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RIEMANN–HILBERT PROBLEM AND SOLVABILITY OF DIFFERENTIAL EQUATIONS

Abstract. In this paper Riemann–Hilbert problem is applied to the solvability of a mixed type Monge-Ampère equation and the index formula of ordinary differential equations. Blowing up onto the torus turns mixed type equations into elliptic equations, to which R-H problem is applied.

1. Introduction

This paper is concerned with the Riemann–Hilbert problem and the (unique) solvability of differential equations. The Riemann–Hilbert problem has many applications in mathematics and physics. In this paper we are interested in the solvability of a mixed type Monge-Ampère equation, a homology equation appearing in a normal form theory of singular vector fields and the index formula of ordinary differential equations. These equations have a singularity at some point, say at the origin. We handle these singular nature of the equations by a kind of blowing up and the Riemann–Hilbert problem.

Our idea is as follows. When we want to solve these degenerate mixed type equations in a class of analytic functions, we transform the equation onto the torus embedded at the origin. This is done by a change of variables similar to a blowing up procedure. Although we transform the local problem for a mixed type equation to a global one on tori, it turns out that, in many cases the transformed equations are elliptic on the torus. This enables us to apply a Riemann–Hilbert problem with respect to tori. Once we can solve the lifted problems we extend the solution on the torus inside the torus analytically by a harmonic (analytic) extension. The extended function is holomorphic in a domain whose Silov boundary is a torus. Moreover, by the maximal principle, the extended function is a solution of a given nonlinear equation since it satisfies the same equation on its Silov boundary, i.e., on tori. The uniqueness on the boundary and the maximal principle also implies the uniqueness of the solution.

This paper is organized as follows. In Section 2 we give examples and a general class of mixed type equations for which the blowing up procedure turns the mixed type equations into elliptic equations on tori. In Section 3 we discuss the relation of the blowing up procedure with a resolution of singularities. In Sections 4 and 5 we study the solvability of ordinary differential equations via blowing up procedure and the Riemann-Hilbert problem. In Section 6 we study the index formula of a system of

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singular ordinary differential operators from the viewpoint of the blowing up procedure and the Riemann-Hilbert problem. In Sections 7 and 8 we apply the R–H problem with respect to $\mathbb{T}^2$ to a construction of a parametrix. In Section 9 we apply the results of Sections 7 and 8 to the unique solvability of a mixed type Monge–Ampère equation of two variables. In Section 10 we study the solvability of a mixed type Monge-Ampère equation of general independent variables. In Section 11 we apply our argument to a system of nonlinear singular partial differential equations arising from the normal form theory of a singular vector field. In Section 12 we extend our argument to the solvability of equations containing a large parameter.

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2. Blowing up and mixed type operators

Let us consider the following Monge-Ampère equation

$$M(u) := \det(u_{x_i x_j}) = f(x),\ u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,$$

where $x = (x_1, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$ (resp. in $\mathbb{C}^n$) for some domain Ω. Let $u_0(x)$ be a smooth (resp. holomorphic) function in Ω, and set

$$f_0(x) = \det((u_0)_{x_i x_j}).$$

Then $u_0(x)$ is a solution of the above equation with $f = f_0$. (f_0 is a so-called Gauss curvature of u_0). Consider a solution $u = u_0 + v$ which is a perturbation of $u_0(x)$, namely

$$(MA) \qquad \det(v_{x_i x_j} + (u_0)_{x_i x_j}) = f_0(x) + g(x) \quad \text{in } \Omega,$$

where g is smooth in Ω(resp. analytic in Ω).

Define

$$W_R(D_R) := \{u = \sum_\eta u_\eta x^\eta;\ \|u\|_R := \sum_\eta |u_\eta| R^\eta < \infty\}.$$

We want to solve (MA) for $g \in W_R(D_R)$.

We shall lift (MA) onto the torus $\mathbb{T}^n$. The function space $W_R(D_R)$ is transformed to $W_R(\mathbb{T}^n)$,

$$W_R(\mathbb{T}^n) = \{u = \sum_\eta u_\eta R^\eta e^{i\eta\theta};\ \|u\|_R := \sum_\eta |u_\eta| R^\eta < \infty\},$$

where $R = (R_1, \ldots, R_n)$, $R^\eta = R_1^{\eta_1} \cdots R_n^{\eta_n}$. In order to calculate the operator on the torus we make the substitution

$$\partial_{x_j} \mapsto \frac{1}{R_j e^{i\theta_j}} \frac{1}{i} \frac{\partial}{\partial \theta_j} \equiv \frac{1}{R_j e^{i\theta_j}} D_j, \quad x_j \mapsto R_j e^{i\theta_j} \equiv z_j.$$

The reduced operator on the torus is given by

$$\det\left(z_j^{-1}z_k^{-1}D_jD_kv + (u_0)_{x_jx_k}(z)\right) = f_0 + g.$$

REMARK 1. The above transformation onto the torus is related with a Cauchy-Riemann equation as follows. For the sake of simplicity we consider the one dimensional case. The same things hold in the general case. We recall the following formula, for $t = re^{i\theta}$

$$t\partial = t\frac{\partial}{\partial t} = \frac{1}{2}\left(r\frac{\partial}{\partial r} - i\frac{\partial}{\partial\theta}\right), \quad \overline{t\partial} = \overline{t}\overline{\frac{\partial}{\partial t}} = \frac{1}{2}\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial\theta}\right),$$

where $\overline{\partial}$ be a Cauchy-Riemann operator. Assume that $\overline{\partial}u = 0$. Then, by the above formula we obtain

$$r\frac{\partial}{\partial r}u = -i\frac{\partial}{\partial\theta}u, \quad t\partial_t u = -i\frac{\partial}{\partial\theta}u = D_\theta u.$$

Note that the second relation is the one which we used in the above.

REMARK 2. (*Relation to Langer's transformation*) The transformation used in the above is essentially $x_j = e^{i\theta_j}$. Similar transformation $x = e^y$ was used by Langer in the study of asymptotic analysis of Schrödinger operator for a potential with pole of degree 2 at $x = 0$

$$-\frac{d^2}{dx^2} + \lambda^2\left(V(x) + \frac{k(k+1)}{x^2\lambda^2}\right)u = Eu,$$

where E is an energy and $V(x)$ is a regular function.

Some examples

Let $n = 2$, and set $x_1 = x$, $x_2 = y$. Consider the Monge-Ampère equation

$$(MA) \qquad M(u) + c(x,y)u_{xy} = f_0(x,y) + g(x,y),$$

where

$$M(u) = u_{xx}u_{yy} - u_{xy}^2, \quad f_0 = M(u_0) + c(x,y)(u_0)_{xy},$$

with $c(x,y)$ and u_0 being analytic in x and y. Let $Pv := M'_{u_0}v = \frac{d}{d\varepsilon}M(u_0+\varepsilon v)|_{\varepsilon=0}$ be a linearization of $M(u)$ at $u = u_0$. By simple calculations we obtain

$$M'_{u_0}v := (u_0)_{xx}\partial_y^2 v + (u_0)_{yy}\partial_x^2 v - 2(u_0)_{xy}\partial_x\partial_y v.$$

EXAMPLE 1. Consider the equation (MA) for

$$u_0 = x^2y^2, \quad c(x,y) = kxy \quad k \in \mathbb{R}.$$

We have $f_0 = 4(k-3)x^2y^2$. The linearized operator is given by

$$P = 2x^2\partial_x^2 + 2y^2\partial_y^2 + (k-8)xy\partial_x\partial_y, \quad \partial_x = \partial/\partial x, \partial_y = \partial/\partial y.$$

The characteristic polynomial is given by (with the standard notation) $-2x^2\xi_1^2 - 2y^2\xi_2^2 - (k-8)xy\xi_1\xi_2$. The discriminant is given by

$$D \equiv (k-8)^2x^2y^2 - 16x^2y^2 = (k-4)(k-12)x^2y^2.$$

It follows that (MA) is (degenerate) hyperbolic if and only if $k < 4$ or $k > 12$, while (MA) is (degenerate) elliptic if and only if $4 < k < 12$. In either case, (MA) degenerates on the lines $xy = 0$, namely the characteristic polynomial vanishes.

By lifting P onto the torus we obtain

$$2D_1(D_1 - 1) + 2D_2(D_2 - 1) + (k-8)D_1D_2.$$

Here, for the sake of simplicity we assume $R_j = 1$. The symbol is given by

$$\sigma(\eta) = 2(\eta_1(\eta_1 - 1) + \eta_2(\eta_2 - 1)) + (k-8)\eta_1\eta_2,$$

where η_j is the covariable of θ_j. Consider now the homogeneous part of degree 2. If this does not vanish on $|\eta| = 1$ we obtain the following

$$2 + (k-8)\eta_1\eta_2 \neq 0 \ \text{ for all } \eta \in \mathbb{R}^2, \ |\eta| = 1.$$

The condition is clearly satisfied if $k = 8$. If $k \neq 8$, noting that $-1/2 \leq \eta_1\eta_2 \leq 1/2$ we obtain $-1/2 \leq -2/(k-8) \leq 1/2$. By simple calculation we obtain $4 < k < 12$. Namely, if the given operator is degenerate elliptic the operator on the torus is an elliptic operator.

EXAMPLE 2. Consider (MA) under the following condition

$$u_0 = x^4 + kx^2y^2 + y^4, \ \ k \in R, \ c \equiv 0.$$

Then we have

$$f_0 = M(u_0) = 12(2kx^4 + 2ky^4 + (12 - k^2)x^2y^2).$$

The linearized operator is given by

$$P = 12y^2\partial_x^2 + 12x^2\partial_y^2 + 2k(x^2\partial_x^2 + y^2\partial_y^2) - 8xy\partial_x\partial_y.$$

The characteristic polynomial is given by

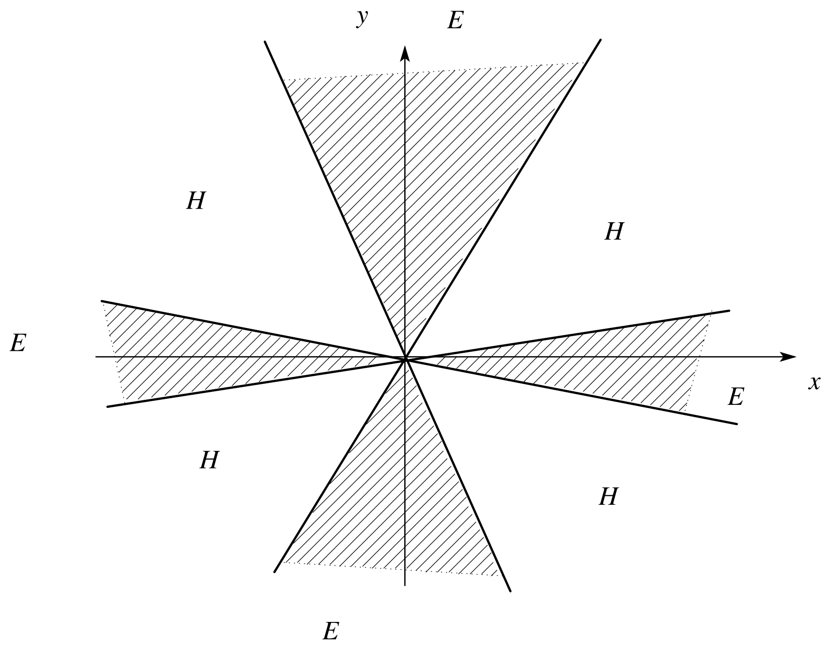
$$-12y^2\xi_1^2 - 12x^2\xi_2^2 - 2k(x^2\xi_1^2 + y^2\xi_2^2) + 8xy\xi_1\xi_2.$$

Since the discriminant is equal to $-f_0$, we study the signature of f_0. The following facts are easy to verify :

$$f_0/12 = 2k\left(x^2 + \frac{12-k^2}{4k}y^2\right)^2 - \frac{D}{8k}y^4, \quad D = (k^2-12)^2 - 16k^2.$$

It follows that $D < 0$ iff $-6 < k < -2$ or $2 < k < 6$, and $D > 0$ iff $k < -6$, $k > 6$ or $-2 < k < 2$. Hence, by the signature of f_0 we obtain:
if $k < -2$ it is hyperbolic and degenerates at the origin,
if $k = -2$ it is hyperbolic and degenerates on the line $x = \pm y$,
if $-2 < k < 0$ it is of mixed type,
if $k = 0$ it is elliptic and degenerates on the lines $x = 0$ and $y = 0$,
if $0 < k < 6$ it is elliptic and degenerates at the origin,
if $k = 6$ it is elliptic and degenerates on the lines $x = \pm y$,
if $k > 6$ it is of mixed type.

More precisely, in the mixed case the set $\{f_0 = 0\} \subset R^2$ consists of four lines intersecting at the origin. The equation changes its type from elliptic to hyperbolic or vice versa when crossed one of these lines. The equation degenerates on this line. (See the following figure of the case $k > 6$, where H and E denote the hyerbolic and elliptic region, respectively.)



In the case $-2 < k < 0$, a similar structure appears. The elliptic and hyperbolic regions are interchanged.

The operator on the torus is given by

$$\begin{aligned}\hat{P} &:= 12(e^{2i\theta_2-2i\theta_1}D_1(D_1-1)+e^{2i\theta_1-2i\theta_2}D_2(D_2-1))\\ &+ 2k(D_1(D_1-1)+D_2(D_2-1))-8D_1D_2.\end{aligned}$$

Here we assume $R_j = 1$ as before. Setting $z_j = e^{i\theta_j}$, the principal symbol is given by

$$\sigma(z, \eta) := 2k(\eta_1^2 + \eta_2^2) - 8\eta_1\eta_2 + 12(z_1^{-2}z_2^2\eta_1^2 + z_1^2 z_2^{-2}\eta_2^2).$$

Hence the condition $\sigma(z, \eta) \neq 0$ on $\mathbb{T}^2$ reads:

$$k - 4\eta_1\eta_2 + 6(\eta_1^2 t^2 + \eta_2^2 t^{-2}) \neq 0 \qquad \forall t \in \mathbb{C},\ |t| = 1\ \forall \eta \in \mathbb{R}^2,\ |\eta| = 1.$$

If $\eta_1 = \eta_2$ we have $\eta_1 = \eta_2 = \pm 1/\sqrt{2}$ in view of $|\eta| = 1$. By substituting this into the above equation we have, for $t = e^{i\theta}$

$$k - 2 + 6\cos 2\theta \neq 0 \quad 0 \leq \theta \leq 2\pi.$$

It follows that $k \notin [-4, 8]$. Similarly, if $\eta_1 = -\eta_2$ it follows that $k \notin [-8, 4]$. In case $\eta_1 \neq \pm\eta_2$ we have

$$2i\ Im\ (\eta_1^2 t^2 + \eta_2^2 t^{-2}) = (\eta_1^2 - \eta_2^2)(t^2 - t^{-2}) \neq 0, \quad \text{if } t^2 \neq \pm 1.$$

Hence the imaginary part of $k - 4\eta_1\eta_2 + 6(\eta_1^2 t^2 + \eta_2^2 t^{-2})$ does not vanish.

If $t^2 = \pm 1$, our condition can be written in $k \neq 4\eta_1\eta_2 \pm 6$. Because $-1/2 \leq \eta_1\eta_2 \leq 1/2$ it follows that $k \notin [-8, -4]$ and $k \notin [4, 8]$. Summing up the above we obtain $k < -8$ or $k > 8$. Under the condition the operator on the torus is elliptic. Especially, we remark that the same property holds in the mixed case $k > 8$.

We will extend these examples to more general equations. Because the problem is an essentially linear problem we study a linear equation. We consider a Grushin type operator

$$P = \sum_{|\alpha| \geq m, |\beta| \leq m} a_{\alpha\beta} x^\alpha \left(\frac{\partial}{\partial x}\right)^\beta,$$

where $a_{\alpha\beta} \in \mathbb{R}$ and $m \geq 1$. For the sake of simplicity we assume $R_j = 1$ $(j = 1, \ldots, n)$. The principal symbol of the lifted operator of P on $\mathbb{T}^n$ is given by, with $e^{i\theta} = (e^{i\theta_1}, \ldots, e^{i\theta_n}) \in \mathbb{T}^n$,

$$p(\theta, \xi) = \sum_{|\alpha| \geq m, |\beta| = m} a_{\alpha\beta} e^{i(\alpha - \beta)\theta} \xi^\beta.$$

Let $p_0(\xi)$ be the averaging of $p(\theta, \xi)$ on $\mathbb{T}^n$

$$p_0(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} p(\theta, \xi) d\theta = \sum_\alpha a_{\alpha\alpha} \xi^\alpha,$$

and define

$$Q(\theta, \xi) = p(\theta, \xi) - p_0(\xi).$$

We assume that $p_0(\xi)$ is elliptic: there exist $C > 0$ and $N > 0$ such that

$$|p_0(\xi)| \geq C|\xi|^m, \text{ for all } \xi \in \mathbb{R}^n, |\xi| \geq N.$$

We define the norm of $\|Q\|$ as the sum of absolute values of all Fourier coefficients of Q. We note that if $\|Q\|$ is sufficiently small compared with C the lifted operator P on the torus is elliptic.

We will show that P may be of mixed type in some neighborhood of the origin for any $C > 0$. We assume that P is hyperbolic with respect to x_1 at the point $x = r(1, 0, \dots, 0)$ for some small $r > 0$ chosen later. We note that this condition is consistent with the ellipticity assumption. Indeed, in P all terms satisfying $\alpha = \beta$ vanish at $x = r(1, 0, \dots, 0)$ except for the term $r^m \partial_{x_1}^m$. On the other hand there appears the term

$$\sum_{|\beta| \le m, \alpha = (\alpha_1, 0, \dots, 0), \alpha_1 > m} a_{\alpha\beta} r^{\alpha_1} \partial_x^\beta$$

from P corresponding to $\alpha \neq \beta$. We note that $\partial_{x_1}^m$ does not appear in the sum. Therefore, by an appropriate choice of the sign of coefficients in the averaging part the hyperbolicity condition is satisfied. This is possible for any large C. Same argument is valid if we consider near the other cordinate axis x_j.

Next we study the type of P near $x = r(1, \dots, 1)$. We can write the principal symbol of $i^{-m} P$ as follows.

$$\begin{aligned} \sum_{|\alpha| \ge m, |\beta| = m} a_{\alpha\beta} r^{|\alpha|} \xi^\beta &= r^m \sum_{|\alpha| = m, |\beta| = m} a_{\alpha\beta} \xi^\beta + \sum_{|\alpha| > m, |\beta| = m} a_{\alpha\beta} r^{|\alpha|} \xi^\beta \\ &= r^m \left(\sum_{|\alpha| = m} a_{\alpha\alpha} \xi^\alpha + \sum_{|\alpha| = m, \alpha \neq \beta} a_{\alpha\beta} \xi^\beta \right) + \sum_{|\alpha| > m, |\beta| = m} a_{\alpha\beta} r^{|\alpha|} \xi^\beta . \end{aligned}$$

The averaging part in the bracket in the right-hand side dominates the second term if $|a_{\alpha\beta}|$ is sufficiently small for $\alpha \neq \beta$, namely if $\|Q\|$ is sufficiently small. The terms corresponding to $|\alpha| > m$, $|\beta| = m$ can be absorbed to the first term if $r > 0$ is sufficiently small. Therefore we see that P is elliptic near $x = r(1, \dots, 1)$ for sufficiently small $r > 0$. Hence P is of mixed type in some neighborhood of the origin, while its blow up to the torus is elliptic. Summing up the above we have

THEOREM 1. *Under the above assumptions, if $\|Q\|$ is sufficiently small and if P is hyperbolic with respect to x_1 at the point $x = r(1, 0, \dots, 0)$ for small $r > 0$ the operator P is of mixed type near the origin, while its blowing up to the torus is elliptic.*

In the following sections we will construct a parametrix for such operators.

3. Relation to a resolution

We will show that the transformation in the previous section can be introduced directly via a resolution of singularities as follows. First we give a definition of a resolution in a special case.

Let $\mathbb{C}\mathbf{P}^1$ be a complex projective space and let $p : \mathbb{C}^2 \setminus O \to \mathbb{C}\mathbf{P}^1$ be a fibration of a projective space. Denote the graph of p by $\Gamma \subset (\mathbb{C}^2 \setminus O) \times \mathbb{C}\mathbf{P}^1$. The set Γ can be regarded as a smooth surface in $\mathbb{C}^2 \times \mathbb{C}\mathbf{P}^1$. The projection $\pi_1 : \mathbb{C}^2 \times \mathbb{C}\mathbf{P}^1 \to \mathbb{C}^2$ maps Γ onto $\mathbb{C}^2 \setminus O$ homeomorphically. The closure of the graph Γ of the map p in $\mathbb{C}^2 \times \mathbb{C}\mathbf{P}^1$ is the surface $\Gamma_1 = \Gamma \cup (O \times \mathbb{C}\mathbf{P}^1)$.

Indeed, let (x, y) be the coordinate in $\mathbb{C}^2$, and let $u = y/x$ be the local coordinate of $\mathbb{C}\mathbf{P}^1$. Then (x, y, u) is a local coordinate of $\mathbb{C}^2 \times \mathbb{C}\mathbf{P}^1$. Γ is given by $y = ux$, $x \neq 0$, and Γ_1 is given by $y = ux$. This is obtained by adding $O \times \mathbb{C}\mathbf{P}^1$ to Γ.

We can show the smoothness of Γ_1 by considering the second coordinate $(x, y, v), x = vy$. The projection $\pi_2 : \mathbb{C}^2 \times \mathbb{C}\mathbf{P}^1 \to \mathbb{C}\mathbf{P}^1$ foliate Γ_1 with a family of lines.

DEFINITION 1. *The procedure from $\mathbb{C}^2$ to Γ_1 is called the blowing up to $O \times \mathbb{C}\mathbf{P}^1$.*

EXAMPLE 3. Consider three lines intersecting at the origin O, $y = \alpha x$, $y = \beta x$, $y = \gamma x$. By $y = ux$, these lines are given by $x = 0, u = \alpha, u = \beta, u = \gamma$. In Γ_1 they intersect with $\mathbb{C}\mathbf{P}^1$ at different points.

We cosider the case $y = x^2$, $y = 0$. By blowing up we see that $u = x, u = 0, x = 0$ on Γ_1. Indeed, $y = 0$ is $0 = ux$, and $y = x^2$ is $ux = x^2$. Hence we are lead to the above case.

In the case $x^2 = y^3$, by setting $x = vy$ we have $v^2 = y$ and $y = 0$. Hence we are reduced the above case.

Grushin type operators

Let us consider a Grushin type operator.

$$P = \sum_{|\alpha|=|\beta|} a_{\alpha\beta} y^\alpha \left(\frac{\partial}{\partial y}\right)^\beta .$$

For the sake of simplicity we assume that $a_{\alpha\beta}$ are constants. We make the blowing up

$$y_j = z_j t, \quad j = 1, \ldots, n$$

where t is a variable which tends to zero and z_j $(j = 1, 2, \ldots, n)$ are variables which remain non zero when $t \to 0$. By introducing these variables we study the properties of P.

EXAMPLE 4. In the case of an Euler operator $\sum_{j=1}^n y_j \frac{\partial}{\partial y_j}$, we obtain

$$\sum_{j=1}^n y_j \frac{\partial}{\partial y_j} = t\frac{\partial}{\partial t} = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}.$$

If we introduce $z_j = \exp(i\theta_j)$, the right hand side is elliptic on a Hardy space on the torus. On the other hand in the radial direction t, it behaves like a Fuchsian operator.

If we assume that t is a parameter we have

$$\frac{\partial}{\partial z_j} = \frac{\partial y_j}{\partial z_j}\frac{\partial}{\partial y_j} = t\frac{\partial}{\partial y_j}.$$

Noting that $|\alpha| = |\beta|$ we obtain

$$y^\alpha \partial_y^\beta = z^\alpha t^{|\alpha|} t^{-|\beta|} \partial_z^\beta = z^\alpha \partial_z^\beta.$$

Hence P is transformed to the following operator on the torus

$$\hat{P} = \sum_{|\alpha|=|\beta|} a_{\alpha\beta} z^\alpha \left(\frac{\partial}{\partial z}\right)^\beta.$$

This is identical with the operator introduced in the previous section if we set $z_j = e^{i\theta_j}$.

4. Ordinary differential operators

Consider the following ordinary differential operator

$$p(t, \partial_t) = \sum_{k=0}^m a_k(t)\partial_t^k,$$

where $\partial_t = \partial/\partial t$ and $a_k(t)$ is holomorphic in $\Omega \subset \mathbb{C}$. For the sake of simplicity, we assume $\Omega = \{|t| < r\}$, where $(r > 0)$ is a small constant. We consider the following map

$$p : \mathcal{O}(\Omega) \mapsto \mathcal{O}(\Omega).$$

The operator p is singular at $t = 0$. Therefore, instead of considering at the origin directly we lift p onto the torus $\mathbb{T} = \{|t| = r\}$. In the following we assume that $r = 1$ for the sake of simplicity. The case $r \neq 1$ can be treated similarly if we consider the weighted space.

Let $L^2(\mathbb{T})$ be the set of square integrable functions on the torus, and define the Hardy space $H^2(\mathbb{T})$ by

$$H^2(\mathbb{T}) := \{u = \sum_{-\infty}^{\infty} u_n e^{in\theta} \in L^2; u_n = 0 \text{ for } n < 0\}.$$

$H^2(\mathbb{T})$ is closed subspace of $L^2(\mathbb{T})$. Let π be the projection on $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$. Namely,

$$\pi\left(\sum_{-\infty}^{\infty} u_n e^{in\theta}\right) = \sum_{0}^{\infty} u_n e^{in\theta}.$$

In this situation, the correspondence between functions on the torus and holomorphic functions in the disk is given by

$$\mathcal{O}(\Omega) \ni \sum_0^\infty u_n z^n \longleftrightarrow \sum_0^\infty u_n e^{in\theta} \in H^2(\mathbb{T}).$$

By the relation $t\partial_t \mapsto D_\theta$ the lifted operator on the torus is given by

$$\hat{p} = \sum_k a_k(e^{i\theta})e^{-ik\theta} D_\theta(D_\theta - 1)\cdots(D_\theta - k + 1),$$

where we used $t^k\partial_t^k = t\partial_t(t\partial_t - 1)\cdots(t\partial_t - k + 1)$. By definition we can easily see that $\pi\hat{p} = \hat{p}$.

For a given equation $Pu = f$ in some neighborhood of the origin we consider $\hat{p}\hat{u} = \hat{f}$ on the torus, where $\hat{f}(\theta) = f(e^{i\theta})$. If we obtain a solution $\hat{u} = \sum_0^\infty u_n e^{in\theta} \in H^2(\mathbb{T})$ of $\hat{p}\hat{u} = \hat{f}$, $u := \sum_0^\infty u_n t^n$ is a holomorphic extension of $\hat{u}$ into $|t| \le 1$. The function $Pu - f$ is holomorphic in the disk $|t| \le 1$, and vanishes on its boundary since $\hat{p}\hat{u} = \hat{f}$. Maximal principle implies that $Pu = f$ in the disk, i.e, u is a solution of a given equation. Clearly, the maximal principle also implies that if the solution on the torus is unique, the analytic solution inside is also unique. Hence it is sufficient to study the solvability of the equation on the torus.

Reduced equation on the torus

Define $\langle D_\theta\rangle$ by the following

$$\langle D_\theta\rangle u := \sum_n u_n\langle n\rangle e^{in\theta},\ \langle n\rangle = (1 + n^2)^{1/2}.$$

This operator also operates on the set of holomorphic functions in the following way

$$\langle t\partial_t\rangle u := (1 + (t\partial/\partial t)^2)^{1/2}u = \sum u_n\langle n\rangle z^n.$$

We can easily see that

$$D_\theta(D_\theta - 1)\cdots(D_\theta - k + 1)\langle D_\theta\rangle^{-k} = Id + K,$$

where K is a compact operator on H^2.

It follows that since $\langle D_\theta\rangle^{-m}$ is an invertible operator we may consider $\hat{p}\langle D_\theta\rangle^{-m}$ instead of $\hat{p}$. Note that $\hat{p}\langle D_\theta\rangle^{-m} = \pi\hat{p}\langle D_\theta\rangle^{-m}$, and the principal part of $\hat{p}\langle D_\theta\rangle^{-m}$ is $a_m(e^{i\theta})e^{-im\theta}$. Hence, modulo compact operators we are lead to the following operator

$$(*) \qquad \pi a_m(e^{i\theta})e^{-im\theta} : H^2 \mapsto H^2.$$

Indeed, the part with order $< m$ is a compact operator if $\langle D_\theta\rangle^{-m}$ is multiplied.

The last operator contains no differentiation, and the coefficients are smooth. It should be noted that although $a_m(t)$ vanishes at $t = 0$, $a_m(e^{i\theta})$ does not vanish on the torus.

DEFINITION 2. *We call the operator* $(*)$ *on* $H^2(\mathbb{T})$ *a Toeplitz operator. The function* $a_m(e^{i\theta})$ *is called the symbol of a Toeplitz operator.*

5. Riemann-Hilbert problem and solvability

DEFINITION 3. *A rational function $p(z) := a(z)z^{-m}$ is said to be Riemann-Hilbert factorizable with respect to $|z| = 1$ if the following factorization*

$$p(z) = p_-(z)p_+(z),$$

holds, where $p_+(z)$, being holomorphic in $|z| < 1$ and continuous up to the boundary, does not vanish in $|z| \leq 1$, and $p_-(z)$, being holomorphic in $|z| > 1$ and continuous up to the boundary, does not vanish in $|z| \geq 1$.

The factorizability is equivalent to saying that the R–H problem for the jump function p and the circle has a solution.

EXAMPLE 5. We consider $p(z) := a(z)z^{-m}$ $(a(0) \neq 0)$ $(m \geq 1)$. Let $a(z)$ be a polynomial of order $m + n$ $(n \geq 1)$. Then we have

$$\begin{aligned} p(z) &= c(z-\lambda_1)\cdots(z-\lambda_m)(z-\lambda_{m+1})\cdots(z-\lambda_{m+n})z^{-m} \\ &= c(1-\frac{\lambda_1}{z})\cdots(1-\frac{\lambda_m}{z})(z-\lambda_{m+1})\cdots(z-\lambda_{m+n}), \end{aligned}$$

where $\lambda_j \in \mathbb{C}$. We can easily see that p is Riemann-Hilbert factorizable with respect to the unit circle if and only if

$$(RH) \qquad |\lambda_1| \leq \cdots \leq |\lambda_m| < 1 < |\lambda_{m+1}| \leq \cdots \leq |\lambda_{m+n}|.$$

THEOREM 2. *Suppose that (RH) is satisfied. Then the kernel and the cokernel of the map $(*)$ vanishes.*

Proof. We consider the kernel of $(*)$. By definition, $\pi p u = 0$ is equivalent to

$$p(e^{i\theta})u(e^{i\theta}) = g(e^{i\theta}),$$

where g consists of negative powers of $e^{i\theta}$. If $|\lambda_j| < 1$ the series $(1 - \lambda_j e^{-i\theta})^{-1}$ consists of only negative powers of $e^{i\theta}$. Hence, if $(1 - \lambda_j e^{-i\theta})U(e^{i\theta}) = F(e^{i\theta})$ for some F consisting of negative powers it follows that $U(e^{i\theta}) = (1 - \lambda_j e^{-i\theta})^{-1}F(e^{i\theta})$ consists of negative powers. By repeating this argument we see that

$$(z - \lambda_{m+1})\cdots(z - \lambda_{m+n})u(z), \quad z = e^{i\theta}$$

consists of only negative powers. On the other hand, since this is a polynomial of z we obtain $u = 0$.

Next we study the cokernel. Let $f \in H^2(\mathbb{T})$ be given. For the sake of simplicity we want to solve

$$\left(1 - \lambda_1 e^{-i\theta}\right)\left(e^{i\theta} - \lambda_2\right)u(e^{i\theta}) \equiv f(e^{i\theta}) \quad \text{modulo negative powers,}$$

where $|\lambda_1| < 1 < |\lambda_2|$. Hence we have

$$\left(e^{i\theta} - \lambda_2\right) u(e^{i\theta}) \equiv \left(1 - \lambda_1 e^{-i\theta}\right)^{-1} f = f_+ + f_- \equiv f_+$$

modulo negative powers. Here f_+ (resp. f_-) consists of Fourier coefficients of non-negative (resp. negative) part. Hence, we have

$$\left(e^{i\theta} - \lambda_2\right) u(e^{i\theta}) = f_+.$$

The solution is given by $u(e^{i\theta}) = (e^{i\theta} - \lambda_2)^{-1} f_+$. Hence the cokernel vanishes. This ends the proof.

□

6. Index formula of an ordinary differential operator

We will give an elementary proof of an index formula. (Cf. Malgrange, Komatsu, Ramis). Let $\Omega \subset C$ be a bounded domain satisfying the following condition.

(A.1) There exists a conformal map $\psi : D_w = \{|z| < w\} \mapsto \Omega$ such that ψ can be extented in some neighborhood of $\overline{D_w} = \{|z| \le w\}$ holomorphically.

Let $w > 0$, $\mu \ge 0$, and define

$$G_w(\mu) = \{u = \sum_n u_n x^n;\ \|u\|^2 := \sum_n (|u_n| \frac{w^n n!}{(n-\mu)!})^2 < \infty\},$$

where $(n-\mu)! = 1$ if $n - \mu \le 0$. Clearly, $G_w(\mu)$ is a Hilbert space. Define $\mathcal{A}_w(\mu)$ as the totality of holomorphic functions $u(x)$ on Ω such that $u(\psi(z)) \in G_w(\mu)$.

Consider an $N \times N$ ($N \ge 1$) matrix-valued differential operator

$$P(x, \partial_x) = (p_{ij}(x, \partial_x)),$$

where p_{ij} is holomorphic ordinary differential operator on $\overline{\Omega}$. For simplicity, we assume that there exist real numbers ν_i, μ_j $(i, j = 1, \dots, N)$ such that

$$ord\ p_{ij} \le \mu_j - \nu_i, \quad ord\ p_{ii} = \mu_i - \nu_i.$$

Hence

$$P(x, \partial_x) : \prod_{j=1}^{N} \mathcal{A}_w(-\mu_j) \longrightarrow \prod_{j=1}^{N} \mathcal{A}_w(-\nu_j). \tag{1}$$

If we write

$$p_{ij}(x, \partial_x) = \sum_{k=0}^{\mu_j - \nu_i} a_k(x) \partial_x^k, \quad a_k(x) \in \mathcal{O}(\overline{\Omega})$$

we obtain, by the substitution $x = \psi(z)$

$$\tilde{p}_{ij}(z, \partial_z) = \sum_{k=\mu_j-\nu_i} a_k(\psi(z))\psi'(z)^{-k}\partial_z^k + \cdots .$$

Here the dots denotes terms of order $< \mu_j - \nu_i$, which are compact operators.

Define a Toeplitz symbol $Q^\Omega(z)$ by $Q^\Omega(z) := (q_{ij}^\Omega(z))$. Here

$$q_{ij}^\Omega(z) = a_{\mu_j-\nu_i}(\psi(z))(z\psi'(z))^{\nu_i-\mu_j}. \tag{2}$$

Then we have

THEOREM 3. *Suppose* (A.1). *Then the map* (1) *is a Fredholm operator if and only if*

$$\det Q^\Omega(z) \neq 0 \qquad \textit{for } \forall z \in \mathbb{C}, |z| = w. \tag{3}$$

If (3) *holds the Fredholm index of (1),* $\chi(:= dim_{\mathbb{C}} \operatorname{Ker} P - codim_{\mathbb{C}} \operatorname{Im} P)$ *is given by the following formula*

$$-\chi = \frac{1}{2\pi}\oint_{|z|=w} d(\log\det Q^\Omega(z)), \tag{4}$$

where the integral is taken in counterclockwise direction.

Proof. Suppose (3). We want to show the Fredholmness of (1). For the sake of simplicity, we suppose that $\mu_j - \nu_i = m$, i.e., $ord\ p_{ij} = m$. If we lift P onto the torus and we multiply the lifted operator on torus with $\langle D_\theta\rangle^{-m}$ we obtain an operator πQ^Ω on H^2 modulo compact operators. It is easy to show that πQ^Ω on H^2 is a Fredholm operator. (cf. [3]). Because the difference of these operators are compact operators the lifted operator is a Fredholm operator.

In order to see the Fredholmness of (1) we note that the kernel of the operator on the boundary coincides with that of the operator inside (under trivial analytic extension) because of a maximal principle. The same property holds for a cokernel. Therefore the Fredholmness of the lifted operator implies the Fredholmness of (1).

Conversely, assume that (1) is a Fredholm operator. We want to show (3). By the argument in the above we may assume that the operator πQ^Ω on H^2 is a Fredholm operator. For the sake of simplicity, we prove in the case $N = 1$, a single case.

We denote πQ^Ω by T. Let K be a finite dimensional projection onto $Ker\ T$. Then there exists a constant $c > 0$ such that

$$\|Tf\| + \|Kf\| \geq c\|f\|, \qquad \forall f \in H^2.$$

It follows that

$$\|\pi Q^\Omega \pi g\| + \|\pi K \pi g\| + c\|(1-\pi)g\| \geq c\|g\|, \qquad \forall g \in L^2.$$

Let U be a multiplication operator by $e^{i\theta}$. Then we have

$$\|\pi Q^{\Omega}\pi U^n g\| + \|\pi K\pi U^n g\| + c\|(1-\pi)U^n g\| \geq c\|U^n g\|, \qquad \forall g \in L^2.$$

Because U preserves the distance we have

$$\|U^{-n}\pi Q^{\Omega}\pi U^n g\| + \|\pi K\pi U^n g\| + c\|U^{-n}(1-\pi)U^n g\| \geq c\|g\|, \qquad \forall g \in L^2.$$

The operator $U^{-n}\pi U^n$ is strongly bounded in L^2 uniformly in n. We have

$$U^{-n}\pi U^n g \to g$$

strongly in L^2 for every trigonometric polynomial g. Therefore it follows that $U^{-n}\pi U^n g \to g$ strongly in L^2. Thus $U^{-n}(1-\pi)U^n g$ converges to 0 strongly, and

$$U^{-n}\pi Q^{\Omega}\pi U^n g = U^{-n}\pi U^n Q^{\Omega} U^{-n}\pi U^n g \to Q^{\Omega}$$

in the strong sense. On the other hand, because U^n converges to 0 weakly $\pi K\pi U^n g$ tends to 0 strongly by the compactness of K. It follows that

$$\|Q^{\Omega} g\| \geq c\|g\|$$

for every $g \in L^2$. If Q^{Ω} vanishes at some point t_0, there exists g with support in some neighborhood of t_0 with norm equal to 1. This contradicts the above inequality. Hence we have proved the assertion.

Next we will show the index formula (4). For the sake of simplicity, we assume that $w = 1$ and $Q^{\Omega}(z)$ is a rational polynomial of z, namely

$$Q^{\Omega}(z) = c(z-\lambda_1)\cdots(z-\lambda_m)(z-\lambda_{m+1})\cdots(z-\lambda_{m+n})z^{-k}.$$

Here

$$|\lambda_1| \leq \cdots \leq |\lambda_m| < 1 < |\lambda_{m+1}| \leq \cdots \leq |\lambda_{m+n}|.$$

We can easily see that the right-hand side of (4) is equal to $m-k$. We will show that the Fredholm index of the operator

$$\pi Q^{\Omega} : H^2 \to H^2$$

is equal to $k-m$. Because $(z-\lambda_{m+1})\cdots(z-\lambda_{m+n})$ does not vanish on the unit disk the multiplication operator with this function is one-to-one on H^2. We may assume that $Q^{\Omega}(z) = (z-\lambda_1)\cdots(z-\lambda_m)z^{-k}$.

We can calculate the kernel and the cokernel of this operator by constructing a recurrence relation. Let us first consider the case $Q^{\Omega}(z) = (z-\lambda)z^{-k}$ ($|\lambda| < 1$). By substituting $u = \sum_{n=0}^{\infty} u_n z^n$ into

$$\pi(z-\lambda)z^{-k}u = 0$$

we obtain
$$(z-\lambda)z^{-k}\sum_{n=0}^{\infty}u_nz^n=\sum_{n=0}^{\infty}(u_{n-1}-\lambda u_n)z^{n-k}\equiv 0,$$
modulo negative powers of z. By comparing the coefficients we obtain the following recurrence relation
$$u_{k-1}-u_k\lambda=0,\quad u_k-\lambda u_{k+1}=0,\ldots$$
Here $u_0,u_1,\ldots u_{k-2}$ are arbitrary. Suppose that $u_{k-1}=c\neq 0$. Then we have
$$u_k=c/\lambda,u_{k+1}=c/\lambda^2,\ldots$$
Because the radius of convergence of the function u constructed from this series is <1, u is not in the kernel. Therefore, the kernel is $k-1$ dimensional.

Next we want to show that the cokernel is trivial, namely the map is surjective. Consider the following equation
$$\pi(z-\lambda)z^{-k}u=f=\sum_{n=0}^{\infty}f_nz^n.$$
By the same arguement as in the above we obtain
$$u_{k-1}-u_k\lambda=f_0,\quad u_k-\lambda u_{k+1}=f_1,\quad u_{k+1}-\lambda u_{k+2}=f_2,\ldots$$
By setting
$$u_0=u_1=\cdots=u_{k-2}=0,$$
we obtain, from the above recurrence relations
$$\begin{aligned}u_{k-1}=\lambda u_k+f_0&=f_0+\lambda f_1+\lambda^2u_{k+1}=f_0+\lambda f_1+\lambda^2f_2+\lambda^3u_{k+2}+\cdots\\&=f_0+\lambda f_1+\lambda^2f_2+\lambda^3f_3+\cdots\end{aligned}$$
The series in the right-hand side converges because $|\lambda|<1$. Similarly we have
$$\begin{aligned}u_k=\lambda u_{k+1}+f_1&=f_1+\lambda f_2+\lambda^2u_{k+2}=f_1+\lambda f_2+\lambda^2f_3+\lambda^3u_{k+3}+\cdots\\&=f_1+\lambda f_2+\lambda^2f_3+\lambda^3f_4+\cdots.\end{aligned}$$
The series also converges. In the same way we can show that u_j $(j=k-1,k,k+1,\ldots)$ can be determined uniquely. Hence the map is surjective. It follows that Ind $=k-1$. This proves the index formula. The general case can be treated in the same way by solving a recurrence relation.

We give an alternative proof of this fact. We recall the following facts.

The operator πz^{-k} has exactly k dimensional kernel given by the basis $1,z,\ldots,z^{k-1}$. The map $\pi(z-\lambda)$ $(|\lambda|<1)$ has one dimensional cokernel. Indeed, the equation $(z-\lambda)\sum u_nz^n=1$ does not have a solution in H^2 because we have

$u_0 = -1/\lambda$, $u_1 = (-1/\lambda)^2$, $u_2 = (-1/\lambda)^3$, ..., which does not converge on the torus. These facts show the index formula for particular symbols.

In order to show the index formula for general symbols we recall the following theorems.

THEOREM 4 (ATKINSON). *If $A : H^2 \to H^2$ and $B : H^2 \to H^2$ are Fredholm operators BA is a Fredholm operator with the index*

$$Ind\, BA = Ind\, B + Ind\, A.$$

THEOREM 5. *For the Toeplitz operators $\pi q : H^2 \to H^2$ and $\pi p : H^2 \to H^2$ the operator $\pi(pq) - (\pi p)(\pi q)$ is a compact operator.*

These theorems show that the index formula for Q^Ω is reduced to the one with symbols given by every factor of the factorization of Q^Ω.

7. Riemann-Hilbert problem - Case of 2 variables

We start with

DEFINITION 4. *A function $a(\theta_1, \theta_2) = \sum_\eta a_\eta e^{i\eta\theta}$ on $T^2 := S \times S$, $S = \{|z| = 1\}$ is Riemann-Hilbert factorizable with respect to T^2 if there exist nonvanishing functions a_{++}, a_{-+}, a_{--}, a_{+-} on T^2 with (Fourier) supports contained repectively in*

$$I := \{\eta_1 \geq 0, \eta_2 \geq 0\}, \quad II := \{\eta_1 \leq 0, \eta_2 \geq 0\},$$

$$III := \{\eta_1 \leq 0, \eta_2 \leq 0\}, \quad IV := \{\eta_1 \geq 0, \eta_2 \leq 0\}$$

such that

$$a(\theta_1, \theta_2) = a_{++}a_{-+}a_{--}a_{+-}.$$

THEOREM 6. *Suppose that the following conditions are verified.*

$$(A.1) \qquad \sigma(z, \xi) \neq 0 \quad \forall z \in \mathbb{T}^2, \forall \xi \in \mathbb{R}^2_+, |\xi| = 1,$$

$$(A.2) \qquad ind_1\, \sigma = ind_2\, \sigma = 0,$$

where

$$ind_1\, \sigma = \frac{1}{2\pi i} \oint_{|\zeta|=1} d_{z_1} \log \sigma(\zeta, z_2, \xi),$$

and $ind_2\, \sigma$ is similarly defined. Then $\sigma(z, \xi)$ is R–H factorizable.

Here the integral is an integer-valued continuous function of z_2 and ξ, which is constant on the connected set $\mathbb{T}^2 \times \{|\xi| = 1\}$. Hence it is constant.

Proof. Suppose that (A1) and (A.2) are verified. Then the function $\log a(\theta)$ is well defined on $\mathbb{T}^2$ and smooth. By Fourier expansion we have

$$\log a(\theta) = b_{++} + b_{-+} + b_{--} + b_{+-}$$

where the supports of $b_{++}, b_{-+}, b_{--}, b_{+-}$ are contained in I, II, III, IV, respectively. The factorization

$$a(\theta) = \exp(b_{++}) \exp(b_{-+}) \exp(b_{--}) \exp(b_{+-})$$

is the desired one. This ends the proof.

□

REMARK 3. The above definition can be extended to a symbol of a pseudodifferential operator $a = a(\theta_1, \theta_2, \xi_1, \xi_2)$. We assume that the factors a_{++}, a_{-+}, a_{--}, a_{+-} are smooth functions of ξ, in addition.

8. Riemann-Hilbert problem and construction of a parametrix

In this section we give a rather concrete construction of a parametrix of an operator reduced on the tori under the R–H factorizability.

Let $L^2(\mathbb{T}^2)$ be a set of square integrable functions, and let us define subspaces H_1, H_2 of $L^2(\mathbb{T}^2)$ by

$$H_1 := \left\{ u \in L^2; u = \sum_{\zeta_1 \geq 0} u_\zeta e^{i\zeta\theta} \right\}, \; H_2 := \left\{ u \in L^2; u = \sum_{\zeta_2 \geq 0} u_\zeta e^{i\zeta\theta} \right\}.$$

We note that $H^2(\mathbb{T}^2) = H_1 \cap H_2$. We define the projections π_1 and π_2 by

$$\pi_1 : L^2(\mathbb{T}^2) \longrightarrow H_1, \quad \pi_2 : L^2(\mathbb{T}^2) \longrightarrow H_2.$$

Then the projection $\pi : L^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2)$ is, by definition, equal to $\pi_1\pi_2$. We define a Toeplitz operator $T_{+\cdot}$ and $T_{\cdot+}$ by

$$T_{+\cdot} := \pi_1 a(\theta, D) : H_1 \longrightarrow H_1, \; T_{\cdot+} := \pi_2 a(\theta, D) : H_2 \longrightarrow H_2.$$

If the Toeplitz symbols of these operators are Riemann-Hilbert factorizable it follows that $T_{+\cdot}$ and $T_{\cdot+}$ are invertible modulo compact operators, and their inverses (modulo compact operators) are given by

$$T_{+\cdot}^{-1} = \pi_1 a_{++}^{-1} a_{+-}^{-1} \pi_1 a_{-+}^{-1} a_{--}^{-1} \pi_1, \; T_{\cdot+}^{-1} = \pi_2 a_{++}^{-1} a_{-+}^{-1} \pi_2 a_{+-}^{-1} a_{--}^{-1} \pi_2, \tag{5}$$

where the equality means the one modulo compact operators.

THEOREM 7. *Let $a(\theta, D)$ be a pseudodifferential operator on the torus. Suppose that $a(\theta, D)$ is R–H factorizable. Then the parametrix R of $\pi a(\theta, D)$ is given by*

$$R = \pi(T_{+\cdot}^{-1} + T_{\cdot+}^{-1} - a(\theta, D)^{-1}), \tag{6}$$

where $a(\theta, D)^{-1}$ is a pseudodifferential operator with symbol given by $a(\theta, \xi)^{-1}$.

These facts are essentially proved in [9] under slightly different situation. We give the proof for the reader's convenience. In the following $A \equiv B$ means that A and B are equal modulo compact operators.

Proof of (5). By comparing the principal symbol of both sides we obtain $a(\theta, D) \equiv a_{++}a_{-+}a_{--}a_{+-}$.

$$T_{+\cdot}\pi_1 a_{++}^{-1}a_{+-}^{-1}\pi_1 a_{-+}^{-1}a_{--}^{-1}\pi_1 \equiv \pi_1 a_{++}a_{-+}a_{--}a_{+-}\pi_1 a_{++}^{-1}a_{+-}^{-1}\pi_1 a_{-+}^{-1}a_{--}^{-1}\pi_1$$

$$\equiv \pi_1 a_{-+}a_{--}a_{++}a_{+-}a_{++}^{-1}a_{+-}^{-1}\pi_1 a_{-+}^{-1}a_{--}^{-1}\pi_1$$

$$+\pi_1 a_{-+}a_{--}a_{++}a_{+-}(I-\pi_1)a_{++}^{-1}a_{+-}^{-1}\pi_1 a_{-+}^{-1}a_{--}^{-1}\pi_1 \equiv \pi_1 a_{-+}a_{--}\pi_1 a_{-+}^{-1}a_{--}^{-1}\pi_1,$$

where we used

$$(I-\pi_1)a_{++}^{-1}a_{+-}^{-1}\pi_1 = 0.$$

Therefore, the right-hand side is equal to

$$\pi_1 a_{-+}a_{--}a_{-+}^{-1}a_{--}^{-1}\pi_1 + \pi_1 a_{-+}a_{--}(I-\pi_1)a_{-+}^{-1}a_{-+}^{-1}\pi_1$$

and hence $\equiv \pi_1$. Here we used $\pi_1 a_{-+}a_{--}(I-\pi_1) = 0$. Similarly, we can show

$$\pi_1 a_{++}^{-1}a_{+-}^{-1}\pi_1 a_{-+}^{-1}a_{--}^{-1}\pi_1 T_{+\cdot} \equiv \pi_1.$$

This ends the proof. □

Proof of (6). Noting that $\pi = \pi_1\pi_2$ we have

$$\pi T_{+\cdot}^{-1}\pi a\pi = \pi T_{+\cdot}^{-1}\pi_1\pi_2 a\pi = \pi T_{+\cdot}^{-1}\pi_1 a\pi - \pi T_{+\cdot}^{-1}\pi_1(I-\pi_2)a\pi$$

$$\equiv \pi - \pi a_{++}^{-1}a_{+-}^{-1}\pi_1 a_{-+}^{-1}a_{--}^{-1}\pi_1(I-\pi_2)a\pi$$

$$= \pi - \pi a_{++}^{-1}a_{+-}^{-1}(\pi_1\pi_2 + \pi_1(I-\pi_2))a_{-+}^{-1}a_{--}^{-1}\pi_1(I-\pi_2)a\pi.$$

Similarly, we have

$$\pi T_{\cdot+}^{-1}\pi a\pi = \pi T_{\cdot+}^{-1}\pi_1\pi_2 a\pi = \pi T_{\cdot+}^{-1}\pi_2 a\pi - \pi T_{\cdot+}^{-1}\pi_2(I-\pi_1)a\pi$$

$$\equiv \pi - \pi a_{++}^{-1}a_{-+}^{-1}\pi_2 a_{+-}^{-1}a_{--}^{-1}\pi_2(I-\pi_1)a\pi$$

$$= \pi - \pi a_{++}^{-1}a_{-+}^{-1}(\pi_1\pi_2 + \pi_2(I-\pi_1))a_{+-}^{-1}a_{--}^{-1}\pi_2(I-\pi_1)a\pi.$$

On the othe hand, since $a^{-1}a \equiv I$ we have

$$-\pi a^{-1}\pi a\pi = -\pi a^{-1}\pi_1\pi_2 a\pi \equiv -\pi - \pi a^{-1}(\pi_1\pi_2 - I)a\pi.$$

By using

$$\pi_1\pi_2 - I = \pi_1(\pi_2 - I) + (\pi_1 - I)\pi_2 - (\pi_1 - I)(\pi_2 - I)$$

we have

$$-\pi a^{-1}\pi a\pi \equiv -\pi_1\pi_2 - \pi_1\pi_2 a^{-1}\pi_1(\pi_2 - I)a\pi$$
$$-\pi a^{-1}(\pi_1 - I)\pi_2 a\pi + \pi a^{-1}(\pi_1 - I)(\pi_2 - I)a\pi.$$

Combining these relations

$$RT \equiv \pi - \pi a_{++}^{-1}a_{+-}^{-1}(\pi + \pi_1(I - \pi_2))a_{-+}^{-1}a_{--}^{-1}\pi_1(I - \pi_2)a\pi$$
$$-\pi a_{++}^{-1}a_{-+}^{-1}(\pi + \pi_2(I - \pi_1))a_{+-}^{-1}a_{--}^{-1}\pi_2(I - \pi_1)a\pi$$
$$-\pi a^{-1}\pi_1(\pi_2 - I)a\pi - \pi a^{-1}(\pi_1 - I)\pi_2 a\pi + \pi a^{-1}(\pi_1 - I)(\pi_2 - I)a\pi.$$

We note

$$\pi + \pi_1(I - \pi_2) = I - (\pi_1 - I)(\pi_2 - I) - \pi_2(I - \pi_1),$$
$$\pi + \pi_2(I - \pi_1) = I - (\pi_1 - I)(\pi_2 - I) - \pi_1(I - \pi_2).$$

It follows that

$$RT - \pi \equiv \pi a_{++}^{-1}a_{+-}^{-1}((\pi_1 - I)(\pi_2 - I)$$
$$+\pi_2(I - \pi_1))a_{-+}^{-1}a_{--}^{-1}\pi_1(I - \pi_2)a\pi + \pi a^{-1}(\pi_1 - I)(\pi_2 - I)a\pi$$
$$+\pi a_{++}^{-1}a_{-+}^{-1}\left((\pi_1 - I)(\pi_2 - I) + \pi_1(I - \pi_2)\right)a_{+-}^{-1}a_{--}^{-1}\pi_2(I - \pi_1)a\pi.$$

In order to show that the right-hand side operators are compact operators we will show that the operators

$$\pi\varphi(\pi_1 - I)(\pi_2 - I),\ \pi_2(I - \pi_1)\varphi\pi_1(I - \pi_2),\ \pi_1(I - \pi_2)\varphi\pi_2(I - \pi_1)$$

are compact. Here φ is an appropriately chosen smooth function. In order to show this let

$$u = \sum_\alpha u_\alpha e^{i\alpha\theta} \in L^2,\ \varphi(\xi) = \sum_\beta \varphi_\beta(\xi)e^{i\beta\theta}$$

be the Fourier expansion of $u \in L^2$ and $\varphi \in C^\infty$, respectively. Because $\varphi(\theta, D)$ is order zero pseudodifferential operator the Fourier coefficients of $\varphi_\beta(\xi)$ is rapidly decreasing in ξ when $|\beta| \to \infty$. Therefore

$$\pi\varphi(\pi_1 - I)(\pi_2 - I)u = \sum_{\mu=\alpha+\beta\in I}\left(\sum_{\alpha+\beta=\mu,\alpha\in III}\varphi_\beta(\mu)u_\alpha\right)e^{i\mu\theta}.$$

Because $\mu \in I$ and $-\alpha \in I$ by the definition of I and III, β satisfies that $|\beta| = |\mu - \alpha| \geq |\mu|$. It follows that, for all $n \geq 1$ and μ

$$|\mu|^n \sum_{\alpha+\beta=\mu,\alpha\in III}|\varphi_\beta(\mu)||u_\alpha| \leq \sum|\beta|^n|\varphi_\beta(\mu)||u_\alpha| < \infty.$$

Indeed, $|\varphi_\beta(\mu)||\beta|^n$ is bounded in μ and β. It follows that the Fourier coefficients converge uniformly in $u \in L^2$. Thus $\pi\varphi(\pi_1 - I)(\pi_2 - I)$ is a compact operator. The compactness of other operators are proved similarly. Hence R is a left regularizer. We can similarly show that R is a right regularizer. This ends the proof.

□

9. Solvability in two dimensional case

Let f be a formal power series, and $k = \operatorname{ord} f$ be the order of f, namely the least degree of monomials which constitute f. Hence it follows that $\partial_x^\alpha f(0) \neq 0$ for some $|\alpha| = k$ and $\partial_x^\beta f(0) = 0$ for all $|\beta| \leq k-1$. For a polynomial u_0 of $\operatorname{ord} u_0 = 4$ we define $f_0 = M(u_0)$. Then we have

THEOREM 8. *Let $n = 2$. Suppose that (A.1) and (A.2) are verified. Then there exist $r > 0$ and an integer $N \geq 4$ depending only on u_0 and the equation such that, for every $g \in W_R$ satisfying $\|g\|_R < r$, $ord\, g \geq N$ the equation* (MA)

$$(MA) \qquad M(v + u_0) := \det(v_{x_i x_j} + (u_0)_{x_i x_j}) = f_0(x) + g(x) \quad in\ \Omega,$$

has a unique solution $v \in W_R$ such that $ord\, v \geq N$.

REMARK 4. The conditions (A.1) and (A.2) are invariant if we replace R with $R\rho$ $(0 < \rho < 1)$. By taking R small, if necessary, we may assume $\|g\|_R < r$. Hence the solution exists in some neighborhood of the origin.

Proof. We linearize M

$$M(u_0 + v) = M(u_0) + \pi P v + R(v),$$

where $R(v)$ is a remainder. It follows that

$$(*) \qquad \pi P v + R(v) = g \quad \text{on} \quad W_R(\mathbb{T}^n).$$

By the argument in the preceeding section there exists a parametrix S of πP. Indeed, we have $S\pi P = \pi + R$, where R is an operator of negative order. It follows that the norm of R on the subspace of W_R with order greater than N can be made arbitrarily small if N is sufficiently large. It follows that $S\pi P = \pi + R$ is invertible on the subspace of W_R with order greater than N for sufficiently large N. Therefore if N is sufficiently large and if the order of g is greater than N we can solve $(*)$ by a standard iteration. Hence, if $\|g\|_R$ is sufficiently small $(*)$ has a unique solution v.

Let $\hat{v}$ be an analytic extension of v to D_R. The function

$$M(u_0 + \hat{v}) - f_0 - g$$

is holomorphic in D_R, and vanishes on the Silov boundary of D_R. By the maximal principle, we have

$$M(u_0 + \hat{v}) = f_0 + g \quad \text{in } D_R.$$

Hence we have the solvability.

Uniqueness. Suppose that there exist two solutions w_1 and w_2 to (MA) such that $\|w_j\| \leq \varepsilon$ for small ε. We blow up the equation to $\mathbb{T}^n$. By the uniqueness of the operator on the boundary we have $w_1 = w_2$ on $\mathbb{T}^n$. By the maximal principle we have $w_1 = w_2$ in D_R.

□

We consider two examples in Section 2. We use the same notations as in Section 2.

EXAMPLE 6. The condition (A.1) reads

$$2 + (k - 8)\eta_1\eta_2 \neq 0 \ \text{ for all } \eta \in \mathbb{R}^2_+, \ |\eta| = 1.$$

This is equivalent to $k > 4$. We can easily see that (A.2) holds if $k > 4$. The condition is weaker than the ellipticity condition in Section 2 because we work on a Hardy space. The same is true in the next example.

EXAMPLE 7. By the same argument as before we can verify that (A.1) is equivalent to $k < -6$ or $k > 8$. We can easily verify (A.2) for $\xi = (0, 1)$ under these conditions.

Convergence of all formal power series solutions We give an application of Theorem 8. Kashiwara-Kawai-Sjöstrand ([5]) gave a subclass of linear Grushin operators for which all formal power series solutions converge. Here we give a class of nonlinear operators for which all formal power series solutions converge.

THEOREM 9. *Assume* (A.1) *and* (A.2). *Then, for every g holomorphic in some neighborhood of the origin such that* $\operatorname{ord} g > 4$ *all formal power series solutions of* (MA) *of the form* $u = u_0 + w$, $\operatorname{ord} w > 4$ *converge in some neighborhood of the origin.*

Proof. Let $w = \sum_{j=5}^{\infty} w_j$ be any formal solution of (MA) for $\operatorname{ord} g \geq 5$, where w_j is a polynomial of homogenous degree j. Let k be an integer determined later, and set $w = w_0 + U$, where $w_0 = \sum_{j=5}^{k} w_j$, $\operatorname{ord} U \geq k + 1$. Determine h by $M(u_0 + w_0) = f_0 + h$, and write the equation in the form

$$M(u_0 + w_0 + U) = f_0 + h + g - h.$$

The order of $g - h$ can be made arbitrarily large if k is sufficiently large. It follows from Theorem 8 that, if k is sufficiently large the formal power series solution U is uniquely determined by $g - h$. The condition (A.1) and (A.2) are invariant if we replace u_0 with $u_0 + w_0$. By Remark 4 the above equation has a unique analytic solution. By the uniqueness of a formal solution U converges.

□

10. Solvability in general independent variables

For a given $u_0(x)$ holomorphic in some neighborhood of the origin such that $\operatorname{ord} u_0 = 4$ we set $f_0(x) = M(u_0) := \det((u_0)_{x_i x_j})$. For an analytic $g(x)$ ($\operatorname{ord} g \geq 5$) we study the equation

$$(MA) \qquad M(u_0 + v) = f_0(x) + g(x).$$

By the argument in Section 2 M may be of mixed type at $u = u_0$, while its blow up onto the torus is elliptic. If we can construct a parametrix of the reduced operator on the torus of the linearized operator of (MA) the argument in the case of two independent variables can be applied to the case of general independent variables. Therefore, in order to show the solvability we construct a parametrix.

Let P be the linearized operator of $M(u)$ at $u = u_0$

$$P := M_{u_0} = \sum_{|\alpha|\le m} (\partial M/\partial z_\alpha)(x, u_0)\partial_x^\alpha \equiv \sum_{\alpha, |\alpha|\le m} a_\alpha(x)\partial_x^\alpha,$$

where $m \in \mathbf{N}$, and $a_\alpha(x)$ is holomorphic in some neighborhood of the origin. We define the symbol $\sigma(z, \xi)$ of the reduced operator on tori by

$$\sigma(z, \xi) := \sum_{|\alpha|\le m} a_\alpha(z) z^{-\alpha} p_\alpha(\xi)\langle\xi\rangle^{-m},$$

where $z_j = R_j e^{i\theta_j}$, $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ and $p_\alpha(\xi) = \prod_{j=1}^n \xi_j(\xi_j - 1)\cdots(\xi_j - \alpha_j + 1)$.

REMARK 5. By elementary calculations we can show that

$$\sigma(z, \xi)\langle\xi\rangle^m = (z_1\cdots z_n)^{-2} \det\Big(\xi_j\xi_k + z_j z_k u^0_{x_j x_k}(z)\Big) \; - f_0(z).$$

We will not use the concrete expression in the following argument.

We decompose $\sigma(z, \xi)$ as follows

$$\sigma(z, \xi) = \sigma'(\xi) + \sigma''(z, \xi),$$

where $\sigma'(\xi) = \int_{\mathbb{T}^n} \sigma(Re^{i\theta}, \xi)d\theta$ is the average over $\mathbb{T}^n$. We assume

(B.1) there exist constant $c \in \mathbb{C}$, $|c| = 1$ and $K > 0$ such that

$$Re\, c\sigma'(\xi) \ge K > 0 \quad \text{for all} \quad \forall\xi \in \mathbb{Z}^n_+.$$

Then we have

THEOREM 10. *Assume* (B.1). *Then there exists K_0 such that for every $K \ge K_0$ the reduced operator of P on $\mathbb{T}^n$ has a parametrix on $W_R(\mathbb{T}^n)$.*

Proof. We lift the operator $P < D_x >^{-m}$ to the torus. Its symbol is given by $\sigma(z, \xi)$. We have, for $u \in W_R(\mathbb{T}^n)$

$$\|(I - \varepsilon c\pi\sigma)u\|_R = \|\pi(1 - \varepsilon c\sigma)u\|_R \le \|(1 - \varepsilon c\sigma)u\|_{\ell^1_R}.$$

Here we used the boundedness of $\pi : \ell^1_R \to \ell^1_{R,+}$. If we can prove that

$$\|(1 - \varepsilon c\sigma)u\|_{\ell^1_R} < \|u\|_{\ell^1_R} = \|u\|_R$$

we have $\|(I - \varepsilon c\pi\sigma)u\|_R < \|u\|_R$. Thus $\varepsilon c\pi\sigma = I - (I - \varepsilon c\pi\sigma)$ is invertible on $W_R(\mathbb{T}^n)$, and $\pi\sigma$ is invertible. Indeed, it follows from (B.1) that there exists $K_1 > 0$ such that if $K > K_1$ we have

$$Re\, c\sigma(z,\xi) = Re\, c\sigma'(\xi) + Re\, c\sigma''(z,\xi) > K - K_1, \quad \forall z \in \mathbb{T}^n, \forall \xi \in \mathbb{Z}_+^n.$$

Hence, if $\varepsilon > 0$ is sufficiently small we have

$$\|1 - \varepsilon c\sigma(\cdot,\xi)\|_{L^\infty} < 1 - \varepsilon(K - K_1), \quad \forall \xi \in \mathbb{Z}_+^n.$$

From this estimate we can prove the desired estimate (cf. [12]).

□

11. Solvability of a homology equation

We want to linearize an analytic singular vector field at a singular point via coordinate change. The transformation satisfies a so-called homology equation

$$\mathcal{L}u = R(x+u), \quad \mathcal{L} = \sum_{j=1}^{n} \lambda_j x_j \frac{\partial}{\partial x_j},$$

where $x = (x_1, \ldots, x_n)$, and $R(y)$ is an analytic function of y given by the vector field, and λ_j are eigenvalues of the linear part of the vector field. Here we assume that the vector field is semi-simple. We say that the Poincaré condition is satisfied if the convex hull of all λ_j in the complex plane does not contain the origin. Let us apply our arguement to this equation. By a blowing up we obtain a nonlinear equation on $H^2(\mathbb{T}^n)$. Then we have

PROPOSITION 1. *The Poincaré condition holds if and only if the lifted operator of $\mathcal{L}$ to $H^2(\mathbb{T}^n)$ is elliptic.*

Proof. The latter condition reads: $\sum_{j=1}^n \lambda_j \xi_j \neq 0\ \forall \xi \in \mathbb{R}_+^n, |\xi| = 1$. One can easily see that Poincaré condition implies the condition. Conversely, if the ellipticity hols we obtain the Poincaré condition. This ends the proof.

□

We remark that, by the solvability on tori we can prove the so-called Poincaré's theorem.

Next we think of the simultaneous reduction of a system of d vector fields $\{\mathcal{X}^\nu\}_\nu$ whose eigenvalues of the linear parts are given by λ_j^ν $(j = 1, \ldots, n)$ $(\nu = 1, \ldots, d)$. By the same way as before we are lead to the system of equations

$$\mathcal{L}_\mu u = R_\mu(x+u), \quad \mathcal{L}_\mu = \sum_{j=1}^{n} \lambda_j^\mu x_j \frac{\partial}{\partial x_j}, \quad \mu = 1, \ldots, d.$$

Define $\lambda_j := (\lambda_j^1, \dots, \lambda_j^d), \quad j = 1, \dots, n$ and

$$\Gamma := \left\{ \sum_{j=1}^{n} \xi_j \lambda_j ;\, \xi_j \geq 0, \xi_1^2 + \cdots + \xi_n^2 \neq 0 \right\}.$$

We say that a system of vector fields satisfies a simultaneous Poincaré condition if Γ does not contain the origin. Set $\xi = (\xi_1, \dots, \xi_n)$. Then the condition can be written in

$$\forall \xi \in \mathbb{R}_+^n \setminus 0, \quad \exists k \ (1 \leq k \leq d) \text{ such that } \sum_{j=1}^{n} \lambda_j^k \xi_j \neq 0.$$

This is equivalent to saying that the lifted operator on tori is an elliptic system.

12. Analysis of equations containing a large parameter

Let $p(x, \partial_x)$ be a pseudodifferential operator of order m with polynomial coefficients, and let $q(x)$ be a rational function. For a given analytic f we consider the asymptotic behaviour when $\lambda \to \infty$ of the solution u of the equation

$$(p(x, \partial_x) + \lambda^2 q(x))u = f(x). \tag{7}$$

By the substitution $x \mapsto e^{i\theta} = (e^{i\theta_1}, \dots, e^{i\theta_n})$ we obtain an equation on $\mathbb{T}^n$.

$$(p(e^{i\theta}, e^{-i\theta} D_\theta) + \lambda^2 q(e^{i\theta}))u = f(e^{i\theta}).$$

We consider the case $n = 1$. Set $z = e^{i\theta}$ and define

$$\sigma(z, \xi, \lambda) := p(z, z^{-1}\xi) + \lambda^2 q(z).$$

Assume the uniform R-H factorization condition

$$(URH) \qquad \sigma(z, \xi, \lambda) \neq 0 \text{ for } \forall z \in \mathbb{T}, \forall (\xi, \lambda) \in \mathbb{R}_+^2, \xi^2 + \lambda^2 = 1,$$

$$\frac{1}{2\pi i} \int_{|z|=1} d_z \log \sigma(z, \xi, \lambda) = 0 \ \ \exists (\xi, \lambda) \in \mathbb{R}_+^2, \xi^2 + \lambda^2 = 1.$$

Let $\| \cdot \|_s$ be a Sobolev norm. We recall that ord f is the least degree of monomials which constitute f. Then we have

THEOREM 11. *Let $s > 0$, and assume (URH). Then there exists $N \geq 1$ such that for any f satisfying ord $f \geq N$ (7) has a unique solution u. Moreover, there exists $C > 0$ such that the estimate*

$$\|u\|_{s+mq} \leq \lambda^{-2p}(C\|f\|_s + C^{-1}\|u\|_0)$$

holds for all $\lambda > 0$, where $p + q = 1$, $0 \leq p \leq 1$.

Proof. We consider the principal part and we neglect the lower order terms. Write

$$\pi\sigma(z, z^{-1}D_\theta, \lambda) = (D_\theta^m + \lambda^2)\pi(D_\theta^m + \lambda^2)^{-1}\sigma(z, z^{-1}D_\theta, \lambda).$$

Then $\pi(D_\theta^m + \lambda^2)^{-1}\sigma(z, z^{-1}D_\theta, \lambda)$ is uniformly invertible for $\lambda > 0$ by virtue of (URH). The estimate for $D_\theta^m + \lambda^2$ follows from direct computation.

□

We consider the case $n = 2$. We define $\sigma(z, \xi, \lambda)$ $(z \in \mathbb{T}^2)$ as in the above, and we assume

(URH) $\qquad \sigma(z, \xi, \lambda) \neq 0$ for $\forall z \in \mathbb{T}, \forall(\xi, \lambda) \in \mathbb{R}_+^3, |\xi|^2 + \lambda^2 = 1,$

$$\frac{1}{2\pi i}\int_{|z|=1} d_{z_j} \log\sigma(z, \xi, \lambda) = 0, \text{ for } j = 1, 2, \exists(\xi, \lambda) \in \mathbb{R}_+^3, |\xi|^2 + \lambda^2 = 1.$$

Under these conditions the operator $\pi(|D_\theta|^m + \lambda^2)^{-1}\sigma(z, D_\theta, \lambda)$ has a regularizer. Therefore, it can be transformed to $|D_\theta|^m + \lambda^2$ modulo compact operators. By solving the transformed equation via Fourier method we obtain the same estimate as $n = 1$.

REMARK 6. If λ moves in a sector, $\lambda = \rho e^{i\alpha}$ $(\theta_1 \leq \alpha \leq \theta_2)$ we can treat (7) similarly if we replace q with $e^{2i\alpha}q$ in (URH).

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