in X$.

Let us find a number $m \in \mathbb{N}$ such that $\left(\frac{A+0.5}{A+1}\right)^m < \frac{1}{2}$. Then $\left(\frac{A+2}{A+1}\right)^m > 2 > \frac{3}{2}$ since $\frac{A+0.5}{A+1} \cdot \frac{A+2}{A+1} \geqslant 1$, therefore the function $g_{1,2} = \left(\frac{g+A\cdot e_C}{A+1}\right)^m - \frac{1}{2} \cdot e_C$ satisfies the following conditions ($\bar{g}_{1,2} = g_{1,2}\pi$):

α') $\bar{g}_{1,2}x \geqslant 1$ for all $x \in F \cap f^{-1}Uy$;
β') $-\frac{1}{2} \leqslant \bar{g}_{1,2}x < 0$ for all $x \in Ux_1 \cup Ux_2$;
γ') $\bar{g}_{1,2}x \geqslant -\frac{1}{2}$ for all $x \in X$.

Let $U_{1,2} = \{x \in X : \bar{g}_{1,2}x < 0\}$. Then we can use the covering $\{U_{1,2}, Ux_3, \ldots, Ux_n\}$ and the functions $g_{1,2}, g_{x_3}, \ldots, g_{x_n}$.

Repeating this reasoning we can decrease the number n to 1. □

5.19. **Lemma.** *The mapping $f\colon X \xrightarrow{\text{onto}} Y$ is perfect and globally completely regular.*

Proof. The mapping f is globally completely regular by Lemma 5.17.

The mapping f is compact by Lemma 5.4 a). Let us prove the closedness of the mapping f. For this end it is sufficient to prove that if $U \subseteq X$ is an open set and $y \in Y$ is a point such that $f^{-1}y \subseteq U$, then there exists a neighborhood $Uy \subseteq Y$ such that $f^{-1}Uy \subseteq U$. Indeed, for a given open set $U \subseteq X$ there exist, due to Lemma 5.18, a neighborhood $U_0 y \subseteq Y$ and a real function $g \in \tilde{C}$ such that $-\frac{1}{2} \leqslant \bar{g}x < 0$ for all $x \in f^{-1}y$ and $\bar{g}x \geqslant 1$ for all $x \in f^{-1}U_0 y \setminus U$. Since $n_y g \leqslant \frac{1}{2}$ by Lemma 5.4 b), the set U_g is a neighborhood of the point y, and the set $Uy = U_g \cap U_0 y$ is a required neighborhood. □

5.20. **Corollary.** a) *For each $y \in Y$ and $g \in C(\mathfrak{M})$ the equality*

$$n_y g = \inf\{\sup\{|g\pi x| : x \in f^{-1}Uy\} : Uy \subseteq Y \text{ is a neighborhood of the point } y\}$$

is fulfilled, and, hence, the set $U_g = \{y \in Y : n_y g < 1\}$ is open for each $g \in C(\mathfrak{M})$.

b) *The topology of the space Y is independent of the choice of the algebra $\tilde{C}$ in the item 5.14.*

F. **Realizations of algebras**

5.21**. Corollary.** *The map $\varphi\colon C(\mathfrak{M}) \to C(f)$* (see 5.16) *is a topological isomorphism onto a subalgebra preserving all semi-norms n_y, $y \in Y$. Moreover, if the mapping π is quotient* (*in particular, if it is closed or open*) *then $\varphi C(\mathfrak{M})$ is a closed subalgebra of the algebra $C(f)$.*

Proof. By Corollary 5.20 a) the isomorphism φ preserves all semi-norms n_y, $y \in Y$. Hence, it is a topological isomorphism onto a subalgebra of $C(f)$.

If the mapping π is quotient then the closedness of $\varphi C(\mathfrak{M})$ in $C(f)$ can be proved in a way analogous to the proof of the statement 2) of Theorem 2.23. □

5.22**. Theorem.** *Let C be a Hausdorff topological commutative complex algebra. Let us suppose that $\{n_y : y \in Y\}$ is a sufficient family of continuous seminorms on the algebra C satisfying the conditions 1)–7) of Theorem 2.7. Then there exist a topology on the set Y, a topological space X and a perfect globally completely regular mapping $f\colon X \xrightarrow{\text{onto}} Y$ such that the algebra C is topologically* (*and isometrically*) *isomorphic to a subalgebra of the algebra $C(f)$.*

Proof follows from the items 5.1, 5.14, 5.16, 5.19 and 5.21. □

5.23**. Lemma.** *If $(y_1, M), (y_2, M) \in X$ and $y_1 \leqslant y_2$, then for each continuous function $g\colon C \to \mathbb{C}$ the equality $g(y_1, M) = g(y_2, M)$ holds.*

Proof. By the definition of the topology of the space X for any couples (y_1, M), $(y_2, M) \in X$ satisfying the condition $y_1 \leqslant y_2$, every neighborhood of the point (y_2, M) contains the point (y_1, M). Therefore a function $g\colon C \to \mathbb{C}$ which is continuous at the point (y_2, M) has to satisfy the condition $g(y_1, M) = g(y_2, M)$. □

5.24**. Assertion.** *Let us assume that there exists a subspace $\mathfrak{M}' \subseteq X$ such that the mapping $\tilde{\pi} = \pi|_{\mathfrak{M}'}\colon \mathfrak{M}' \xrightarrow{\text{onto}} \mathfrak{M}$ is a homeomorphism. Then the map $h\colon C(f) \to C(\mathfrak{M})$ defined by the formula $hg = g\tilde{\pi}^{-1}$ for $g \in C(f)$ is a continuous homomorphism such that $h\varphi$ is the identity map and φh is a retraction. In particular, the algebra $\varphi C(\mathfrak{M})$ is closed in $C(f)$.*

5.25**. Corollary.** *Let the assumption of Assertion 5.24 be valid and let for each elements $(y_1, M), (y_2, M) \in X$ there exists a couple $(y, M) \in X$ such that the element y is comparable with y_1 and y_2 in the sense of 5.5. Then the homomorphisms h and φ are mutually inverse topological isomorphisms preserving all semi-norms $\{n_y : y \in Y\}$.*

5.26**. Lemma.** a) *The correspondence $M \to y_M$ for $M \in \mathfrak{M}$* (see 5.7 b)) *defines a homeomorphic embedding $i_C\colon \mathfrak{M} \to Y_C$.*

b) *The mappings $\tilde{f} = f|_{\mathfrak{M}'}\colon \mathfrak{M}' \xrightarrow{\text{onto}} i_C\mathfrak{M}$ and $\tilde{\pi} = \pi|_{\mathfrak{M}'}\colon \mathfrak{M}' \xrightarrow{\text{onto}} \mathfrak{M}$, where $\mathfrak{M}' = f^{-1}i_C\mathfrak{M}$, are homeomorphisms.*

Proof. a) Let us note that all sets of the form

$$U(g, \varepsilon, c) = \{M \in \mathfrak{M} : |gM - c| < \varepsilon\} = \{M \in \mathfrak{M} : \tilde{g}M < \varepsilon^2\},$$

where $g \in \tilde{C}$, $\varepsilon > 0$, $c \in \mathbb{C}$ and $\tilde{g} = (g - c \cdot e_C) \cdot (g - c \cdot e_C)^*$ (e_C is the unit), constitute a subbase for the topology of the space $\mathfrak{M}$.

Analogously, all sets of the form

$$U_g = \{y \in i_C\mathfrak{M} : n_y g < 1\} = \{y_M : M \in \mathfrak{M}, |gM| < 1\} = \{y_M : M \in \mathfrak{M}, \tilde{g}M < 1\},$$

where $g \in \tilde{C}$ and $\tilde{g} = g \cdot g^*$, constitute a subbase for the topology of the space $i_C\mathfrak{M}$.

Comparing these definitions, we get that $i_C U(g,\varepsilon,c) = U_{\tilde g}$, where $\tilde g = \frac{1}{\varepsilon}\cdot(g - c\cdot e_C)$, and $i_C^{-1} U_g = U(g,1,0)$ for all $g \in \tilde C$, $\varepsilon > 0$ and $c \in \mathbb{C}$. Hence, the map i_C is a homeomorphism onto $i_C\mathfrak{M}$ since i_C is one-to-one.

b) This statement is obvious. □

5.27. **Corollary.** *Let one of the following conditions holds:*

a) *the set Y has the largest element $y_0 \in Y$ (that is, $y_0 \geqslant y$ for all $y \in Y$);*

b) $j_Y Y \supseteq i_C\mathfrak{M}$ (see 5.5).

Then $\varphi C(\mathfrak{M}) = C(f)$.

5.28. **Corollary.** *The map φ_C* (see 5.16) *is a topological isomorphism of the algebra $C(\mathfrak{M})$ onto the algebra $C(f_C)$ preserving all semi-norms.*

G. Homomorphisms and mappings

5.29. **Proposition.** a) *Let $j_X\colon X \to X_C$ be a map defined by the formula $j_X(y,M) = (j_Y y, M)$ for all $(y,M) \in X$. This map is continuous and satisfies the conditions $f_C j_X = j_Y f$ and $\pi = \pi_C j_X$. Moreover, for every point $y \in Y$ the mapping $j_X|_{f^{-1}y}\colon f^{-1}y \xrightarrow{\text{onto}} f_C^{-1} j_Y y$ is a homeomorphism.*

b) *If for any $y_1, y_2 \in Y$ such that $y_1 \neq y_2$ the inequality $n_{y_1} \neq n_{y_2}$ is valid, then the mapping j_X is a homeomorphic embedding.*

c) *The map $\psi\colon C(f_C) \to C(f)$ defined by the formula $\psi g = g j_X$ for all $g \in C(f_C)$ is a topological isomorphism onto a subalgebra which preserves all semi-norms n_y, $y \in j_Y Y$ (that is, $n_{j_Y y} = n_y$ for all $y \in Y$), and satisfies the condition $\varphi = \psi\varphi_C$.*

Proof follows from the constructions of mappings f, f_C, π, π_C, φ, φ_C and j_Y in the items 5.3, 5.5 and 5.16. □

5.30. **Proposition.** *Let $f_0\colon X_0 \to Y_0$ be a mapping satisfying the condition $[f_0 X_0]_{Y_0} = Y_0$, and let $f\colon X \xrightarrow{\text{onto}} Y$ be the mapping constructed in the item 5.16 by using the algebra $C = C(f_0)$ with the family of semi-norms $\{n_y : y \in Y_0\}$. Then*

1) *the identity map $j_0\colon Y_0 \xrightarrow{\text{onto}} Y$ is continuous; the mapping j_0 is a homeomorphism iff for each point $y \in Y_0$ and its neighborhood $Uy \subseteq Y_0$ there exists a function $g \in C(f_0)$ such that $y \in U_{g,1} \subseteq Uy$* (see 2.11)*;*
2) *the map $j\colon X_0 \to X$ defined by the formula $jx = (j_0 f_0 x, M_x)$, where $M_x = \{g \in C(f_0) : gx = 0\}$ for all $x \in X_0$* (see Lemma 3.5 a))*, is continuous and satisfies the condition $[jX_0]_X = X$;*
3) *if the mapping f_0 is globally completely regular, and for each point $y \in f_0 X_0$ and its neighborhood $Uy \subseteq Y_0$ there exists a function $g \in C(f_0)$ such that $y \in U_{g,1} \subseteq Uy$, then the mapping j is a homeomorphic embedding;*
4) *if the mapping f_0 is perfect then $jX_0 = X$; the mapping j is a homeomorphism iff the mapping f_0 is perfect and globally completely regular and the mapping j_0 is a homeomorphism;*
5) *the map $\psi\colon C(f) \xrightarrow{\text{onto}} C(f_0)$ defined by the equality $\psi g = gj$ for all $g \in C(f)$ is a topological isomorphism preserving all semi-norms n_y, $y \in Y_0$.*

Proof. 1). This follows from Corollary 2.11 and the definition of the topology of the space Y (see 5.14).

2). The continuity of the map j follows from the definition of the space X (see 5.16 and Lemma 3.5 a)). The equality $[jX_0]_X = X$ follows from the statement 4) of Theorem 2.20 (let us note that $\varphi M_{q_f x} = M_x$ for all $x \in X_0$).

3). This statement follows from Lemma 3.5 b), the definition of the space X and Definition 2.18.

4). The equality $jX_0 = X$ follows from the definition of the space X, Lemma 3.5 and properties of perfect mappings (see [11], Lemma 8, and [3], Chapter I, §10, Proposition 5).

5). This is obvious. □

5.31. Theorem. *Let C_1 be a commutative complex algebra with the unit e_C equipped with a Hausdorff topology generated by a family of semi-norms $\{n_y : y \in \in Y\}$ satisfying the conditions 1)–7) of Theorem 2.7, and let $C_2 \subseteq C_1$ be a subalgebra with an involution containing the unit e_C (with the same family of semi-norms). Let $\mathfrak{M}_k$, Y_k, X_k, f_k, π_k, φ_k, $C(\mathfrak{M}_k)$ and $C(f_k)$ be topological spaces, mappings and algebras constructed by using of the algebras C_k, $k = 1, 2$, in the items 5.2, 5.3, 5.14 and 5.16. Then*

1) *there exists a unique map $h\colon \mathfrak{M}_1 \to \mathfrak{M}_2$ satisfying the condition $gh = \varphi g$ for all $g \in C_2$* (see 5.3), *where $\varphi\colon C_2 \xrightarrow{\subseteq} C_1$ is an identity embedding; the map h is continuous; the map $\psi\colon C(\mathfrak{M}_2) \to C(\mathfrak{M}_1)$ defined by the equality $\psi g = gh$ for all $g \in C(\mathfrak{M}_2)$ is a continuous homomorphism satisfying the conditions $\psi|_{C_2} = \varphi$* (see 5.13) *and $n_y(\psi g) \leqslant n_y g$ for all $y \in Y$ and $g \in \in C(\mathfrak{M}_2)$;*
2) *the identity map $h_2\colon Y_1 \xrightarrow{\text{onto}} Y_2$ is continuous and preserves the preorder* (see 5.5)*;*
3) *there exists a unique map $h_1\colon X_1 \to X_2$ satisfying the conditions $f_2h_1 = = h_2f_1$ and $(\varphi_2 g)h_1 = \varphi_1\varphi g$ for all $g \in C_2$; the map h_1 is continuous and satisfies the condition $h\pi_1 = \pi_2 h_1$; the map $\psi_f\colon C(f_2) \to C(f_1)$ defined by the formula $\psi_f g = gh_1$ for all $g \in C(f_2)$ is a continuous homomorphism satisfying the conditions $\psi_f\varphi_2 = \varphi_1\psi$ and $n_y(\psi_f g) \leqslant n_{h_2y} g$ for all $y \in Y_1$ and $g \in C(f_2)$.*

$$\begin{tikzcd}
 & X_2 \arrow[rr, "\pi_2"] \arrow[dr] \arrow[ddd, "f_2"] & & \mathfrak{M}_2 \arrow[dl] \\
 & & \mathbb{C} & \\
X_1 \arrow[uur, "h_1"] \arrow[urr] \arrow[rr, "\pi_1"'] \arrow[dd, "f_1"] & & \mathfrak{M}_1 \arrow[u] \arrow[uur, "h"'] & \\
 & Y_2 & & \\
Y_1 \arrow[ur, "h_2"'] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
C_2 \arrow[r, "\subseteq"] \arrow[d, "\cap|"', "\varphi"] & C(\mathfrak{M}_2) \arrow[r, "\subseteq", "\varphi_2"'] \arrow[d, "\psi"] & C(f_2) \arrow[d, "\psi_f"] \\
C_1 \arrow[r, "\subseteq"] & C(\mathfrak{M}_1) \arrow[r, "\subseteq", "\varphi_1"'] & C(f_1)
\end{tikzcd}$$

Proof. 1). It is possible to define the map $h\colon \mathfrak{M}_1 \to \mathfrak{M}_2$ by the equality $hM = = M \cap C_2$ for all $M \in \mathfrak{M}_1$. Due to the equality $p_{hM} = p_M\varphi$ (see 5.3) we get $ghM = p_{hM}g = p_M\varphi g = (\varphi g)M$ for all $M \in \mathfrak{M}_1$ and $g \in C_2$. The uniqueness of the map h follows from the fact that for any $M_1, M_2 \in \mathfrak{M}_2$ satisfying the condition $M_1 \neq M_2$ there exists a function $g \in C_2$ such that $gM_1 \neq gM_2$. The continuity of the map h follows from the definition of the topologies of the spaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$. The remaining statements about the homomorphism ψ are obvious.

2). The continuity of the map h_2 follows from the definition of the topologies of the spaces Y_1 and Y_2 since the algebra C_2 has an involution. The fact that the mapping h_2 preserves the preorder follows from the continuity of the map h_2 and Lemma 5.15 d).

3). The map $h_1\colon X_1 \to X_2$ can be defined by the formula $h_1(y, M) = (h_2 y, M \cap \cap C_2)$ for all $(y, M) \in X_1$. The equalities $h\pi_1 = \pi_2 h_1$ and $f_2 h_1 = h_2 f_1$ follow from the definitions of the maps h and h_1. The remaining statements can be proved analogously to the proof of the statement 1). □

5.32. *Remark.* The topology of the space Y defined in the item 5.14 is the smallest topology which is compatible with the algebra C, but it is possible to use any larger topology. Indeed, let $p\colon \tilde{Y} \xrightarrow{\text{onto}} Y$ be a (one-to-one) mapping. Let $\tilde{X}$ be the fan product of the spaces X and $\tilde{Y}$ relative to the mappings f and p, and let $q\colon \tilde{X} \to X$ and $\tilde{f}\colon \tilde{X} \to \tilde{Y}$ be the projections of the fan product to its factors. Then due to the lemmas "on Parallels", the map q is continuous, "onto" (and one-to-one), and the map $\tilde{f}$ is perfect and "onto" ([1], §2 of Supplement to Chapter I). Moreover, it is easy to verify using the definition of the fan product that the space $\tilde{X}$ is a subspace of the product $\tilde{Y} \times \mathfrak{M}$ (see 5.16) and the mapping $\tilde{f}$ is globally completely regular. By Theorem 2.23 and Proposition 2.9 the map $\tilde{\psi}\colon C(f) \to C(\tilde{f})$, defined by the formula $\tilde{\psi} g = gq$ for all $g \in C(f)$, is a topological isomorphism onto a subalgebra such that $n_{py} g = n_y(\tilde{\psi} g)$ for all $y \in \tilde{Y}$ and $g \in C(f)$.

The statements 5.17–5.22, 5.24, 5.29 a) and 5.29 c) will remain correct if the mapping f is replased by the mapping $\tilde{f}$.

5.33. **Theorem.** *Let C_1 and C_2 be commutative complex algebras with the units e_{C_1} and e_{C_2} equipped with Hausdorff topologies generated by families of semi-norms $\{n_y : y \in Y_1\}$ and $\{n_y : y \in Y_2\}$ satisfying the conditions 1)–7) of Theorem 2.7. Let, moreover, a homomorhism $\varphi\colon C_2 \to C_1$ and a map $h_2\colon Y_1 \to Y_2$, satisfying the conditions $\varphi e_{C_2} = e_{C_1}$ and $n_y(\varphi g) \leqslant n_{h_2 y} g$ for all $y \in Y_1$ and $g \in C_2$, be given. Let $\mathfrak{M}_k$, X_k, f_k, π_k, φ_k, $C(\mathfrak{M}_k)$ and $C(f_k)$ be topological spaces, mappings and algebras constructed by using the algebras C_k, $k = 1, 2$, in the items 5.2, 5.3, 5.14 and 5.16. Let us suppose that a perfect globally completely regular mapping $\tilde{f}_2\colon \tilde{X}_2 \xrightarrow{\text{onto}} \tilde{Y}_2$ and one-to-one mappings $p_2\colon \tilde{Y}_2 \xrightarrow{\text{onto}} Y_2$ and $q_2\colon \tilde{X}_2 \xrightarrow{\text{onto}} X_2$ satisfying the condition $p_2 \tilde{f}_2 = f_2 q_2$* (see Remark 5.32) *are given. Then*

1) *there exists a unique map $h\colon \mathfrak{M}_1 \to \mathfrak{M}_2$ satisfying the condition $gh = \varphi g$ for all $g \in C_2$* (see 5.3)*; the map h is continuous; the map $\psi\colon C(\mathfrak{M}_2) \to \to C(\mathfrak{M}_1)$ defined by the formula $\psi g = gh$ for all $g \in C(\mathfrak{M}_2)$ is a continuous homomorphism satisfying the conditions $\psi|_{C_2} = \varphi$* (see 5.13) *and $n_y(\psi g) \leqslant \leqslant n_{h_2 y} g$ for all $y \in Y_1$ and $g \in C(\mathfrak{M}_2)$;*
2) *there exist a perfect globally completely regular mapping $\tilde{f}_1\colon \tilde{X}_1 \xrightarrow{\text{onto}} \tilde{Y}_1$ and one-to-one mappings $p_1\colon \tilde{Y}_1 \xrightarrow{\text{onto}} Y_1$ and $q_1\colon \tilde{X}_1 \xrightarrow{\text{onto}} X_1$, satisfying the condition $p_1 \tilde{f}_1 = f_1 q_1$, such that the unique map $\tilde{h}_2\colon \tilde{Y}_1 \to \tilde{Y}_2$, satisfying the condition $p_2 \tilde{h}_2 = h_2 p_1$, is continuous;*
3) *there exists a unique map $\tilde{h}_1\colon \tilde{X}_1 \to \tilde{X}_2$ satisfying the conditions $\tilde{f}_2 \tilde{h}_1 = = \tilde{h}_2 \tilde{f}_1$ and $(\tilde{\psi}_2 \varphi_2 g)\tilde{h}_1 = \tilde{\psi}_1 \varphi_1 \varphi g$ for all $g \in C_2$; the map $\tilde{h}_1$ is continuous and satisfies the condition $h\pi_1 q_1 = \pi_2 q_2 \tilde{h}_1$; the map $\psi_{\tilde{f}}\colon C(\tilde{f}_2) \to C(\tilde{f}_1)$, defined by the equality $\psi_{\tilde{f}} g = g\tilde{h}_1$ for all $g \in C(\tilde{f}_2)$, is a continuous homomorphism satisfying the conditions $\psi_{\tilde{f}} \tilde{\psi}_2 \varphi_2 = \tilde{\psi}_1 \varphi_1 \psi$ and $n_y(\psi_{\tilde{f}} g) \leqslant n_{\tilde{h}_2 y} g$ for all $y \in \tilde{Y}_1$ and $g \in C(\tilde{f}_2)$.*

$$\begin{tikzcd}
 & \tilde{X}_2 \arrow[rr, "q_2"] \arrow[dr] \arrow[ddd, "\tilde{f}_2"] & & X_2 \arrow[rr, "\pi_2"] \arrow[dl] \arrow[ddd, "f_2"] & & \mathfrak{M}_2 \arrow[dlll] \\
 & & \mathbb{C} & & & \\
\tilde{X}_1 \arrow[ur, "\tilde{h}_1"] \arrow[urr] \arrow[rr, "q_1"'] \arrow[dd, "\tilde{f}_1"] & & X_1 \arrow[u] \arrow[rr, "\pi_1"'] \arrow[dd, "f_1"] & & \mathfrak{M}_1 \arrow[ull] \arrow[ur, "h"] & \\
 & \tilde{Y}_2 \arrow[rr, "p_2"] & & Y_2 & & \\
\tilde{Y}_1 \arrow[ur, "\tilde{h}_2"] \arrow[rr, "p_1"'] & & Y_1 \arrow[ur, "h_2"] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
C_2 \arrow[r, "\subseteq"] \arrow[d, "\varphi"] & C(\mathfrak{M}_2) \arrow[r, "\subseteq", "\varphi_2"'] \arrow[d, "\psi"] & C(f_2) \arrow[r, "\subseteq", "\tilde{\psi}_2"'] & C(\tilde{f}_2) \arrow[d, "\psi_{\tilde{f}}"] \\
C_1 \arrow[r, "\subseteq"] & C(\mathfrak{M}_1) \arrow[r, "\subseteq", "\varphi_1"'] & C(f_1) \arrow[r, "\subseteq", "\tilde{\psi}_1"'] & C(\tilde{f}_1)
\end{tikzcd}$$

The continuity of the map h_2 is not necessary, even if this map preserves the preorder.

Proof. 1). This statement can be proved analogously to the statement 1) of Theorem 5.31.

2). We can equip the space $\tilde{Y}_1$ with the smallest topology such that the maps p_1 and $\tilde{h}_2$ are continuous; then we can use Remark 5.32.

3). The proof of this statement is analogous to the proof of the statement 3) of Theorem 5.31. □

§ 6. A completeness of algebras of functions on mappings

A. The standard completeness

6.1. We shall assume that $f\colon X \to Y$ is a mapping of topological spaces such that $[fX]_Y = Y$ (the latter condition is not essential; see the foot-note in the item 2.25). Further for $g \in C(f)$ we shall write U_g instead of $U_{g,1}$ (see 2.11).

Let us remind that a net[7] $\{g_t : t \in T\} \subseteq C(f)$ is a Cauchy net (with respect to the natural uniformity of the algebra $C(f)$) if for each neighborhood $U \subseteq C(f)$ of the zero of the algebra $C(f)$ there exists an element $t_0 \in T$ such that for any $t_1, t_2 \in T$ satisfying the conditions $t_1 \geqslant t_0$ and $t_2 \geqslant t_0$ we have $g_{t_1} - g_{t_2} \in U$.

6.2. **Proposition.** *If for each subset $G \subseteq C(f)$ the set $\bigcap\{U_g : g \in G\}$ is open in the space Y then the algebra $C(f)$ is complete.*

Proof. Let $\{g_t : t \in T\} \subseteq C(f)$ be a Cauchy net. For any elements $t_1, t_2 \in T$ and a number $\varepsilon > 0$ let $U(t_1, t_2, \varepsilon) = U_g$ where $g = \frac{1}{\varepsilon} \cdot (g_{t_1} - g_{t_2})$.

Let us choose an arbitrary point $y_0 \in Y$ and for every $n \in \mathbb{N}$ construct a neighborhood $W_n y_0 \subseteq Y$ of the point y_0 and an element $t_{0n} \in T$ satisfying the

[7]See [6], §20, the item IX, or [5], §1.6.

following condition:

$$\text{if } t_1, t_2 \in T,\ t_1 \geqslant t_{0n},\ t_2 \geqslant t_{0n} \text{ and } y \in W_n y_0, \text{ then } n_y(g_{t_1} - g_{t_2}) < \frac{1}{n}.$$

To begin let $W_0 y_0 = Y$ and let $t_{00} \in T$ be an arbitrary element.

Let us suppose now that for some $n \in \mathbb{N}$ the set $W_{n-1} y_0$ and the element t_{0n-1} have been constructed. Since $\{g_t : t \in T\}$ is a Cauchy net, there exists an element $t_{0n} \in T$, $t_{0n} \geqslant t_{0n-1}$, such that for any $t_1, t_2 \in T$ satisfying the conditions $t_1 \geqslant t_{0n}$ and $t_2 \geqslant t_{0n}$ the inequality $n_{y_0}(g_{t_1} - g_{t_2}) < \frac{1}{n}$ is fulfilled (we put $U = V_C\left(0_C, \frac{1}{n}, \{y_0\}\right)$ in the definition of a Cauchy net, where 0_C is the zero of $C(f)$). Further, we set

$$W_n y_0 = W_{n-1} y_0 \cap \bigcap \left\{ U\left(t_1, t_2, \frac{1}{n}\right) : t_1, t_2 \in T,\ t_1 \geqslant t_{0n},\ t_2 \geqslant t_{0n} \right\}.$$

This set is an open neighborhood of the point y_0 due to the condition of our Proposition. It is possible to continue the construction.

Finally, let $W y_0 = \bigcap \{W_n y_0 : n \in \mathbb{N}\}$. This set is an open neighborhood of the point y_0 since this set is an intersection of sets of the form U_g, $g \in C(f)$.

Let us set $T' = \{t \in T : t \geqslant t_{01}\}$ and consider the net $\{(g_t - g_{t_{01}})|_{f^{-1} W y_0} : t \in T'\}$. It is easily seen that this net is a Cauchy net in the algebra $C^*(f^{-1} W y_0)$ with respect to its standard norm (see 2.3). Hence this net converges to some function $g'_{W y_0} \in C^*(f^{-1} W y_0)$. Therefore by Proposition 2.16 the net $\{g_t|_{f^{-1} W y_0} : t \in T\}$ converges to the function $g_{W y_0} = g'_{W y_0} + g_{t_{01}}|_{f^{-1} W y_0} \in C(f|_{f^{-1} W y_0})$.

Being made for all points $y_0 \in Y$, the preceding reasonings give us an open covering $\{W y : y \in Y\}$ of the space Y and a family of continuous functions $\{g_{W y} : y \in Y\}$ such that for all $y_1, y_2 \in Y$ and $x \in f^{-1}(W y_1 \cap W y_2)$ the equality $g_{W y_1} x = g_{W y_2} x$ holds. Therefore (see [5], Proposition 2.1.11) there is a continuous function $g \colon X \to \mathbb{C}$ such that $g|_{W y} = g_{W y}$ for all $y \in Y$. Since for each $y \in Y$ the set $W y$ is open and $g_{W y} \in C(f|_{f^{-1} W y})$, the function g belongs to $C(f)$. Analogously, the net $\{g_t : t \in T\}$ converges to the function g. Hence, the algebra $C(f)$ is complete. $\square$

6.3. **Proposition.** *Let the mapping f be closed. Let for each point $y_0 \in Y$ and for each point $y \in Y \setminus \bar{y}_0$, where $\bar{y}_0 = \bigcap \{U_g : g \in C(f),\ y_0 \in U_g\}$, there exists a function $g \in C(f)$ such that $gx \leqslant 0$ for all $x \in f^{-1} y_0$ and $gx \geqslant 1$ for all $x \in f^{-1} y$.*[8] *If the algebra $C(f)$ is complete then the set $\bigcap \{U_g : g \in G\}$ is open for every subset $G \subseteq C(f)$.*

Proof. Let us assume that there exists a subset $G \subseteq C(f)$ such that the set $A = \bigcap \{U_g : g \in G\}$ is not open. Let us choose any point $y_0 \in A \cap [Y \setminus A]_Y$. Of course, we have $y_0 \in \bar{y}_0 \cap [Y \setminus \bar{y}_0]_Y$. Moreover, for each function $g \in C(f)$ and every point $y \in \bar{y}_0$ the inequality $n_y g \leqslant n_{y_0} g$ is satisfied.

Let $T = \{M : M \subseteq Y \setminus \bar{y}_0 \text{ is finite}\}$ be the set with the following partial order: $M_1 \leqslant M_2$ if $M_1 \subseteq M_2$ for all $M_1, M_2 \in T$.

Let $M \in T$. For each point $y \in M$ there is a continuous function $g_y \colon X \to [0,1]$ such that $g_y x = 0$ for all $x \in f^{-1} \bar{y}_0$ and $g_y x = 1$ for all $x \in f^{-1} y$: if $g' \in C(f)$ is a real function such that $g' x \leqslant 0$ for $x \in f^{-1} y_0$ and $g' x \geqslant 1$ for $x \in f^{-1} y$ then the function $g_y \colon X \to [0,1]$, defined for $x \in X$ by the formula

$$g_y x = \begin{cases} 0 \text{ if } g' x \leqslant \frac{1}{2}, \\ 2 \cdot g' x - 1 \text{ if } \frac{1}{2} < g' x < 1, \\ 1 \text{ if } g' x \geqslant 1, \end{cases}$$

[8] For example, it is valid if for every different points $y_0, y \in Y$ there is a continuous function $g \colon Y \to \mathbb{R}$ such that $g y_0 \neq g y$ (in this case the space Y is called *functionally Hausdorff*).

has the required properties since $\bar{y}_0 \subseteq U_{g_y}$. Let us define the function $g_M \in C(f)$ by the equality $g_M x = \max\{g_y x : y \in M\}$ for all $x \in X$.

Thus we have a net $\{g_M : M \in T\}$ in the algebra $C(f)$. Let us prove that it is a Cauchy net. Let $V_C(0_C, \varepsilon, M_0)$ be an arbitrary neighborhood of the zero 0_C of the algebra $C(f)$ ($\varepsilon > 0$, $M_0 \subseteq Y$ is a finite set). Let us set $M = M_0 \setminus \bar{y}_0$. Then for any $M', M'' \in T$ such that $M' \geqslant M$ and $M'' \geqslant M$ we have $g_{M'}x - g_{M''}x = 0 - 0 = 0$ for $x \in f^{-1}\bar{y}_0$ and $g_{M'}x - g_{M''}x = 1 - 1 = 0$ for $x \in f^{-1}M$; therefore due to Proposition 2.9 the equality $n_y(g_{M'} - g_{M''}) = 0$ holds for all $y \in M \cup \bar{y}_0 \supseteq M_0$ and, hence, $g_{M'} - g_{M''} \in V_C(0_C, \varepsilon, M_0)$. Hence, $\{g_M : M \in T\}$ is a Cauchy net.

Let us suppose that the net $\{g_M : M \in T\}$ converges to some function $g\colon X \to \mathbb{C}$. Then the equalities $gx = 0$ for all $x \in f^{-1}\bar{y}_0$ and $gx = 1$ for all $x \in f^{-1}(Y \setminus \bar{y}_0)$ hold, but this function is not continuous since the mapping f is closed and the set $Y \setminus \bar{y}_0 = fg^{-1}1$ is not closed (hence, the set $g^{-1}1$ is not closed). Therefore due to Proposition 2.15 the net $\{g_M : M \in T\}$ does not converge to the function g. Hence, the algebra $C(f)$ is not complete. This contradiction proves that the set A must be open. □

6.4. **Corollary.** *If the mapping f is closed, the space Y is functionally Hausdorff and the algebra $C(f)$ is complete then the space Y is discrete.*

B. The convergence in the algebra $C(f)$

6.5. **Assertion.** *Let a net $\{g_t : t \in T\} \subseteq C(f)$ converges to a function $g \in C(f)$. Then for each point $y \in Y$ the net $\{g_t|_{f^{-1}y} : t \in T\}$ uniformly converges to the function $g|_{f^{-1}y}$ in the algebra $C^*(f^{-1}y)$.*

Proof is obvious. □

6.6. **Proposition.** *Let the mapping f be closed and $\Phi \subseteq C(f)$ be a closed subset. Let $\{g_t : t \in T\} \subseteq \Phi$ be a net such that for each point $y \in Y$ the net $\{g_t|_{f^{-1}y} : t \in T\}$ uniformly converges to a function $g_y \in C^*(f^{-1}y)$. If the function $g\colon X \to \mathbb{C}$, defined by the equality $gx = g_{fx}x$ for all $x \in X$, is continuous, then $g \in \Phi$ and the net $\{g_t : t \in T\}$ converges to the function g with respect to the topology of the algebra $C(f)$.*

Proof follows from Proposition 2.9. □

6.7. **Corollary.** *Let the mapping f be perfect and $C \subseteq C(f)$ be a closed subalgebra. Let us suppose that $g \in C$ and that $\sum\limits_{n=1}^{\infty} a_n \cdot z^n$ is a series with complex members which converges for all $z \in gX$. Then the series*[9] $\sum\limits_{n=1}^{\infty} a_n \cdot g^n$ *converges to some element of C with respect to the topology of the algebra C.*

6.8. **Proposition.** *Let $C \subseteq C(f)$ be a closed linear subspace and let $g\colon [a,b] \to C$ be a mapping. Then there is a function $g_0 \in C$ such that for any $x \in X$ the following equality holds:*

$$g_0 x = \int\limits_a^b (gt)x \cdot dt.$$

Proof is standard. □

[9]Here and further $g^n = \underbrace{g \cdot g \cdot \ldots \cdot g}_{n \text{ times}}$.

6.9. *Remark.* If we use the usual definition of a Riemann type integral for $\int_a^b gt\cdot dt$ then its integral sums converge to the function g_0 with respect to the topology of the space C due to Proposition 6.6. Therefore we can write $g_0 = \int_a^b gt \cdot dt$.

6.10. **Lemma.** *For a non-empty set $A \subseteq C(f)$ and $y \in Y$ let us define the function $d_y^A \colon C(f) \to \mathbb{R}$ by the formula*

$$d_y^A g = \inf\{n_y(g - g') : g' \in A\}$$

for $g \in C(f)$. Then

1) *$|d_y^A g_1 - d_y^A g_2| \leqslant n_y(g_1 - g_2)$ for all $g_1, g_2 \in C(f)$;*
2) *the function d_y^A is continuous;*
3) *if A is a linear subspace then $d_y^A(\lambda\cdot g) = |\lambda|\cdot d_y^A g$ and $|d_y^A g_1 - d_y^A g_2| \leqslant d_y^A(g_1 + g_2) \leqslant d_y^A g_1 + d_y^A g_2$ for all $g, g_1, g_2 \in C(f)$ and $\lambda \in \mathbb{C}$;*
4) *if A is a linear subspace and $g\colon [a,b] \to C(f)$ is a mapping then $d_y^A \int_a^b gt\cdot dt \leqslant \int_a^b d_y^A(gt) \cdot dt$;*
5) *if A is a subalgebra then $d_y^A(g_1 \cdot g_2) \leqslant d_y^A g_1 \cdot n_y g_2 + n_y g_1 \cdot d_y^A g_2 + d_y^A g_1 \cdot d_y^A g_2$ for all $g_1, g_2 \in C(f)$;*
6) *if A is an ideal then $d_y^A(g_1 \cdot g_2) \leqslant d_y^A g_1 \cdot n_y g_2$ for all $g_1, g_2 \in C(f)$.*

Proof is very simple. □

6.11. **Lemma.** *Let $\Phi \subseteq C(f)$ be a closed subset and let a net $\{g_t : t \in T\}$ be such that for every number $\varepsilon > 0$ and every point $y \in Y$ there exists an element $t_0 \in T$ satisfying the condition $d_y^\Phi g_t < \varepsilon$ for all $t \geqslant t_0$, $t \in T$. If the net $\{g_t : t \in T\}$ converges to some function $g \in C(f)$ then $g \in \Phi$.*

Proof is obvious. □

C. A $\tilde{Y}$-completeness

6.12. Further on let C be a commutative complex algebra with a family of semi-norms $\{n_y : y \in Y\}$ which satisfy the conditions 1)–7) of Theorem 2.7 and determine a Hausdorff topology (see 5.1). Moreover, let the set Y be equipped with the topology which is compatible with the algebra C (see 5.14[10] and 5.32); the set Y with this topology will be denoted by the symbol $\tilde{Y}$. Let $\tilde{\varphi} = \tilde{\psi}\varphi|_C \colon C \to C(\tilde{f})$ be the topological isomorphism onto a subalgebra constructed in the items 5.3, 5.16 and 5.32. This isomorphism satisfies the conditions $\tilde{\varphi}g = g\pi q$ and $n_y(\tilde{\varphi}g) = n_y g$ for all $g \in C$ (see 5.4, 5.21 and 5.32). Let $\tilde{C} = \tilde{\varphi}C$.

6.13. **Definition.** The algebra C will be called *$\tilde{Y}$-complete* if the embedding $\tilde{\varphi}\colon C \to C(\tilde{f})$ is closed, or, equivalently, if the algebra $\tilde{C}$ is closed in $C(\tilde{f})$.

6.14. In particular, if the algebra C is $\tilde{Y}$-complete then C is a closed subalgebra of $C(\mathfrak{M})$ since $\tilde{\psi}\varphi\colon C(\mathfrak{M}) \to C(\tilde{f})$ is a topological isomorphism onto a subalgebra.

6.15. **Theorem.** *Let $\{n_y : y \in Y_1\}$ and $\{n_y : y \in Y_2\}$ be families of semi-norms on the algebra C which satisfy the conditions 1)–7) of Theorem 2.7 and determine the same Hausdorff topology. Moreover, let the sets Y_1 and Y_2 be equipped with topologies which are compatible with the algebra C (see 5.32; the sets Y_1 and Y_2 with these topologies will be denoted by $\tilde{Y}_1$ and $\tilde{Y}_2$). Let us suppose that there exists a mapping $h\colon \tilde{Y}_1 \to \tilde{Y}_2$ such that $n_y g \leqslant n_{hy} g$ for all $y \in \tilde{Y}_1$ and $g \in C$. Let $\mathfrak{M}$, X_k,*

[10]Let us note that the algebra $\tilde{C}$ in the item 5.14 must contain the unit of the algebra $C(\mathfrak{M})$ due to the assumptions 5.1.

$\tilde{X}_k$, Y_k, f_k, $\tilde{f}_k$, p_k, q_k, π_k, φ_k, $\tilde{\psi}_k$, $\tilde{\varphi}_k$, $C(\mathfrak{M})$, $C(f_k)$, $C(\tilde{f}_k)$ and $\tilde{C}_k$ be topological spaces, mappings and algebras constructed by using the algebra C, the sets Y_k and the spaces $\tilde{Y}_k$, $k = 1, 2$, in the items 5.2, 5.3, 5.14, 5.16, 5.32 and 6.12. Then

1) *there is a unique map $\tilde{h}\colon \tilde{X}_1 \to \tilde{X}_2$ such that $\tilde{f}_2\tilde{h} = hf_1$ and $\pi_2 q_2 \tilde{h} = \pi_1 q_1$; the map $\tilde{h}$ is continuous;*
2) *the map $\psi_{\tilde{f}}\colon C(\tilde{f}_2) \to C(\tilde{f}_1)$, defined by the formula $\psi_{\tilde{f}} g = g\tilde{h}$ for all $g \in$ $\in C(\tilde{f}_2)$, is a continuous homomorphism satisfying the conditions $\psi_{\tilde{f}}\tilde{\psi}_2\varphi_2 =$ $= \tilde{\psi}_1\varphi_1$ and $n_y(\psi_{\tilde{f}} g) \leqslant n_{hy} g$ for $y \in \tilde{Y}_1$ and $g \in C(\tilde{f}_2)$;*
3) *if the algebra C is $\tilde{Y}_1$-complete then it is $\tilde{Y}_2$-complete.*

$$\begin{tikzcd}
 & \tilde{X}_2 \arrow[rrr, "q_2"] \arrow[drr] \arrow[ddd, "\tilde{f}_2"'] & & & X_2 \arrow[dl] \arrow[dr, "\pi_2"] \arrow[ddd, "f_2"] & \\
 & & & \mathbb{C} & & \mathfrak{M} \arrow[ll] \\
\tilde{X}_1 \arrow[uur, "\tilde{h}"] \arrow[urrr] \arrow[rr, "q_1"'] \arrow[dd, "\tilde{f}_1"'] & & X_1 \arrow[ur] \arrow[urrr, "\pi_1"] \arrow[dd, "f_1"] & & & \\
 & \tilde{Y}_2 \arrow[rrr, "p_2"] & & & Y_2 & \\
\tilde{Y}_1 \arrow[ur, "h"] \arrow[rr, "p_1"'] & & Y_1 & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & C(f_2) \arrow[r, "\subseteq", "\tilde{\psi}_2"'] & C(\tilde{f}_2) \arrow[dd, "\psi_{\tilde{f}}"] \\
C \arrow[r, "\subseteq"] & C(\mathfrak{M}) \arrow[ur, "\subseteq", "\varphi_2"'] \arrow[dr, "\subseteq", "\varphi_1"'] & & \\
 & & C(f_1) \arrow[r, "\subseteq", "\tilde{\psi}_1"'] & C(\tilde{f}_1)
\end{tikzcd}$$

Proof. 1). Due to Assertion 5.6 and the constructions 5.16 and 5.32 we can define the map $\tilde{h}$ by the formula $\tilde{h}(y, M) = (hy, M)$ for $(y, M) \in \tilde{X}_1$. The continuity of the map $\tilde{h}$ follows from the definition of the spaces $\tilde{X}_1$ and $\tilde{X}_2$.

2). Obviously, the map $\psi_{\tilde{f}}$ is a homomorphism and $\psi_{\tilde{f}}\tilde{\psi}_2\varphi_2 = \tilde{\psi}_1\varphi_1$. The continuity of the map $\psi_{\tilde{f}}$ follows from the inequality $n_y(\psi_{\tilde{f}} g) \leqslant n_{hy} g$ for all $y \in \tilde{Y}_1$ and $g \in C(\tilde{f}_2)$.

3). Let us denote $\tilde{\varphi}_k = \tilde{\psi}_k\varphi_k|_C$, $k = 1, 2$ (see 6.12). Let the algebra C be $\tilde{Y}_1$-complete and $\{g_t : t \in T\} \subseteq \tilde{C}_2 = \tilde{\varphi}_2 C$ be a net which converges to some element $g \in C(\tilde{f}_2)$. We have to prove that $g \in \tilde{C}_2$.

Since the map $\psi_{\tilde{f}}$ is continuous, the net $\{\psi_{\tilde{f}} g_1 : t \in T\} \subseteq \tilde{C}_1 = \tilde{\varphi}_1 C$ converges to $\psi_{\tilde{f}} g \in C(\tilde{f})$. We have $\psi_{\tilde{f}} g \in \tilde{C}_1$ because of the set $\tilde{C}_1$ is closed in $C(\tilde{f}_1)$. By the construction the map $\psi_{\tilde{f}}|_{\tilde{C}_2}$ is a topological isomorphism onto $\tilde{C}_1$, therefore there exists an element $g' \in \tilde{C}_2$ such that the net $\{g_t : t \in T\}$ converges to g'. Since the algebra $C(\tilde{f}_2)$ is Hausdorff, we have $g' = g$ and, hence, $g \in \tilde{C}_2$, that is, the algebra $\tilde{C}_2$ is closed in $C(\tilde{f}_2)$. □

6.16. **Definition.** A net $\{g_t : t \in T\} \subseteq C$ will be called *$\tilde{Y}$-fundamental* if for each number $\varepsilon > 0$ there exists a family $\{O_t : t \in T\}$ of open sets $O_t \subseteq \tilde{Y}$, $t \in T$, satisfying the following conditions:

1) for every $y \in \tilde{Y}$ there is $t_y \in T$ such that $y \in O_t$ for all $t \geqslant t_y$, $t \in T$;
2) if $t_1, t_2 \in T$ and $y \in O_{t_1} \cap O_{t_2}$ then $n_y(g_{t_1} - g_{t_2}) < \varepsilon$.

6.17. **Theorem.** *The algebra C is $\tilde{Y}$-complete iff every $\tilde{Y}$-fundamental net $\{g_t : t \in T\} \subseteq C$ converges to some element $g \in C$.*

Proof. Necessity. Let the algebra C be $\tilde{Y}$-complete and $\{g_t : t \in T\}$ be a $\tilde{Y}$-fundamental net in C. Let us denote $\tilde{g}_t = \tilde{\varphi} g_t$ for $t \in T$. Then the net $\{\tilde{g}_t : t \in \in T\} \subseteq \tilde{C}$ is $\tilde{Y}$-fundamental in $C(\tilde{f})$ because of the homomorphism $\tilde{\varphi}$ preserves all semi-norms n_y, $y \in \tilde{Y}$.

It follows from Definition 6.16 and Proposition 2.9 that for each $y \in \tilde{Y}$ the net $\{\tilde{g}_t|_{\tilde{f}^{-1}y} : t \in T\}$ is a Cauchy net in the complete algebra $C^*(\tilde{f}^{-1}y)$ and, hence, one converges to some function $\tilde{g}_y \in C^*(\tilde{f}^{-1}y)$. Let us define the function $\tilde{g} \colon \tilde{X} \to \mathbb{C}$ by the equality $\tilde{g}x = \tilde{g}_{\tilde{f}x} x$ for all $x \in \tilde{X}$.

We must prove that $\tilde{g} \in C(\tilde{f})$, that is, that the function $\tilde{g}$ is continuous. Let $\varepsilon > 0$ be an arbitrary number. By Definition 6.16 there is a family $\{O_t : t \in T\}$ of open subsets of the space $\tilde{Y}$ satisfying the following conditions:

1) for each $y \in \tilde{Y}$ there exists an element $t_y \in T$ such that $y \in O_t$ for all $t \geqslant t_y$, $t \in T$;
2) if $t_1, t_2 \in T$ and $y \in O_{t_1} \cap O_{t_2}$ then $n_y(g_{t_1} - g_{t_2}) < \frac{\varepsilon}{3}$.

Then in a usual way we get the inequality $|\tilde{g}x - \tilde{g}_t x| \leqslant \frac{\varepsilon}{3}$ for all $t \in T$ and $x \in \tilde{f}^{-1} O_t$.

For a given point $x_0 \in \tilde{X}$ let $y_0 = \tilde{f} x_0$ and let $t_0 \in T$ be any element such that $y_0 \in O_{t_0}$. Since the function $\tilde{g}_{t_0}$ is continuous, there is a neighborhood $Ux_0 \subseteq \subseteq \tilde{f}^{-1} O_{t_0}$ such that $|\tilde{g}_{t_0} x - \tilde{g}_{t_0} x_0| < \frac{\varepsilon}{3}$ for all $x \in Ux_0$. Then for every $x \in Ux_0$ we have the inequality

$$|\tilde{g}x - \tilde{g}x_0| = |\tilde{g}x - \tilde{g}_{t_0}x + \tilde{g}_{t_0}x - \tilde{g}_{t_0}x_0 + \tilde{g}_{t_0}x_0 - \tilde{g}x_0| \leqslant \\ \leqslant |\tilde{g}x - \tilde{g}_{t_0}x| + |\tilde{g}_{t_0}x - \tilde{g}_{t_0}x_0| + |\tilde{g}_{t_0}x_0 - \tilde{g}x_0| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

that is, the function $\tilde{g}$ is continuous.

By Proposition 6.6 the net $\{\tilde{g}_t : t \in T\}$ converges to the function $\tilde{g}$ in the algebra $C(\tilde{f})$. Since the set $\tilde{C}$ is closed in $C(\tilde{f})$, we have $\tilde{g} \in \tilde{C}$. Therefore the net $\{g_t : t \in \in T\}$ converges in the algebra C to the element $g = \tilde{\varphi}^{-1}\tilde{g}$ because of $\tilde{\varphi} \colon C \xrightarrow{\text{onto}} \tilde{C}$ is a topological isomorphism.

Sufficiency. Let us suppose that each $\tilde{Y}$-fundamental net in the algebra C converges. We have to prove that the algebra $\tilde{C}$ is closed in $C(\tilde{f})$.

Let $\tilde{g} \in [\tilde{C}]_{C(\tilde{f})}$. Then there exists a net $\{\tilde{g}_t : t \in T\} \subseteq \tilde{C}$ which converges to $\tilde{g}$. To prove that this net is $\tilde{Y}$-fundamental, let us define for a given number $\varepsilon > 0$ and any $t \in T$ the set $O_t = \left\{y \in \tilde{Y} : n_y(\tilde{g} - \tilde{g}_t) < \frac{\varepsilon}{2}\right\}$. The sets O_t, $t \in T$, are open by Corollary 2.11. It is easily seen that the conditions 1) and 2) of Definition 6.16 are valid.

Thus, the net $\{\tilde{\varphi}^{-1}\tilde{g}_t : t \in T\}$ is $\tilde{Y}$-fundamental since the isomorphism $\tilde{\varphi} \colon C \xrightarrow{\text{onto}} \tilde{C} \subseteq C(\tilde{f})$ preserves all semi-norms n_y, $y \in \tilde{Y}$. Hence, this net converges to some element $g \in C$. Since the algebra $C(\tilde{f})$ is Hausdorff, we have $\tilde{g} = \tilde{\varphi}g$, that is, the algebra $\tilde{g}$ is closed in $C(\tilde{f})$. □

6.18**. Corollary.** *If the algebra C is complete with respect to the natural uniformity* (see 6.1) *then it is $\tilde{Y}$-complete* (see also 6.3 and 6.2).

Proof. It is sufficient to note that each $\tilde{Y}$-fundamental net is a Cauchy net. □

D. Some properties of $\tilde{Y}$-complete algebras

6.19**. Theorem.** *Let the algebra C have not a unit and let C' be the algebra which is obtained by the addition of the unit e* (see the foot-note in the item 5.1). *If the algebra C is $\tilde{Y}$-complete then the algebra C' is $\tilde{Y}$-complete*[11] *also, and C is closed in C'.*

Proof. We have to prove the $\tilde{Y}$-completeness of the algebra C' only.

Let $\{\lambda_t \cdot e + g_t : t \in T\} \subseteq C'$ be a $\tilde{Y}$-fundamental net ($\lambda_t \in \mathbb{C}$ and $g_t \in C$ for $t \in T$). First of all we shall prove that there exists an element $t_0 \in T$ such that the set $\{\lambda_t : t \geqslant t_0,\ t \in T\}$ is bounded.

Let us suppose that the set $\{\lambda_t : t \geqslant t_0,\ t \in T\}$ is unbounded for all $t_0 \in T$. We shall write $t_1 \gg t_2$ for $t_1, t_2 \in T$ if $t_1 \geqslant t_2$ and $|\lambda_{t_1}| \geqslant |\lambda_{t_2}|$; the set T with the relation "$\gg$" will be denoted by $\tilde{T}$. It is easily seen that the set $\tilde{T}$ is directed under our assumption. Let us prove that the net $\left\{e + \frac{1}{\lambda_t} \cdot g_t : t \in \tilde{T}\right\}$ converges to the zero $0_C \in C'$.

Let $\varepsilon > 0$ and $y \in \tilde{Y}$ be arbitrary. Since the net $\{\lambda_t \cdot e + g_t : t \in T\}$ is $\tilde{Y}$-fundamental, there are $t'_y \in T$ and $B_y \in \mathbb{R}$ such that $n_y(\lambda_t \cdot e + g_t) \leqslant B_y$ for all $t \geqslant t'_y$, $t \in T$. Since the set $\{\lambda_t : t \geqslant T'_y,\ t \in T\}$ is unbounded, there is $t_y \geqslant t'_y$, $t_y \in T$, such that $|\lambda_{t_y}| > \frac{1}{\varepsilon} \cdot B_y$. Then for all $t \gg t_y$, $t \in \tilde{T}$, we have the inequality $n_y\left(e + \frac{1}{\lambda_t} \cdot g_t\right) = \frac{1}{|\lambda_t|} \cdot n_y(\lambda_t \cdot e + g_t) \leqslant \frac{1}{|\lambda_t|} \cdot B_y < \varepsilon$.

Thus for every neighborhood $V_{C'}(0_C, \varepsilon, A)$ of the zero ($\varepsilon > 0$, $A \subseteq \tilde{Y}$ is finite) and every $t \gg t_A$, $t \in \tilde{T}$, where $t_A \in \tilde{T}$ is an element such that $t_A \gg t_y$ for all $y \in A$, we have $e + \frac{1}{\lambda_t} \cdot g_t \in V_{C'}(0_C, \varepsilon, A)$, that is, the net $\left\{e + \frac{1}{\lambda_t} \cdot g_t : t \in \tilde{T}\right\}$ converges to 0_C. It means that the net $\left\{-\frac{1}{\lambda_t} \cdot g_t : t \in \tilde{T}\right\}$ converges to e; in particular, this net is $\tilde{Y}$-fundamental. Hence, $e \in C$ because of the algebra C is $\tilde{Y}$-complete, and we have got a contradiction.

Thus, there exists an element $t_0 \in T$ such that the set $\{\lambda_t : t \geqslant t_0,\ t \in T\}$ is bounded. Therefore the net $\{\lambda_t : t \in T\}$ has a cluster point $\lambda_0 \in \mathbb{C}$ (see [5], Theorem 3.1.23). By Proposition 1.6.1 of the book [5] there exists a net $\{\lambda_t \cdot e + g_t : t \in T'\}$ which is finer than the net $\{\lambda_t \cdot e + g_t : t \in T\}$ such that the net $\{\lambda_t : t \in T'\}$ converges to the point λ_0. It is easily seen that the nets $\{\lambda_t \cdot e + g_t : t \in T'\}$ and $\{\lambda_t \cdot e : t \in T'\}$ are $\tilde{Y}$-fundamental, therefore the net $\{g_t : t \in T'\}$ is $\tilde{Y}$-fundamental too. Since the algebra C is $\tilde{Y}$-complete, the latter net converges to some element $g_0 \in C$. Hence, the net $\{\lambda_t \cdot e + g_t : t \in T'\}$ converges to $\lambda_0 \cdot e + g_0$.

Using the fact that the net $\{\lambda_t \cdot e + g_t : t \in T\}$ is $\tilde{Y}$-fundamental, we can easily verify that this net converges to $\lambda_0 \cdot e + g_0$ too. Hence, the algebra C' is $\tilde{Y} - complete$. □

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[11] Let us note that the topology of the space $\tilde{Y}$ is compatible with the algebra C' by the assumptions 5.1: we have to add the unit before to define the topology.

6.20. **Theorem.** *Let the algebra C be $\tilde{Y}$-complete and have the unit e_C. An element $g \in C$ has inverse element $\frac{e_C}{g} \in C$ iff it does not belong to any closed maximal ideal $M \subset C$.* [12]

Proof. Necessity is obvious (see [8], §7.4, the statement III).

Sufficiency. Let $g_0 \in C$ be an element such that $g_0 \notin M$ for every closed maximal ideal $M \subset C$. Let us consider the set $J_0 = \{g \cdot g_0 : g \in C\}$. The following cases are imaginable.

1). If $e_C \in J_0$ then the element g_0 has the inverse element in C.

2). If $e_C \notin J = [J_0]_C$ then J is a closed ideal, and we get a contradiction using Lemma 4.5, since $g_0 \in J_0 \subseteq J \subseteq M$ for some closed maximal ideal $M \subset C$.

3). If $e_C \in [J_0]_C \setminus J_0$ then there is a net $\{g_t \cdot g_0 : t \in T\} \subseteq J_0$ which converges to e_C ($g_t \in C$ for $t \in T$). Let us denote $\tilde{g}_0 = \tilde{\varphi} g_0$, $\tilde{g}_t = \varphi g_t$ for $t \in T$ (see 6.12). Of course, $\tilde{\varphi} e_C = \tilde{g}_e$ is the unit of the algebra $C(\tilde{f})$ by the construction.

Since $g_0 \notin M$ for every $M \in \mathfrak{M}$, we have $\tilde{g}_0 x \neq 0$ for all $x \in \tilde{X}$. The mapping $\tilde{f}$ is perfect and the function $\tilde{g}_0$ is continuous, therefore for each point $y \in \tilde{Y}$ there exists the number

$$l_y = \inf\{|\tilde{g}_0 x| : x \in f^{-1} y\} =$$
$$= \inf\{\sup\{|\tilde{g}_0 x| : x \in f^{-1} U y\} : Uy \subseteq \tilde{Y} \text{ is a neighborhood of the point } y\} > 0$$

(compare with Proposition 2.9). Moreover, we can define a function $\tilde{g}_0' \in C(\tilde{f})$ by the equality $\tilde{g}_0' x = \frac{1}{\tilde{g}_0 x}$ for all $x \in \tilde{x}$. Obviously, $n_y \tilde{g}_0' = \frac{1}{l_y}$ for all $y \in \tilde{Y}$. We shall prove that the net $\{\tilde{g}_t : t \in T\}$ converges to the function $\tilde{g}_0'$.

Let a number $\varepsilon > 0$ and a finite set $A \subseteq \tilde{Y}$ be given. We have to find an element $t_0 \in T$ such that $\tilde{g}_t \in V_C(\tilde{g}_0', \varepsilon, A)$ for all $t \geqslant t_0$, $t \in T$. Let us define a number $l_A = \min\{l_y : y \in A\}$. Since the set A is finite, the inequality $l_A > 0$ holds.

Since the net $\{\tilde{g}_t \cdot \tilde{g}_0 : t \in T\}$ converges to the function $\tilde{g}_e$, there is an element $t_0 \in T$ such that $\tilde{g}_t \cdot \tilde{g}_0 \in V_C(\tilde{g}_e, \varepsilon \cdot l_A, A)$, that is, $n_y(\tilde{g}_t \cdot \tilde{g}_0 - \tilde{g}_e) < \varepsilon \cdot l_A$ for all $y \in A$ and $t \geqslant t_0$, $t \in T$. Then for all $y \in A$ and $t \geqslant t_0$, $t \in T$, we get the inequality

$$n_y(\tilde{g}_t - \tilde{g}_0') = n_y((\tilde{g}_t \cdot \tilde{g}_0 - \tilde{g}_e) \cdot \tilde{g}_0') \leqslant n_y(\tilde{g}_t \cdot \tilde{g}_0 - \tilde{g}_e) \cdot n_y \tilde{g}_0' < \varepsilon \cdot l_A \cdot \frac{1}{l_y} \leqslant \varepsilon,$$

that is, $\tilde{g}_t \in V_C(\tilde{g}_0', \varepsilon, A)$ for all $t \geqslant t_0$, $t \in T$. Hence, the net $\{\tilde{g}_t : t \in T\}$ converges to the function $\tilde{g}_0'$. The algebra C is $\tilde{Y}$-complete, therefore the algebra $\tilde{C} = \tilde{\varphi} C$ is closed in $C(\tilde{f})$ and, hence, $\tilde{g}_0' \in \tilde{C}$.

Thus, the net $\{g_t : t \in T\}$ converges to some element $g_0' = \tilde{\varphi}^{-1} \tilde{g}_0' \in C$ because of $\tilde{\varphi}$ is a topological isomorphism onto $\tilde{C}$. Since the multiplication is continuous, we have $e_C = g_0' \cdot g_0 \in J_0$, but this contradicts the assumption $e_C \in [J_0]_C \setminus J_0$.

Thus, the case 1) is possible only, and the proof is completed. □

6.21. Let the algebra C be $\tilde{Y}$-complete and have the unite e_C. Theorem 6.20 means that element $g \in C$ has the inverse element $\frac{e_C}{g} \in C$ iff $gM \neq 0$ for all $M \in \mathfrak{M}$ (or, that is the same, iff $(\tilde{\varphi} g)x \neq 0$ for all $x \in \tilde{X}$). In particular, for each element $g \in C$ its spectrum

$$Sp_C g = \{\lambda \in \mathbb{C} : \text{ the element } g - \lambda \cdot e_C \text{ has not an inverse element in } C\}$$

coinsides with the set $g\mathfrak{M} = (\tilde{\varphi} g)\tilde{X}$.

6.22. **Corollary.** *If the algebra C is $\tilde{Y}$-complete and has the unit e_C, and an element $g \in C$ satisfies the condition $n_y g < B$ for all $y \in \tilde{Y}$, where $B \in \mathbb{R}$ is some*

[12] It means that the algebra C is advertibly complete (see [7], Chapter III, the statement (5.12)).

number, then the element $\frac{e_C}{g-\lambda\cdot e_C}$ *exists for all* $\lambda\in\mathbb{C}$ *such that* $|\lambda|\geqslant B$*; moreover,*

$$\frac{e_C}{g-\lambda\cdot e_C}=-\frac{1}{\lambda}\cdot e_C-\sum_{n=1}^{\infty}\lambda^{-n-1}\cdot g^n$$

(*the series converges with respect to the topology of the algebra* C)*; if, in addition,* $A\subseteq C$ *is a subalgebra such that* $e_C\in A$ *and* $d_y^A g=0$ *for some* $y\in\tilde{Y}$ *then* $d_y^A\frac{e_C}{g-\lambda\cdot e_C}=0$ (see 6.10).

6.23. Let the algebra C be $\tilde{Y}$-complete and have not a unit. Let C' be the algebra described in 6.19. Then by a definition we shall consider that $Sp_C g=Sp_{C'}g$ for $g\in C$.

The set $M_C=C$ is a closed maximal ideal of the algebra C' and $gM_C=0$ for all $g\in C$, therefore $0\in Sp_C g$ for any $g\in C$; of course, the equality $Sp_C g=g\mathfrak{M}=$ $=(\tilde{\varphi}g)\tilde{X}$ holds in this case also, if $\mathfrak{M}$ is the space of closed maximal ideals of the algebra C'.

It follows from Theorem 6.20 that for each element $g\in C$ such that $gM\neq 1$ for all $M\in\mathfrak{M}$ (or, that is the same, $(\tilde{\varphi}g)x\neq 1$ for all $x\in\tilde{X}$) there exists an element $g^-\in C$ satisfying the equality $g^-M=\frac{gM}{gM-1}$ for all $M\in\mathfrak{M}$ (the element g^- is called *quasi-inverse* to g). Indeed, the element g^- exists in C' since $1\notin Sp_{C'}g$, and $g^-M_C=0$, therefore $g^-\in C$.

Let us note that the element $\left(\frac{1}{\lambda}\cdot g\right)^-$ exists for $g\in C$ and $\lambda\in\mathbb{C}$ iff $\lambda\notin Sp_C g$.

By the same reason for every $g,g'\in C$ and $\lambda\in\mathbb{C}$ such that $\lambda\notin Sp_C g$ there is an element $g''\in C$ satisfying the condition $g''M=\frac{g'M}{gM-\lambda}$ for all $M\in\mathfrak{M}$. We shall write $g''=\frac{g'}{g-\lambda\cdot e}$ in this case. In particular, $\left(\frac{1}{\lambda}\cdot g\right)^-=\frac{g}{g-\lambda\cdot e}$.

6.24. **Corollary.** *If the algebra* C *is* $\tilde{Y}$*-complete and has not a unit, and an element* $g\in C$ *satisfies the condition* $n_y g<B$ *for all* $y\in\tilde{Y}$*, where* $B\in\mathbb{R}$ *is some number, then the element* $\left(\frac{1}{\lambda}\cdot g\right)^-$ *exists for all* $\lambda\in\mathbb{C}$ *such that* $|\lambda|\geqslant B$*; moreover,*

$$\left(\frac{1}{\lambda}\cdot g\right)^-=-\sum_{n=1}^{\infty}\lambda^{-n}\cdot g^n$$

(*the series converges with respect to the topology of the algebra* C)*; if, in addition,* $A\subseteq C$ *is a subalgebra such that* $d_y^A g=0$ *for some* $y\in\tilde{Y}$ *then* $d_y^A\left(\frac{1}{\lambda}\cdot g\right)^-=0$.

6.25. **Lemma.** *Let the algebra* C *be* $\tilde{Y}$*-complete and* $g\in C$ *be an element such that* $Sp_C g\subseteq\mathbb{R}(\subset\mathbb{C})$ (*that is,* gM *is real for all* $M\in\mathfrak{M}$)*. Then there exists an element* $g'\in C$ *such that* $g'M=|gM|$ *for all* $M\in\mathfrak{M}$ (it is natural to write $g'=$ $=|g|$)*; if, in addition,* $A\subseteq C$ *is a subalgebra and* $d_y^A g=0$ *for some* $y\in\tilde{Y}$ *then* $d_y^A g'=0$.

Proof. It is well known that the series

$$|t|=\sqrt{1-(1-t^2)}=1-\frac{1}{2}\cdot(1-t^2)-\frac{1}{2\cdot 4}\cdot(1-t^2)^2-\frac{1\cdot 3}{2\cdot 4\cdot 6}\cdot(1-t^2)^3-\dots$$

uniformly converges for $t\in[-\sqrt{2},\sqrt{2}]$. In particular, for each $n\in\mathbb{N}$ there is a number $m_n\in\mathbb{N}$ such that $||t|-\bar{p}_n t|<\frac{1}{n^2}$ for all $t\in[-\sqrt{2},\sqrt{2}]$, where

$$\bar{p}_n t=1-\frac{1}{2}\cdot(1-t^2)-\frac{1}{2\cdot 4}\cdot(1-t^2)^2-\dots-\frac{1\cdot 3\cdot\ldots\cdot(2\cdot m_n-3)}{2\cdot 4\cdot 6\cdot\ldots\cdot(2\cdot m_n)}\cdot(1-t^2)^{m_n}.$$

Then the function $p_n\colon\mathbb{R}\to\mathbb{R}$ defined by the formula $p_n t=\bar{p}_n t-\bar{p}_n 0$ for $t\in T$ is a polynomial and $p_n 0=0$, therefore $g_n=n\cdot p_n\left(\frac{1}{n}\cdot g\right)$ is an element of the algebra C for all $n\in\mathbb{N}$. Moreover, we have $||gM|-g_nM|<n\cdot\left(\frac{1}{n^2}+\frac{1}{n^2}\right)=\frac{2}{n}$ for all $M\in\mathfrak{M}$ and $n\in\mathbb{N}$ such that $|gM|\leqslant n\cdot\sqrt{2}$. It easily follows from this fact that

the sequence $\{g_n : n \in \mathbb{N}\}$ is $\tilde{Y}$-fundamental (one can take $O_n = \varnothing$ for $n < \frac{2}{\varepsilon}$ and $O_n = \{y \in \tilde{Y} : n_y g < n \cdot \sqrt{2}\}$ for $n \geqslant \frac{2}{\varepsilon}$, $n \in \mathbb{N}$, in Definition 6.16, where $\varepsilon > 0$ is a given number). Since the algebra C is $\tilde{Y}$-complete, this sequence converges to some element $g' \in C$. Of course, $g'M = |gM|$ for all $M \in \mathfrak{M}$.

The last statement follows from Lemma 6.10 5). □

6.26. **Corollary.** *Let the algebra C be $\tilde{Y}$-complete and $g_1, g_2 \in C$ be elements such that $Sp_C g_k \subseteq \mathbb{R}$ for $k = 1, 2$. Then there exist elements $g', g'' \in C$ such that $g'M = \max\{g_1M, g_2M\}$ and $g''M = \min\{g_1M, g_2M\}$ for all $M \in \mathfrak{M}$* (it is natural to write $g' = \max\{g_1, g_2\}$ and $g'' = \min\{g_1, g_2\}$)*; if, in addition, $A \subseteq C$ is a subalgebra such that $d_y^A g_k = 0$ for $k = 1, 2$ and some $y \in \tilde{Y}$, then $d_y^A g' = 0$ and $d_y^A g'' = 0$.*

6.27. **Theorem.** *Let the algebra C be $\tilde{Y}$-complete and have the unit e_C and an involution. Then the following conditions are equivalent:*

1) *the space $\mathfrak{M}$ of closed maximal ideals of the algebra C is compact;*
2) *all maximal ideals of the algebra C are closed.*

Proof. Let the space $\mathfrak{M}$ be compact, and let $M_0 \subset C$ be some maximal ideal. For each $g \in C$ we can define the closed set $Z_g = \{M \in \mathfrak{M} : gM = 0\}$. If we assume that $\bigcap\{Z_g : g \in M_0\} = \varnothing$ then there is a finite family $\{g_k : k = 1, 2, \ldots, n\} \subseteq M_0$ such that $\bigcap\limits_{k=1}^{n} Z_{g_k} = \varnothing$ because of the space $\mathfrak{M}$ is compact, but then the element $g_1 \cdot g_1^* + g_2 \cdot g_2^* + \ldots + g_n \cdot g_n^* \in M_0$ has the inverse element by Theorem 6.20, but this is impossible by the statement III of §7.4 of the book [8]. Hence, there exists a closed maximal ideal $M \in \bigcap\{Z_g : g \in M_0\}$. We have $M_0 \subseteq M$; moreover, $M_0 = M$ since the ideal M_0 is maximal, therefore the ideal M_0 is closed.

Let all maximal ideals of the algebra C be closed. Let us suppose that the space $\mathfrak{M}$ is not compact. Then there exists a filter $\mathcal{F}$ of closed subsets of $\mathfrak{M}$ such that $\bigcap\{F : F \in \mathcal{F}\} = \varnothing$. Let

$$J = \{g \in C : \text{ there is } F \in \mathcal{F} \text{ such that } Z_g \supseteq F\}.$$

It is clear that J is an ideal. We shall prove that for every $M \in \mathfrak{M}$ there is an element $g \in J$ such that $gM \neq 0$.

Let $M_0 \in M$. There exists a closed set $F \in \mathcal{F}$ such that $M_0 \notin F$. By the definition of the topology of the space $\mathfrak{M}$ there are functions $g_1, g_2, \ldots, g_n \in C$ and open sets $U_1, U_2, \ldots, U_n \subseteq \mathbb{C}$ such that

$$M_0 \in \bigcap_{k=1}^{n} g_k^{-1} U_k \subseteq \mathfrak{M} \setminus F.$$

For $k = 1, 2, \ldots, n$ we have the number $c_k = \inf\{|c - g_k M_0|^2 : c \in \mathbb{C} \setminus U_k\} > 0$. Therefore we can define the function

$$g_0 = \sum_{k=1}^{n} \frac{1}{c_k} \cdot (g_k - g_k M_0 \cdot e_C) \cdot (g_k - g_k M_0 \cdot e_C)^*.$$

It is obvious that the element $g_0 \in C$ has the following properties:

α) $g_0 M$ is real for all $M \in \mathfrak{M}$;
β) $g_0 M_0 = 0$;
γ) $g_0 M \geqslant 1$ for all $M \in F$.

Due to Corollary 6.26 the algebra C contains the element $g = \max\{0_C, e_C - g_0\}$; since $gM = 0$ for all $M \in F$, we get $g \in J$ and $gM_0 = 1$.

Thus, $J \nsubseteq M$ for each $M \in \mathfrak{M}$, therefore every maximal ideal containing J is not closed, and we have a contradiction with the condition 2). Hence, the space $\mathfrak{M}$ is compact. □

6.28. **Theorem.** *Let the algebra C be $\tilde{Y}$-complete and have the unit e_C and an involution. Then the algebra C is topologically* (*and isometrically*) *isomorphic to the algebra $C(\mathfrak{M})$ of all continuous functions on the space of closed maximal ideals with the topology defined in the item 5.13. Hence, the algebra C is topologically* (*and isometrically*) *isomorphic to the algebra $C(f_C)$* (see the end of the item 5.16).

Proof. We shall prove that the algebra C is dense in $C(\mathfrak{M})$ (we suppose that $C \subseteq C(\mathfrak{M})$; see 5.3).

Let $g_0 \in C(\mathfrak{M})$ be any function and $V_{C(\mathfrak{M})}(g_0, \varepsilon, A)$ be its arbitrary neighborhood ($V_{C(\mathfrak{M})}(g_0, \varepsilon, A) = \{g \in C(\mathfrak{M}) : n_y(g - g_0) < \varepsilon$ for all $y \in A\}$, $\varepsilon > 0$, $A \subseteq \tilde{Y}$ is a finite set). The set $\mathfrak{M}_A = \bigcup\{\mathfrak{M}_y : y \in A\}$ is compact in consequence of Lemma 5.4. The algebra $C_A = \{g|_{\mathfrak{M}_A} : g \in C\}$ has the unit and the involution and separates points of the space $\mathfrak{M}_A$, therefore it follows from Stone - Weierstrass theorem that C_A is a dense subset of the algebra $C^*(\mathfrak{M}_A)$. In particular, it means that there exists an element $g \in C$ such that $|gM - g_0M| < \varepsilon$ for all $M \in \mathfrak{M}_A$. It is easily seen that $g \in V_{C(\mathfrak{M})}(g_0, \varepsilon, A)$, that is, the algebra C is dense in $C(\mathfrak{M})$.

Thus we have $C = C(\mathfrak{M})$ since the algebra C is closed in $C(\mathfrak{M})$ (see 6.14). The equality $\varphi_C C = C(f_C)$ follows from Corollary 5.28. □

6.29. **Corollary.** *Let the algebra C be $\tilde{Y}$-complete and have an involution but have not a unit. Then the algebra C is topologically* (*and isometrically*) *isomorphic to the algebra $C_0(\mathfrak{M}) = \{g \in C(\mathfrak{M}) : gM_C = 0\}$ where the closed maximal ideal $M_C \subset C'$ have been defined in the item 6.23.*

6.30. **Theorem.** *Let C_1 and C_2 be commutative complex algebras with the units e_{C_1} and e_{C_2} equipped with Hausdorff topologies generated by families of semi-norms $\{n_y : y \in Y_1\}$ and $\{n_y : y \in Y_2\}$ which satisfy the conditions 1)–7) of Theorem 2.7. Let the sets Y_1 and Y_2 be equipped with topologies which are compatible* (see 5.14) *with algebras C_1 and C_2* (*these topological spaces will be denoted by $\tilde{Y}_1$ and $\tilde{Y}_2$*). *Moreover, let $C \subseteq C_2$ be a dense subalgebra containing the unit e_{C_2}, $h\colon \tilde{Y}_1 \to \tilde{Y}_2$ be a mapping and $\psi\colon C \to C_1$ be a homomorphism such that $n_y(\psi g) \leqslant n_{hy} g$ for all $y \in \tilde{Y}_1$ and $g \in C$.*

If the algebra C_1 is $\tilde{Y}_1$-complete then there is a unique continuous homomorphism $\bar{\psi}\colon C_2 \to C_1$ such that $\bar{\psi}|_C = \psi$; besides that, $n_y(\bar{\psi} g) \leqslant n_{hy} g$ for all $y \in \tilde{Y}_1$ and $g \in C_2$.

Proof. Let $\{g_t : t \in T\} \subseteq C$ be a net converging to a given element $g_0 \in C_2$. We shall prove that the net $\{g'_t : t \in T\} \subseteq C_1$, where $g'_t = \psi g_t$ for $t \in T$, is $\tilde{Y}_1$-fundamental.

Let us take an arbitrary number $\varepsilon > 0$. For every element $t \in T$ let us define the sets $O_t = \left\{y \in \tilde{Y}_2 : n_y(g_t - g_0) < \frac{\varepsilon}{2}\right\}$ and $O'_t = h^{-1} O_t$. It is easily seen that the family $\{O_t : t \in T\}$ satisfies the conditions 1) and 2) of Definition 6.16. The inequality $n_y(\psi g) \leqslant n_{hy} g$ for all $y \in \tilde{Y}_1$ and $g \in C$ forces that the net $\{g'_t : t \in T\}$ and the family $\{O'_t : t \in T\}$ satisfy the above mentioned conditions, therefore the latter net is $\tilde{Y}_1$-fundamental.

Since the algebra C_1 is $\tilde{Y}_1$-complete, there is an element $g'_0 \in C_1$ such that the net $\{g'_t : t \in T\}$ converges to g'_0. It follows from the above-mentioned inequality that we can define the map $\bar{\psi}\colon C_2 \to C_1$ by letting $\bar{\psi} g_0 = g'_0$ for $g_0 \in C_1$, and this map is a continuous homomorphism possessing all required properties. □

6.31. *Remark.* It follows from Remark 5.32 that the algebra C (see 6.12) is isometrically isomorphic to a dense subalgebra of some $\tilde{Y}$-complete algebra $\bar{C}$. This algebra $\bar{C}$ can be called *a $\tilde{Y}$-completion of the algebra C*. The uniqueness of a $\tilde{Y}$-completion (up to an isometrical isomorphism) is a simple consequence of Theorem 6.30.

6.32. Let $A \subseteq C$ be a subset of the algebra C. For each $g \in A$ let $g\colon \mathfrak{M} \to \mathbb{C}$ be the corresponding function (see 5.3; if the algebra C has not a unit then we add the unit e to define the space $\mathfrak{M}$). Then we can define the diagonal mapping

$$g_A = \Delta\{g : g \in A\}\colon \mathfrak{M} \xrightarrow{\text{onto}} \mathfrak{M}^A \subseteq \prod\{\mathbb{C}_g : g \in A\},$$

where $\mathbb{C}_g = \mathbb{C}$ for $g \in A$ (see [5], §2.3), and the sets $\mathfrak{M}^A_y = g_A\mathfrak{M}_y$ for all $y \in \tilde{Y}$ (see 5.2). Let $C(\mathfrak{M}^A)$ be the algebra of all continuous complex functions with the topology[13] generated by the family of semi-norms $\{n_y : y \in \tilde{Y}\}$, which are defined by the formula $n_y g = \sup\{|gz| : z \in \mathfrak{M}^A_y\}$ for all $g \in C(\mathfrak{M}^A)$ and $y \in \tilde{Y}$ (the sets $\mathfrak{M}^A_y$, $y \in \tilde{Y}$, are compact by Lemma 5.4 and Theorem 3.1.10 of the book [5]).

For each $g \in A$ there exists the unique element $\bar{g} \in C(\mathfrak{M}^A)$ such that $g = \bar{g}g_A$ (see [5], §2.3; the function $\bar{g}$ is the restriction of the natural projection $\prod\{\mathbb{C}_g : g \in \in A\} \to \mathbb{C}_g = \mathbb{C}$ to the set $\mathfrak{M}^A$). The unit of the algebra $C(\mathfrak{M}^A)$ will be denoted by e_A.

Let C_A be the smallest closed subalgebra of $C(\mathfrak{M}^A)$ which contains all functions $\bar{g}$, $g \in A$, and satisfies the following conditions:

1) if the algebra C has the unit e_C then C_A contains e_A;
2) if the algebra C has an involution then C_A also has an involution (the complex conjugation; see Lemma 5.12);
3) if the algebra C has the unit then C_A contains all functions of the form $\frac{e_A}{\bar{g}-\lambda\cdot e_A}$, where $g \in A$ and $\lambda \in \mathbb{C} \setminus Sp_C g$ (see 6.20, 6.21);
4) if the algebra C has not a unit then C_A contains all functions of the form $\frac{\bar{g}'}{\bar{g}-\lambda\cdot e_A}$, where $g, g' \in A$ and $\lambda \in \mathbb{C} \setminus Sp_C g$ (see 6.23).

6.33. **Theorem.** a) *If the algebra C has an involution then $C_A = C(\mathfrak{M}^A)$ (if C has the unit e_C) or $C_A = C_0(\mathfrak{M}^A) = \{g \in C(\mathfrak{M}^A) : g\bar{0} = 0\}$ (if C has not a unit) where $\bar{0} \in \mathfrak{M}^A \subseteq \prod\{\mathbb{C}_g : g \in A\}$ is the (unique) point such that $\bar{g}\bar{0} = 0$ for all $g \in A$.*

b) *The map $\varphi_A\colon C(\mathfrak{M}^A) \to C(\mathfrak{M})$ defined by the equality $\varphi_A g = gg_A$ for all $g \in C(\mathfrak{M}^A)$ is an isometrical isomorphism satisfying the condition $\varphi_A\bar{g} = g$ for all $g \in A$.*

c) *If the mapping $g_A\colon \mathfrak{M} \xrightarrow{\text{onto}} \mathfrak{M}^A$ is quotient then φ_A is an isomorphism onto a closed subalgebra of $C(\mathfrak{M})$.*

d) *If the algebra C is $\tilde{Y}$-complete then $\varphi_A C_A \subseteq C$* (compare with [4], §1.4 and §1.5).

Proof. a) This can be proved in the same way as Theorem 6.28.

b) This is obvious.

c) This can be proved by analogy with the proof of the statement 2) of Theorem 2.23.

d) Due to the definition of the algebra C_A and the statements 6.20, 6.23 the set $C \cap \varphi_A C_A$ is dense in $\varphi_A C_A$. Since the algebra C is closed in $C(\mathfrak{M})$ (see 6.14), we get $\varphi_A C_A \subseteq [C \cap \varphi_A C_A]_{C(\mathfrak{M})} \subseteq C$. □

6.34. **Example.** Let the algebra C be $\tilde{Y}$-complete and $g_0 \in C$ be any element. Let $A = \{g_0\}$. Then $\mathfrak{M}^A = Sp_C g_0$ (see 6.21, 6.23) and $g_A = g_0$. Theorem 6.33 d) means that for each function $h \in C_A$ there is an element $g_h \in C$ such that $g_h M = hg_0 M$ for all $M \in \mathfrak{M}$.

6.35. Let Y_d be the set Y (see 6.12) equipped with the discrete topology and $\tilde{p}\colon Y_d \xrightarrow{\text{onto}} \tilde{Y}$ be the identity mapping. Let X_d be the fan product of the spaces

[13] If we would use the topology of the uniform convergence on compact subsets (that is, the compact-open topology) then the algebra C_A defined below would be lesser.

$\tilde{X}$ and Y_d relative to the mappings $\tilde{f}$ and $\tilde{p}$, and let $\tilde{q}\colon X_d \to \tilde{X}$ and $f_d\colon X_d \to Y_d$ be the projections of the fan product to its factors. By the same reasons as in 5.32 the mapping f_d is perfect, globally completely regular and "onto", the mapping $\tilde{q}$ is "onto" and one-to-one, the space X_d is a subspace of the product $Y_d \times \mathfrak{M}$. The map $\psi_d\colon C(\tilde{f}) \to C(f_d)$, defined by the formula $\psi_d g = g\tilde{q}$ for $g \in C(\tilde{f})$, is an isometrical isomorphism onto a subalgebra.

Let us define the isometrical isomorphism $\varphi_d\colon C \to C(f_d)$ (onto a subalgebra) by the equality $\varphi_d = \psi_d\tilde{\varphi}$ (see 6.17).

6.36. The algebra $C(f_d)$ is complete due to Proposition 6.2, therefore the algebra $\hat{C} = [\varphi_d C]_{C(f_d)}$ is a completion of the algebra C (since the completion is unique up to a topological isomorphism). Let us note that $\bar{C} = \psi_d^{-1}\hat{C} = \psi_d^{-1}(\hat{C} \cap \psi_d C(\tilde{f}))$ is a closed subalgebra of $C(\tilde{f})$ such that $\tilde{C} = \tilde{\varphi} C \subseteq \bar{C}$ is a dense subalgebra of $\bar{C}$. Hence, the algebra $\bar{C}$ is the $\tilde{Y}$-completion of the algebra C. In particular, if the algebra C is $\tilde{Y}$-complete then $\tilde{C} = \bar{C}$, and we get the following useful tool:

if the algebra C is $\tilde{Y}$-complete and $g \in \hat{C} \subseteq C(f_d)$ is an element such that the function $\tilde{g} = g\tilde{q}^{-1}\colon \tilde{X} \to \mathbb{C}$ is continuous[14] then $g \in \varphi_d C$.

6.37. **Example.** Let the algebra C be $\tilde{Y}$-complete, a set $A \subset C$ be finite, and let $g_A\colon \mathfrak{M} \xrightarrow{\text{onto}} \mathfrak{M}^A \subseteq \mathbb{C}^{|A|} = \prod\{\mathbb{C}_g : g \in A\}$ be the mapping defined in 6.32.

Moreover, let V be a manifold, $\xi\colon V \to \mathbb{C}^{|A|}$ be a local homeomorphism (that is, for each point $z \in V$ there exists a neighborhood $Oz \subseteq V$ such that $\xi|_{Oz}\colon Oz \xrightarrow{\text{onto}}$ $\xrightarrow{\text{onto}} \xi OZ \subseteq \mathbb{C}^{|A|}$ is a homeomorphism onto an open subset of $\mathbb{C}^{|A|}$) and $\theta\colon \mathfrak{M} \to V$ be a mapping such that $\xi\theta = g_A$. The manifold V is equipped with a complex structure by means of the local homeomorphism ξ (see [7], Chapter VI).

$$\begin{array}{ccccccccc}
X_d & \xrightarrow{\tilde{q}} & \tilde{X} & \xrightarrow{q} & X & \xrightarrow{\pi} & \mathfrak{M} & \xrightarrow{\theta} & V \\
\downarrow{\scriptstyle f_d} & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g_A} & & \downarrow{\scriptstyle \xi} \\
Y_d & \xrightarrow{\tilde{p}} & \tilde{Y} & \xrightarrow{p} & Y & & \mathfrak{M}^A & \xrightarrow{\subseteq} & \mathbb{C}^{|A|}
\end{array}$$

Let us equip the algebra $C(\theta\mathfrak{M})$ with the topology generated by the family of semi-norms $\{n_y : y \in Y\}$, which are defined by the formula $n_y g = \sup\{|g\theta M| : M \in$ $\in \mathfrak{M}_y\}$ for all $y \in Y$ and $g \in C(\theta\mathfrak{M})$. The map $\eta\colon C(\theta\mathfrak{M}) \to C(\mathfrak{M})$ defined by the equality $\eta g = g\theta$ for $g \in C(\theta\mathfrak{M})$ is an isometrical isomorphism onto a subalgebra.

For each neighborhood $U \subseteq V$ of the set $\theta\mathfrak{M}$ let $Hol(U)$ be the set of all holomorphic functions $h\colon U \to \mathbb{C}$ equipped with the compact-open topology, and $\varphi_U\colon Hol(U) \to C(\theta\mathfrak{M})$ be the restriction homomorphism ($\varphi_U h = h|_{\theta\mathfrak{M}}$ for $h \in$ $\in Hol(U)$). The restriction homomorphism φ_U is continuous.

Let $Hol(\theta\mathfrak{M}) = \bigcup\{\varphi_U Hol(U) : U \subseteq V$ is a neighborhood of the set $\theta\mathfrak{M}\}$ and $C_H = [Hol(\theta\mathfrak{M})]_{C(\theta\mathfrak{M})}$ if the algebra C has the unit; otherwise let $Hol_0(\theta\mathfrak{M}) =$ $= \{h \in Hol(\theta\mathfrak{M}) : h\theta M_C = 0\}$, where $M_C \in \mathfrak{M}$ is the ideal defined in the item 6.23 ($gM_C = 0$ for all $g \in C$), and $C_H = [Hol_0(\theta\mathfrak{M})]_{C(\theta\mathfrak{M})}$.

We can define the isometrical isomorphism $\varphi_{\hat{C}}\colon C_H \to C(f_d)$ by the formula $\varphi_{\hat{C}} = \psi_d\tilde{\psi}\varphi\eta|_{C_H}$, that is, $\varphi_{\hat{C}} h = h\theta\pi q\tilde{q}$ for $h \in C_H$. It follows from Theorem 4.1 and Scholium 4.2 of Chapter VI of the book [7] that $\varphi_{\hat{C}} C_H \subseteq [\hat{C}]_{C(f_d)} =$ $= [\varphi_d C]_{C(f_d)} = \hat{C}$. Since for each $h \in C_H$ the function $(\varphi_{\hat{C}} h)\tilde{q}^{-1} = h\theta\pi q$ is continuous, we get $\varphi_{\hat{C}} C_H \subseteq \hat{C} \cap \psi_d C(\tilde{f}) = \psi_d \tilde{C} = \varphi_d C$ (the latter equalities hold because the algebra C is $\tilde{Y}$-complete). Thus, we get $\eta C_H \subseteq C$, that is, there is the isometrical isomorphism $\varphi_C = \eta|_{C_H}\colon C_H \to C$ onto a subalgebra.

[14] This definition is correct since the mapping $\tilde{q}$ is one-to-one and "onto".

$$
\begin{tikzcd}
 & & C(\theta\mathfrak{M}) \arrow[dr, "\eta"] & \\
C_H \arrow[urr, "\subseteq"] \arrow[rr, "\varphi_C"] \arrow[d, "\varphi_{\hat{C}}"'] \arrow[dr, "\varphi_{\hat{C}}"] & & C \arrow[r, "\subseteq"'] \arrow[dll, "\varphi_d"'] \arrow[d, "\tilde{\varphi}"'] & C(\mathfrak{M}) \arrow[dd, "\cap\!|\ \varphi"] \\
\hat{C} \arrow[d, "\cap\!|"'] & \hat{C} \cap \psi_d C(\tilde{f}) \arrow[l, "\supseteq"] \arrow[dr, "\psi_d^{-1}"] & \tilde{C} \arrow[l, "\psi_d|_{\tilde{C}}"'] \arrow[d, "\cap\!|"'] & \\
C(f_d) & & C(\tilde{f}) \arrow[ll, "\supseteq", "\psi_d"'] & C(f) \arrow[l, "\supseteq", "\tilde{\psi}"']
\end{tikzcd}
$$

In particular, for each neighborhood $U \subseteq V$ of the set $\theta\mathfrak{M}$ and each function $h \in Hol(U)$ (such that $h\theta M_C = 0$ if the algebra C has not a unit) there exists an element $g_h \in C$ such that $g_h = h\theta$. Hence, there is the "functional calculus" in the $\tilde{Y}$-complete algebra C.

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[15] The translation of this article into English contains significant errors which do not exist in the Russian text. For the correction it is necessary

1) to omit the word "open" in the second line of §1;
2) to replace the word "continuous" by "irreducible" in Proposition 2;
3) to replace the word "compact" by "Hausdorff compact" in Corollaries 4 and 9 (the Russian term "бикомпакт" means "Hausdorff compact space").

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