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DIFFERENTIAL EQUATIONS WITH INDEFINITE WEIGHT: BOUNDARY VALUE PROBLEMS AND QUALITATIVE PROPERTIES OF THE SOLUTIONS

Abstract. We describe the qualitative properties of the solutions of the second order scalar equation $\ddot{x} + q(t)g(x) = 0$, where q is a changing sign function, and consider the problem of existence and multiplicity of solutions which satisfy various different boundary conditions. In particular we outline some difficulties which arise in the use of the shooting approach.

1. Introduction

We discuss the second-order scalar nonlinear ordinary differential equation:

$$\ddot{x} + q(t)g(x) = 0, \tag{1}$$

where:

- $g : \mathbb{R} \to \mathbb{R}$ is continuous (maybe locally Lipschitz continuous on $\mathbb{R}$ or on $\mathbb{R}\setminus\{0\}$) and such that $g(s) \cdot s > 0$ for every $s \neq 0$
- the "weight" $q : \mathbb{R} \to \mathbb{R}$ is continuous (sometimes more stronger regularity assumptions will be needed and, in some applications, like the two point boundary value problem, it will be enough that q is defined in an interval I).

EXAMPLE 1. A simple case of (1) is the nonlinear Hill's equation:

$$\ddot{x} + q(t)|x|^{\gamma-1}x = 0, \qquad \gamma > 0 \tag{2}$$

(recall that the classical Hill's equation is the one with $\gamma = 1$).

The expression "indefinite weight" means that the function q changes sign.

Waltman [86] in a paper of 1965 studied the oscillating solutions of

$$\ddot{x} + q(t)x^{2n+1} = 0 \qquad n \in \mathbb{N},$$

*Lecture notes of the course held by F. Zanolin during the "Third Turin Forthnight on Nonlinear Analysis", Sept. 23 – Oct. 6 2001, Turin, Italy when the weight q is allowed to change sign. Many authors studied the oscillatory properties of equations like (1): Bhatia [16], Bobisud [17], Butler [19, 20], Kiguradze [52], Kwong and Wong [54], Onose [68], Wong [94, 95, 96].

The existence of periodic solutions for a large class of equations (including (2)) was considered by Butler: in the superlinear case ($\gamma > 1$) he found infinitely many "large solutions" [22] while in the sublinear one ($\gamma < 1$) he found infinitely many "small solutions" [23]. In both cases there are periodic solutions with an arbitrarily large number of zeros and Butler's results, which are valid with respect to a quite wide class of nonlinearities, have been improved in the superlinear case by Papini in [69, 70] and in the sublinear case by Bandle Pozio and Tesei [12], for the existence of small solutions, and by Liu and Zanolin [59] for what concerns large solutions.

Recently, many authors considered generalizations of (1) both in the direction of Hamiltonian systems (with respect to the problem of finding periodic or homoclinic solutions) and elliptic partial differential equations with Dirichlet boundary conditions. In particular Hamiltonian systems with changing sign weights were studied by Lassoued [55, 56], Avila and Felmer [10], Antonacci and Magrone [9], Ben Naoum, Troestler and Willem [13], Caldiroli and Montecchiari [25], Fei [37], Ding and Girardi [33], Girardi and Matzeu [41], Le and Schmitt [57], Liu [58], Schmitt and Wang [76], Felmer and Silva [39], Felmer [38], Ambrosetti and Badiale [8], Jiang [50]. On the other hand, the partial differential case was developed by Alama and Del Pino [1], Alama and Tarantello [2, 3], Amann and Lopez-Gómez [7], Badiale and Nabana [11], Berestycki, Capuzzo-Dolcetta and Nirenberg [14, 15], Khanfir and Lassoued [51], Le and Schmitt [57], Ramos, Terracini and Troestler [74]. Equations of the form:

$$\ddot{x} + q(t)x^{2n+1} = m(t)x + h(t),$$

with a changing sign q, were considered by Terracini and Verzini in [85] paired with either Dirichlet or periodic boundary conditions. They applied a suitable version of the Nehari method [67] in order to find solutions of the boundary value problem with prescribed nodal behavior. More precisely, if the domain $[0, T]$ of q is decomposed into the union of consecutive and adjacent closed intervals $I_1^+, I_1^-, I_2^+, I_2^-, \dots, I_k^+$ such that:

$$q \geq 0,\ q \not\equiv 0 \quad \text{in } I_i^+ \qquad \text{and} \qquad q \leq 0,\ q \not\equiv 0 \quad \text{in } I_i^-,$$

then they found k natural numbers $m_1^*, \dots, m_k^*$, one for each interval of positivity I_i^+, in such a way that, for every choice of k natural numbers $m_1, \dots, m_k$, with $m_i \geq m_i^*$ for all $i = 1, \dots, k$, there are two solutions of the boundary value problem which have exactly m_i zeros in I_i^+ and one zero in I_i^-.

An analogous situation was considered in [70, 71, 72] where, via a shooting approach, boundary value problems associated to (1) were studied, with a general nonlinearity g which has to be superlinear at infinity in some sense. In this case, after having arbitrarily chosen the natural numbers $m_i \geq m_i^*$ and a $(k-1)$-tuple $(\delta_1, \dots, \delta_{k-1})$, with $\delta_i \in \{0, 1\}$, we found two solutions with m_i zeros in I_i^+ and δ_i zeros in I_i^-.

On the other hand Capietto, Dambrosio and Papini [26] focused their attention on the existence of globally defined solutions of (1) with prescribed nodal behavior again in the case of g superlinear at infinity and q changing sign. They showed that the Poincaré map associated to (1) exhibits chaotic features.

It is the aim of these lectures to discuss some qualitative properties of the solutions and some difficulties which arise in the use of the shooting approach.

2. The shooting method

Equation (1) can be written as a first order system in the phase plane:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -q(t)g(x) \end{cases} \tag{3}$$

If we assume that the uniqueness for the Cauchy problems for (3) holds, then we denote by $z(t; t_0, p) = (x(t; t_0, p), y(t; t_0, p))$ the solution of (3) with $z(t_0; t_0, p) = p = (x_0, y_0) \in \mathbb{R}^2$. The shooting method is based on the theorem on the continuous dependence of the solutions with respect to the initial data: if $z(t; t_0, p)$ is defined on an interval $[\alpha, \beta] \ni t_0$ for some $t_0 \in \mathbb{R}$ and some $p \in \mathbb{R}^2$, then $z(t; t_0, p_1)$ is defined on $[\alpha, \beta]$ for each p_1 "near" p and we have that $z(\cdot; t_0, p_1) \to z(\cdot; t_0, p)$ uniformly on $[\alpha, \beta]$ as $p_1 \to p$.

Therefore there is a couple of problems if we wish to apply this method for the study of boundary value problems associated to (1) and (3). The first one is about the uniqueness, which is granted whenever q is locally integrable and g is locally Lipschitz continuous: in particular, if g behaves like $|x|^{\gamma-1}x$ near zero and $0 < \gamma < 1$, we might loose the uniqueness at zero.

The second problem is the global existence of the solutions, since the sole continuity of q does not imply that all the maximal solutions of (1) are globally defined, even if q is assumed to be greater than a positive constant, as shown by Coffman and Ullrich in [28]. Indeed they produce a weight $q(t) = 1 + \delta(t)$, with a function $\delta : [0, +\infty[\to \mathbb{R}$ which is positive and continuous, but has unbounded variation in every left neighborhood of some $\hat{t} > 0$, and they show that the equation:

$$\ddot{x} + (1 + \delta(t))x^3 = 0$$

has a solution which starts from $t_0 = 0$ and blows up as t tends to $\hat{t}$ from the left. On the other hand they prove that, if q is positive, continuous and has bounded variation in an interval $[a, b]$, then every solution of:

$$\ddot{x} + q(t)x^{2n+1} = 0$$

has $[a, b]$ as maximal interval of definition. If we consider a positive weight q which is continuously differentiable on $[a, b]$ and a function g such that $g(x) \cdot x > 0$ for $x \neq 0$, it is not difficult to show that the same conclusion holds for (1). Indeed, let us consider a solution x of (1) starting from $t = a$ and define the auxiliary function:

$$v(t) = \frac{1}{2}\dot{x}^2(t) + q(t)G(x(t)),$$

where $G(x) = \int_0^x g(s)ds$ is nonnegative by the sign assumption on g. The function v is surely defined in an interval $J \subseteq [a, b]$ with a as left end-point. For every t in J we have:

$$\begin{aligned}\dot{v}(t) &= \ddot{x}(t)\dot{x}(t) + q(t)g(x(t))\dot{x}(t) + \dot{q}(t)G(x(t)) \\ &= \dot{q}(t)G(x(t)).\end{aligned}$$

Since q is strictly positive and $\dot{q}$ is continuous on $[a, b]$ there is a constant $M \geq 1$ such that:

$$\dot{q}(t) \leq Mq(t) \qquad \forall\, t \in [a, b],$$

so that we obtain:

$$\dot{v}(t) \leq Mq(t)G(x(t)) \leq Mv(t) \qquad \forall\, t \in J.$$

Hence v satisfies the inequality:

$$v(t) \leq Me^{M(t-a)} \qquad \forall\, t \in J$$

and turns out to be bounded in J. This implies that $|\dot{x}(t)|$ and, therefore, $|x(t)|$ are bounded, too, and, thus, x must be defined up to b.

The argument just employed can be modified in order to cover also some cases in which q is nonnegative and vanishes somewhere. Indeed Butler observed that if one starts from $t = a$ then the solution is defined up to (and including) the first zero $t_0 > a$ of q provided that $\dot{q} \leq 0$ (or, more generally, q is decreasing) in a left neighborhood of t_0. Then the solution surely proceeds further t_0 simply by Peano's theorem about local existence. Similarly, if one looks for backward continuability, every solution starting from $t = b$ reaches the first zero $t_1 < b$ of q provided that q is monotone increasing in a right neighborhood of t_1. Therefore, if every interval $[a, b]$ in which q is nonnegative can be expressed as the union of a finite number of closed intervals (possibly degenerating to a single point) where q vanishes and of a finite number of open sub-intervals $]t_0, t_1[$, such that q is strictly positive in such intervals and is monotone increasing in a right neighborhood of t_0 and decreasing in a left neighborhood of t_1, then the argument above can be repeated a finite number of times in order to obtain the continuability of the solutions across $[a, b]$.

EXAMPLE 2. Let us see how the shooting method can be used to solve a Dirichlet boundary value problem associated to a superlinear Hill's equation like (2) with a nonnegative weight. To be precise we look for solutions of:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -q(t)|x|^{\gamma-1}x \end{cases} \qquad t \in [0, T]$$
$$x(0) = x(T) = 0$$

assuming that $\gamma > 1$ and that q is a nonnegative continuous function in $[0, T]$ which also satisfies the regularity assumptions discussed above in such a way that all the solutions of the differential equation are continuable along the interval $[0, T]$.

The idea is to consider all the solutions which have value 0 and slope $k \in \mathbb{R}$ for $t = 0$, that is $z(\cdot; 0, (0, k))$ with the notation previously introduced, and to determine for which values of k we have $x(T; 0, (0, k)) = 0$. In other words, we are considering the set of the solutions which start at $t = 0$ from the y-axis of the phase plane and we wish to select those which come back to the y-axis at $t = T$. One way to do this is to measure the angle spanned in the phase plane by the solution vector $z(t; 0, p)$ as t runs in $[0, T]$; indeed, if p lies on the y-axis, then $z(T; 0, p)$ is again on the x-axis if and only if the angle spanned by $z(t; 0, p)$ on $[0, T]$ is an integer multiple of π. Now, if $z(t) = (x(t), y(t))$ is a nontrivial solution of the differential equation, then $z(t) \neq (0, 0)$ for every $t \in [0, 1]$ by the uniqueness of the constant solution $(0, 0)$; hence we can define an angular function $\theta(t)$ such that:

$$x(t) = |z(t)| \cos\theta(t) \qquad \text{and} \qquad y(t) = |z(t)| \sin\theta(t)$$

and it is easy to see that it satisfies:

$$-\dot{\theta}(t) = \frac{y^2(t) + q(t)|x(t)|^{\gamma+1}}{y^2(t) + x^2(t)}.$$

Therefore the measure of the angle spanned by $z(t)$ can be obtained by integrating the last expression and it is given by:

$$\operatorname{rot}(p) = \frac{1}{\pi} \int_0^T \frac{y^2(t; 0, p) + q(t)|x(t; 0, p)|^{\gamma+1}}{y^2(t; 0, p) + x^2(t; 0, p)} dt.$$

Thus $z(\cdot; 0, (0, k))$ is a solution of the Dirichlet boundary value problem if and only if $\operatorname{rot}((0, k)) \in \mathbb{Z}$. Now, $\operatorname{rot}(p)$ is clearly a continuous function of p and in this case it can be proved that:

$$\operatorname{rot}(p) \to +\infty \qquad \text{as} \qquad |p| \to +\infty,$$

therefore, by the intermediate values theorem, our boundary value problem has infinitely many solutions. Moreover, the value $\operatorname{rot}(p)$ clearly gives information about how many times the curve $z(t; 0, p)$ crosses the y-axis in the phase plane as t runs from 0 to T and, more precisely, we have that if $\operatorname{rot}((0, k)) = j \in \mathbb{N}$ then $x(\cdot; 0, (0, k))$ has exactly j zeros in $[0, T[$.

The same technique can be used to solve Sturm–Liouville boundary value problems like the following one:

$$\begin{cases} \begin{cases} \dot{x} = y \\ \dot{y} = -q(t)|x|^{\gamma-1}x \end{cases} & t \in [0, T] \\ a_1 x(0) + b_1 y(0) = 0 \\ a_2 x(T) + b_2 y(T) = 0 \end{cases}$$

where $a_i^2 + b_i^2 \neq 0$, $i = 1, 2$, since the boundary conditions just mean that one looks for solutions which start at $t = 0$ on the straight line $a_1 x + b_1 y = 0$ in the phase plane and end at $t = T$ on the straight line $a_2 x + b_2 y = 0$.

On the other hand other boundary value problems, like the periodic one, are more difficult to be solved by the shooting method, as one has to use less trivial fixed points theorems.

What about if we do not have the continuability of the solutions? Hartman [44] avoids the use of the global continuability for an equation of the form:

$$\ddot{x} + f(t,x) = 0 \qquad \text{with} \qquad \lim_{x\to\pm\infty} \frac{f(t,x)}{x} = +\infty \text{ uniformly w.r.t. } t,$$

by assuming that:

$$f(t,0) \equiv 0 \qquad \text{and that} \qquad \frac{f(t,x)}{x} \text{ is bounded in a neighborhood of } x = 0.$$

The idea is that, if, on one hand, small solutions (that are those starting at a point suitably near to the origin of the phase plane) are continuable up to T by the theorem on continuous dependence on initial data, on the other, if a solution blows up before $t = T$, then it oscillates infinitely many times. Therefore $\mathrm{rot}(p)$ can be defined at least in a neighborhood of $p = (0,0)$, and it becomes unbounded either as $|p| \to +\infty$ for the superlinearity assumption on f or for those p's nearby some blowing up solution and, thus, the shooting argument can be still used.

Now we come to the general situation of (1). We denote by $G(x) = \int_0^x g(s)ds$ the primitive of the nonlinearity g and we assume that $G(x) \to +\infty$ as $s \to \pm\infty$. Let $G_l^{-1} : [0,+\infty[\to]-\infty,0]$ and $G_r^{-1} : [0,+\infty[\to [0,+\infty[$ be, respectively, the left and the right inverse functions of G. We describe the phase plane portrait of two autonomous equations which model the situation of $q \geq 0$ and $q \leq 0$, respectively.

Consider a constant weight $q \equiv 1$; then equation (1) becomes:

$$\ddot{x} + g(x) = 0$$

or, equivalently:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g(x) \end{cases} \tag{4}$$

Each non trivial solution (x,y) of (4) satisfies:

$$\frac{1}{2}y^2(t) + G(x(t)) = c \qquad \forall t$$

for some constant $c > 0$. Since the level sets of the function $(x,y) \mapsto \frac{1}{2}y^2 + G(x)$ are closed curves around the origin, every solution of (4) is periodic with a period $\tau^+(c)$ which depends only on the "energy" c of the solution and can be explicitly evaluated:

$$\tau^+(c) = \sqrt{2}\int_{G_l^{-1}(c)}^{G_r^{-1}(c)} \frac{ds}{\sqrt{c - G(s)}}, \qquad c > 0.$$

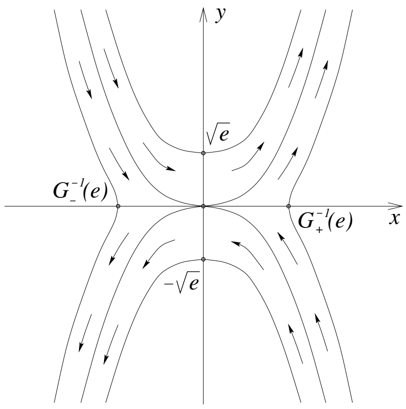

Figure 1: The phase portrait for (5)
with $e = |c|$.

It is well known that the following facts hold:

- $\lim_{s \to \pm\infty} \frac{g(s)}{s} = +\infty \Longrightarrow \lim_{c \to +\infty} \tau^+(c) = 0$;
- if the ratio $g(s)/s$ *monotonically increases* to $+\infty$ as $s \to \pm\infty$ then $\tau^+(c)$ *monotonically decreases* to 0 as $c \to +\infty$.

On the other hand, if we take a constant weight $q \equiv -1$, then (1) becomes:

$$\ddot{x} - g(x) = 0$$

or, equivalently:

$$\begin{cases} \dot{x} = y \\ \dot{y} = g(x) \end{cases} \tag{5}$$

and each solution (x, y) of (5) satisfies:

$$\frac{1}{2} y^2(t) - G(x(t)) = c \qquad \forall\, t$$

for some real constant c.

The phase portrait is that of a saddle (see Figure 1) in which the four nontrivial and unbounded trajectories with "energy" $c = 0$ correspond to the stable (II and IV quadrants) and to the unstable (I and III quadrants) manifolds with respect to the only critical point $(0, 0)$. For each negative value c there are two unbounded trajectories with energy c : one of them lies in the half plane $x > 0$, crosses the positive x-axis at $(G_r^{-1}(-c), 0)$ and corresponds to convex and positive solutions x, and the other lies in $x < 0$, crosses the negative x-axis at $(G_l^{-1}(-c), 0)$ and corresponds to concave and negative solutions x. On the other hand, for each positive c there are two unbounded trajectories with energy c : one of them lies in $y > 0$, crosses the positive y-axis at $(0, \sqrt{2c})$ and corresponds to solutions x which are monotone increasing and have exactly one zero, while the other lies in $y < 0$, crosses the negative y-axis at $(0, -\sqrt{2c})$ and corresponds to solutions x which are monotone decreasing and have exactly one zero.

In this case we do not have any nontrivial periodic solution and, therefore, any period to evaluate; however, when g grows in a superlinear way towards infinity, *all* solutions with nonzero energy have a blow-up in finite time, both in the future and in the past (see [18]). Then we can compute the length of the maximal interval of existence of each trajectory and it turns out to be a function of the energy of the trajectory itself. Indeed, in the case of each of the two trajectories with positive energy c, that length is:

$$\frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} \frac{ds}{\sqrt{c + G(s)}},$$

while for the trajectories with negative energy we have to distinguish between that on $x > 0$, whose maximal interval length is:

$$\sqrt{2} \int_{G_r^{-1}(-c)}^{+\infty} \frac{ds}{\sqrt{c + G(s)}},$$

and the other on $x < 0$, for which the length is:

$$\sqrt{2} \int_{-\infty}^{G_l^{-1}(-c)} \frac{ds}{\sqrt{c + G(s)}}.$$

If for every nonzero c we sum the length of the maximal intervals of the two corresponding trajectories, we obtain the following function:

$$\tau^{-}(c) = \begin{cases} \sqrt{2} \displaystyle\int_{-\infty}^{+\infty} \frac{ds}{\sqrt{c + G(s)}} & \text{if } c > 0 \\ \sqrt{2} \displaystyle\int_{-\infty}^{G_l^{-1}(-c)} \frac{ds}{\sqrt{c + G(s)}} + \sqrt{2} \int_{G_r^{-1}(-c)}^{+\infty} \frac{ds}{\sqrt{c + G(s)}} & \text{if } c < 0 \end{cases}$$

which, like τ^{+}, is infinitesimal for $c \to \pm\infty$ in the superlinear case:

$$\lim_{s \to \pm\infty} \frac{g(s)}{s} = +\infty, \ \left| \int^{\pm\infty} \frac{ds}{G(s)} \right| < +\infty, \ \liminf_{s \to +\infty} \frac{G(ks)}{G(s)} > 1 \Longrightarrow \lim_{c \to \pm\infty} \tau^{-}(c) = 0$$

(k is some constant larger than 1).

EXAMPLE 3. Consider again Hill's equation (2) with exponent $\gamma > 1$ and a piecewise constant weight function q which changes sign:

$$q(t) = \begin{cases} +1 & \text{if } 0 \le t \le t_0 \\ -1 & \text{if } t_0 < t \le T \end{cases}$$

for some $t_0 \in \,]0, T[$. Let us consider the behavior on $[0, T]$ of the solution x_k such that $x_k(0) = 0$ and $\dot{x}_k(0) = k > 0$ as the initial slope k increases. The problem is: is x_k defined on $[0, T]$ and which is its shape? Clearly a blow-up can appear only in $]t_0, T]$ and it depends on which trajectory of (5) the point $(x_k(t_0), \dot{x}_k(t_0))$ belongs to. Indeed we have that $\tau^-(c)$ tends to zero, as c tends to $\pm\infty$, for our $g(s) = |s|^{\gamma-1}s$ and, thus, all the orbits of (5) with an energy c such that $|c| \gg 0$ have a very small maximal interval of existence and are not defined on the whole $[t_0, T]$. On the other hand, all the solutions of (5) passing sufficiently near the two stable manifolds (that are the trajectories of (5) with zero energy which lie in the second and in the fourth quadrant) have a maximal interval of existence which is larger than $[t_0, T]$.

Now, we observe that all the trajectories of (4) intersect the stable manifolds of (5), but for some values of k the point $(x_k(t_0), \dot{x}_k(t_0))$ will be near to the stable manifolds, while for others it will lie far: it depends essentially on the value $\mathrm{rot}((0, k))$, that is on the measure of the angle spanned by the vector $(x(t), \dot{x}(t))$ as t goes from 0 to t_0. Since $\mathrm{rot}((0, k))$ tends to $+\infty$ together with k, it is possible to select a sequence of successive and disjoint intervals:

$$I_0 = [0, k_0[\,,\ \ I_1 = \,]h_1, k_1[\,, \ldots, I_j = \big]h_j, k_j\big[\,, \ldots$$

such that:

- if $k \in I_j$ then $(x_k(t_0), \dot{x}_k(t_0))$ lies near the stable manifolds of (5) and, hence, x_k is defined on $[0, T]$;

- initial slopes belonging to the same I_j determine solutions with the same number of zeros in $[0, t_0]$, but such a number increases together with j.

Moreover, since the stable manifolds separates the trajectories of (5) with positive and negative energy, it is possible to distinguish inside each I_j those initial slopes k such that x_k is monotone in $[t_0, T]$ and with exactly one zero therein, from those such that x_k has constant sign and is convex/concave in $[t_0, T]$. A generalization of this example is given by Lemma 4.

We remark that, when g is superlinear at infinity and q is an arbitrary function, the blow-up always occurs in the intervals where $q < 0$ at least for some "large" initial conditions, no matter how much q and g are regular. This was shown by Burton and Grimmer in [18]: they actually proved that, if $q < 0$, the convergence of one of the following two integrals:

$$\int_{-\infty} \frac{ds}{\sqrt{G(s)}} \qquad \text{and} \qquad \int^{+\infty} \frac{ds}{\sqrt{G(s)}},$$

is a necessary and sufficient condition for the existence of at least one exploding solution of (1).

3. Butler's theorems

In [21] Butler considers the problem of finding periodic solutions of equation (1), or of its equivalent first order system (3), assuming that:

- $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function such that $g(s) \cdot s > 0$ for $s \neq 0$;
- $\lim_{s\to\pm\infty} \frac{g(s)}{s} = +\infty$;
- $\left| \int^{\pm\infty} \frac{1}{\sqrt{G(s)}} ds \right| < +\infty$.

For example, a function g satisfying the first condition and:

$$|g(s)| \geq k|s| \log^{\alpha} |s| \qquad \text{if } s \gg 1,$$

for some $k > 0$ and $\alpha > 2$, satisfies also the other two assumptions. With respect to the weight function q, he supposes that it is a T-periodic and continuous function changing sign a finite number of times and that it is enough regular in the intervals in which it is nonnegative (e.g. q is piecewise monotone), in such a way that in these intervals the solutions cannot blow up; therefore, up to a time-shift, there are j zeros of q, $0 < t_1 < t_2 < \cdots < t_j < T$, such that:

- $q \leq 0$ and $q \not\equiv 0$ in $[0, t_1]$ and in $[t_{j-1}, t_j]$;
- $q \geq 0$ and $q \not\equiv 0$ in $[t_j, T]$;
- $q \not\equiv 0$ and either $q \geq 0$ or $q \leq 0$ in each other interval $[t_i, t_{i+1}]$.

Using the notation introduced at the beginning of Section 2 and recalling what has been said in [18], the value $z(t; t_0, p)$ is surely not defined for some $p \in \mathbb{R}^2$ if the interval between t_0 and t contains points in which q is negative. Therefore Butler introduces the following set of "good" initial conditions with respect to a fixed time interval:

$$\Omega_a^b = \{p \in \mathbb{R}^2 : z(t; a, p) \text{ is defined in the closed interval between } a \text{ and } b\}.$$

In general very little can be said about the shape of Ω_a^b : the theorem about the continuous dependence on initial data implies that it is open and our assumptions guarantee that it always contains the origin, since (1) admits the constant solution $x \equiv 0$. If $q \geq 0$ in $[a, b]$, then $\Omega_a^b = \mathbb{R}^2$, of course, and in particular one has that $\Omega_0^T = \Omega_0^{t_j}$. Clearly, if b lies between a and c then $\Omega_a^c \subset \Omega_a^b$.

One way to find T-periodic solutions of (1) or (3) is to write the T-periodic boundary condition $z(T) = z(0)$ in a way which puts in evidence the dependence on the initial value $z(0) = p$; indeed, we are essentially looking for initial conditions $p \in \mathbb{R}^2$

such that $z(T; 0, p) = p$, that is we search points in the plane where the vector field $p \mapsto z(T; 0, p) - p$ vanishes. If we introduce the following two auxililary functions:

$$\begin{aligned} \phi(p) &= \|z(T; 0, p)\| - \|p\| \\ \psi(p) &= \int_0^T \frac{y^2(t; 0, p) + q(t)g(x(t; 0, p))x(t; 0, p)}{y^2(t; 0, p) + x^2(t; 0, p)} dt, \end{aligned}$$

then the solution departing from p at $t = 0$ is T-periodic if and only if:

$$\begin{cases} \phi(p) = 0 \\ \psi(p) = 2k\pi & \text{for some } k \in \mathbb{Z}. \end{cases}$$

Before entering more in the details of Butler's technique, let us fix some notation. If r is a positive number, then C_r will denote the circumference $\{p \in \mathbb{R}^2 : \|p\| = r\}$ with radius r; if, moreover, $R > r$, then $A[r, R]$ will be the closed annulus with boundary $C_r \cup C_R$. By the word "continuum" we mean, as usual, a compact and connected set.

Here is a first lemma about what happens in any interval of positivity for q.

LEMMA 1. *Assume that $q \geq 0$ and $q \not\equiv 0$ in $[a, b]$. Then for every $M > 0$ and $n \in \mathbb{N}$ there exist $r = r(M, n)$ and $R = R(M, n)$, with $0 < r < R$, such that:*

1. *$\|z(t; a, p)\| \geq M$ for all $t \in [a, b]$ and $\|p\| \geq r$;*
2. *$\Gamma \ni p \mapsto \arg z(b; a, p)$ is a n-fold covering of S^1, for any continuum $\Gamma \subset A[r, R]$ which does not intersect both axes and satisfies $\Gamma \cap C_r \neq \emptyset \neq \Gamma \cap C_R$.*

Roughly speaking, the second statement just means that the map $p \mapsto z(b; a, p)$ transforms any continuum crossing the annulus $A[r, R]$ into a continuum which turns around the origin at least n times. Observe that it is required that Γ "does not intersect both axes", that is it must be contained in one of the four half-planes generated by the coordinate axes: this prevents Γ itself from turning around the origin and escaping the twisting effect of the map $p \mapsto z(b; a, p)$.

REMARK 1. If q is nonnegative in $[a, b]$, then the mappings $p \mapsto z(b; a, p)$ and $p \mapsto z(a; b, p)$ are defined on $\mathbb{R}^2$, continuous and each one is the inverse of the other. Thus they are homeomorphisms of $\mathbb{R}^2$ onto itself and, in particular, map bounded sets into bounded sets.

Even if little can be said in general about the structure of a set Ω_a^b (it might be disconnected and its boundary might not be a continuous arc), Butler actually proved the following, when $[a, b]$ is an interval of negativity for q.

LEMMA 2. *Assume that $q \leq 0$ and $q \not\equiv 0$ on $[a, b]$. If $J \subset \mathbb{R}$ is any compact interval, then $\Omega_a^b \cap J \times \mathbb{R}$ is non-empty and bounded.*
The same holds for Ω_b^a.

This result is simple if $q \equiv -1$, since in this case it turns out that, on one hand, the set Ω_a^b must contain the stable manifolds $y = -\sqrt{2G(x)}$, for $x > 0$, and $y = \sqrt{2G(x)}$, for $x < 0$, while, on the other, it cannot contain any point from the trajectories $y = \pm\sqrt{2(c + G(x))}$, if $c > 0$ is such that $\tau^-(c) < b - a$ and this happens for every sufficiently large c.

LEMMA 3. *There are* $\alpha < 0 < \beta$ *and a continuous arc* $\gamma = (\gamma_1, \gamma_2) :]\alpha, \beta[\to \Omega_0^T$ *such that:*

1. $\gamma(0) = (0, 0)$;
2. $\lim_{s\to\alpha^+} \gamma_1(s) = \lim_{s\to\alpha^+} \gamma_2(s) = \lim_{s\to\beta^-} \gamma_1(s) = \lim_{s\to\beta^-} \gamma_2(s) = \infty$;
3. $\|z(t_j; 0, \gamma(s))\|$ *and* $\|z(T; 0, \gamma(s))\|$ *are uniformly bounded for* $s \in]\alpha, \beta[$.

Proof. Let us consider just the case $j = 1$, in which $t_{j-1} = 0$ and $t_1 = t_j$. The intersection between $\Omega_{t_j}^0$ and the y-axis $\{0\} \times \mathbb{R}$ determines, by Lemma 2, a bounded and open (relatively to the topology of the straight line) set which contains the origin. Therefore there are $\alpha < 0 < \beta$ such that the segment $\{0\} \times]\alpha, \beta[$ is contained in $\Omega_{t_j}^0$ while its end-points $(0, \alpha)$ and $(0, \beta)$ belong to $\partial\Omega_{t_j}^0$. By construction each solution departing from $\{0\} \times]\alpha, \beta[$ at time t_j is defined at least up to 0, hence we can set:

$$\gamma(s) = z(0; t_j, (0, s)) \qquad \text{for } s \in]\alpha, \beta[.$$

Since $\gamma(s)$ is the value at time 0 of a solution defined on $[0, t_j]$, we have that the support of γ lies in $\Omega_0^{t_j}$, which in turn coincides with Ω_0^T because in the last interval $[t_j, T]$ q is nonnegative and, therefore, solutions cannot blow up therein by our assumptions on q.

Clearly statement 1 is satisfied and Statement 2 follows from the fact that the points $(0, \alpha)$ and $(0, \beta)$ do not belong to $\Omega_{t_j}^0$: hence $z(t; t_j, (0, \alpha))$ and $z(t; t_j, (0, \beta))$ blow up somewhere in $[0, t_j]$ and an argument based on the continuous dependence on initial data shows that $\gamma_i(s)$ is unbounded when s ranges near α and β.

The definition of γ implies that:

$$z(t_j; 0, \gamma(s)) = (0, s) \qquad \text{for } s \in]\alpha, \beta[,$$

thus $\|z(t_j; 0, \gamma(s))\|$ is bounded by $\max\{-\alpha, \beta\}$. Finally, observe that $z(T; 0, \gamma(s)) = z(T; t_j, (0, s))$ and that q is nonnegative on $[t_j, T]$; then also Statement 3 holds by Remark 1.

$\square$

THEOREM 1. *Equation* (1) *has infinitely many* T*-periodic solutions.*

Proof. We start fixing some constants. By Lemma 2 the intersection of the y-axis with the set $\Omega_{t_j}^{t_{j-1}}$ is bounded by a constant A_1; therefore, if $z(t_j; 0, p)$ lies on the y-axis then $\|z(t_j; 0, p)\| = |y(t_j; 0, p)| \le A_1$. Moreover, by Remark 1 the following constant:

$$A_2 = \max\{\|z(T; t_j, p)\| : \|p\| \le A_1\}$$

exists and is finite. In particular, if γ is the curve given in Lemma 3, we have that:

$$\|z(t_j; 0, \gamma(s))\| \leq A_1 \qquad \text{and} \qquad \|z(T; 0, \gamma(s))\| \leq A_2 \qquad \text{for } s \in \,]\alpha, \beta[\,.$$

Now let A_3 be any real number such that $A_3 > A_2$ and let L_1 be the vertical straight line $\{A_3\} \times \mathbb{R}$. Let Ω be the connected component of Ω_0^T which contains the support of γ. By Lemma 2 and the fact that $\Omega \subset \Omega_0^{t_1}$, we have that the set $L_1 \cap \Omega$ is bounded and we can define:

$$A_4 = \sup\{\|p\| : p \in L_1 \cap \Omega\} < +\infty \qquad (\Longrightarrow A_4 > A_3).$$

Now take $M = 2A_4$ and *any natural number n* and consider the two radii $r = r(M, n+1)$ and $R = R(M, n+1)$ which are obtained applying Lemma 1 in the interval $[t_j, T]$. We set:

$$A_5 = \max\{\|z(T; t_j, p)\| : \|p\| \leq R\}$$

and call L_2 the vertical straight line $\{A_3 + A_5\} \times \mathbb{R}$. Now, Statement 2 in Lemma 3 guarantees that γ crosses at least one of the two vertical strips $[A_3, A_3 + A_5] \times \mathbb{R}$ and $[-A_3 - A_5, -A_3] \times \mathbb{R}$: assume that it crosses the first one (if it crosses the other one, one can argue in a similar way) and call it $S[L_1, L_2]$. By Lemma 2, the intersection of Ω with the vertical strip $S[L_1, L_2]$ is bounded, therefore:

$$A_6 = \sup\{\|p\| : p \in \Omega \cap [A_3, A_3 + A_5] \times \mathbb{R}\} < +\infty.$$

The curve γ, passing from L_1 to L_2, divides $\Omega \cap S[L_1, L_2]$ into two bounded regions. If $p \in S[L_1, L_2]$ belongs to the support of γ, then $\|z(T; 0, p)\| \leq A_2 < A_3 \leq \|p\|$; hence:

$$p \in \gamma(]\alpha, \beta[) \cap S[L_1, L_2] \Longrightarrow \phi(p) \leq 0.$$

On the other hand, if p lies in $\Omega \cap S[L_1, L_2]$ near $\partial\Omega$, then $\|p\|$ remains bounded by A_6, but $\|z(T; 0, p)\|$ can be made arbitrarily large, since p is near "bad" points with respect to the interval $[0, T]$; thus:

$$\phi(p) \to +\infty \quad \text{if } p \to \partial\Omega,\ p \in \Omega \cap S[L_1, L_2].$$

Therefore, on every curve σ contained in $\Omega \cap S[L_1, L_2]$ and such that it connects a point of γ with a point of $\partial\Omega$, we can find a point p in which $\phi(p) = 0$. This implies (but it is not a trivial topological fact) that there exists a continuum Γ_0 contained in $\Omega \cap S[L_1, L_2]$ and intersecting also L_1 and L_2 such that:

$$p \in \Gamma_0 \Longrightarrow \|z(T; 0, p)\| = \|p\|. \tag{6}$$

The set:

$$\Gamma_j = \{z(t_j; 0, p) : p \in \Gamma_0\}$$

is still a continuum since it is the image of Γ_0 through the continuous map $\Omega_0^T \ni p \mapsto z(t_j; 0, p)$. Γ_j does not intersect the y-axis, since, if $x(t_j; 0, p) = 0$, then one has $\|z(t_j; 0, p)\| \leq A_1$, by the definition of A_1, and $\|z(T; 0, p)\| =$

$\|z(T; t_j, z(t_j; 0, p))\| \leq A_2 < A_3$, by the definition of A_2, while $\|z(T; 0, p)\| = \|p\| \geq A_3$ if $p \in \Gamma_0$. Moreover, fix $p \in L_1 \cap \Gamma_0$ and $q \in L_2 \cap \Gamma_0$; we have:

$$\|z(T; t_j, z(t_j; 0, p)\| = \|z(T; 0, p)\| = \|p\| \leq A_4 < M \Longrightarrow \|z(t_j; 0, p)\| < r$$

by the definition of A_4 and Statement 1 in Lemma 1; and also:

$$\|z(T; t_j, z(t_j; 0, q)\| = \|z(T; 0, q)\| = \|q\| \geq A_3 + A_5 > A_5 \Longrightarrow \|z(t_j; 0, q)\| > R$$

by the definition of A_5. Thus, we have found one point of Γ_j inside the ball of radius r and another point of Γ_j outside that of radius R and we can say that Γ_j is a continuum crossing the annulus $A[r, R]$. Hence Γ_j fulfills all the requirements of Statement 2 in Lemma 1. In particular we have that the map $p \mapsto \arg z(T; 0, p)$ covers S^1 at least $n + 1$ times as p ranges in Γ_0. Let us see what it means in terms of angles and of the function ψ. We can select a continuous angular coordinate $\theta : [0, T] \times \Gamma_0 \to \mathbb{R}$ such that:

1. $z(t; 0, p) = (\|z(t; 0, p)\| \cos\theta(t, p), \|z(t; 0, p)\| \sin\theta(t, p))$ for $(t, p) \in [0, T] \times \Gamma_0$;
2. $-\frac{\pi}{2} < \theta(0, p) < \frac{\pi}{2}$ for $p \in \Gamma_0$ (recall that Γ_0 is contained in the right half-plane).

With this choices, the function ψ can be written as:

$$\psi(p) = \theta(0, p) - \theta(T, p).$$

The fact that $\Gamma_0 \ni p \mapsto \arg z(T; 0, p)$ covers S^1 at least $n + 1$ times, means that the image of $\theta(T, \cdot)$ contains a $2(n + 1)\pi$-long interval. Since $\theta(0, \cdot)$ is forced in a π-long interval, we have that $\psi(p)$ reaches at least n successive integer multiples of 2π as p ranges in Γ_0. Therefore (1) has at least n T-periodic solutions, with n arbitrarily chosen.

□

We have seen that the superlinear growth at infinity of the nonlinear term g in (1) leads to the blow-up of solutions in the intervals where q attains negative values. On the other hand, if g is sublinear around 0, there is the possibility of solutions reaching the origin in finite time, since the uniqueness of the zero solution is no more guaranteed. This case was studied by Butler in [23].

EXAMPLE 4. Let us consider the autonomous system (4) with a function g which is sublinear in zero, that is:

$$\lim_{x\to 0} \frac{g(x)}{x} = +\infty.$$

The uniqueness of the solution of Cauchy problems is still guaranteed, but, now, smaller solutions oscillate more and more.

On the other hand, if we look to the zero-energy solutions of (5), which satisfy $\frac{1}{2}\dot{x}^2 - G(x) = 0$, we found that the time that they take to reach $(0, 0)$ from the value $x = x_0 > 0$ is given by the integral:

$$\int_0^{x_0} \frac{ds}{\sqrt{2G(s)}}$$

which is *finite* when $g(s)$ is a sublinear function like $s|s|^{\gamma-1}$, with $0 < \gamma < 1$. Therefore the uniqueness of the zero-solution holds no more.

In [48] Heidel gives conditions that prevent solutions of (1) from reaching the origin in finite time in the case of a nonnegative weight q. In particular assuming $q \in C^1$ and q being piecewise monotone around its zeros turns out to be sufficient to this aim and is what Butler needs in [23]. Indeed, Butler proves that, if g is sublinear around the origin and q is a T-periodic weight which changes sign and is enough regular, then (1) has infinitely many T-periodic solutions with an arbitrarily large number of small oscillations in the intervals of positivity of q.

On the other hand such solutions may be identically zero in some subintervals of the intervals of negativity of q. Indeed, let us consider a weight q_ϵ such that $q_\epsilon \equiv -1$ in $[0, 2[$, $q_\epsilon \equiv \epsilon > 0$ in $[2, 4[$ and which is 4-periodic. Then Butler shows that ϵ can be chosen sufficiently small in such a way that every solution of:

$$\ddot{x} + q_\epsilon(t)x^{\frac{1}{3}} = 0$$

which is nowhere trivial must be strictly monotone (and, hence, nonperiodic) on some half line.

4. Another possible approach: generalized Sturm–Liouville conditions

Let us consider a situation in which $q : [a, c] \to \mathbb{R}$ is such that:

$$q \geq 0 \text{ in } [a, b] \qquad \text{and} \qquad q \leq 0 \text{ in } [b, c],$$

and assume that g in (1) is superlinear at infinity in the sense that:

$$\lim_{c \to +\infty} \tau^+(c) = \lim_{c \to \pm\infty} \tau^-(c) = 0.$$

Let $Q_1 = [0, +\infty[\times [0, +\infty[$, $Q_2 =]-\infty, 0] \times [0, +\infty[$, $Q_3 =]-\infty, 0] \times]-\infty, 0]$ and $Q_4 = [0, +\infty[\times]-\infty, 0]$ be the four closed quadrants of the plane. Then we have the following result.

LEMMA 4. *There exists $R^* > 0$ (depending only on g and $q|_{[b,c]}$) such that, for every $R > 0$ there is a natural number $n^* = n^*_R$ with the property that for every natural numbers $n > n^*$ and $\delta \in \{0, 1\}$ and for any path $\gamma : [\alpha, \beta[\to [0, +\infty[\times \mathbb{R}$, with $\|\gamma(\alpha)\| \leq R$ and $\|\gamma(s)\| \to +\infty$ as $s \to \beta$, we can select an interval $I \subset]\alpha, \beta[$,* *with* $I =]\alpha_n, \beta_n]$, *if* $\delta = 0$, *and* $I = [\beta_n, \alpha_n[$, *if* $\delta = 1$, *in such a way that for each* $s \in I$ *we have:*

- $z(c; a, \gamma(s))$ *is defined*
- $x(\cdot; a, \gamma(s))$ *has exactly* n *zeros in* $]a, b[$, δ *zeros in* $]b, c[$ *and exactly* $1 - \delta$ *changes of sign of the derivative in* $]b, c[$
- *the curve* $\gamma_n(s) = z(c; a, \gamma(s))$, $s \in I$, *satisfies* $\|\gamma_n(\beta_n)\| \leq R^*$, $\|\gamma_n(s)\| \to +\infty$ *as* $s \to \alpha_n$ *and its support lies either in* Q_1 *(if* $n + \delta$ *is even) or in* Q_3 *(if* $n + \delta$ *is odd).*

The same holds when the support of the curve γ *lies in the left half plane* $]-\infty, 0] \times \mathbb{R}$ *by simply interchanging the role of* Q_1 *and* Q_3.

Let us see how to use Lemma 4 in order to find multiple solutions of (1) satisfying the two-point boundary condition:

$$x(0) = x(T) = 0. \tag{7}$$

We assume that there are t_i, with $i = 0, \ldots, 2j + 1$, such that $0 = t_0 < t_1 < \cdots < t_{2j+1} = T$ and:

$$q \geq 0, \quad q \not\equiv 0 \text{ in } [t_{2i-2}, t_{2i-1}] \qquad \text{and} \qquad q \leq 0, \quad q \not\equiv 0 \text{ in } [t_{2i-1}, t_{2i}],$$

for $i = 1, \ldots, j + 1$, so q is positive near both 0 and T. Let us apply Lemma 4 in the interval $[0, t_2]$ to the unbounded curve $\gamma_0(s) = (0, s)$, for $s \geq 0$, which parametrizes the positive y-axis in the phase plane: each solution x of (1)–(7) with $\dot{x}(0) > 0$ should start from the support of γ_0 at time $t = 0$. Let $R_1^* > 0$ and $n_1^* \in \mathbb{N}$ be respectively the numbers R^* and n_R^* given by Lemma 4 with an arbitrarily small $R > 0$ (since $\gamma_0(0) = (0, 0)$) and fix any $n_1 > n_1^*$ and $\delta_1 \in \{0, 1\}$: then, we obtain an interval $I_1 =]\alpha_1, \beta_1[\subset [0, +\infty[$ such that the solution of (1) starting at $t = 0$ from $\gamma_0(s)$ has nodal behavior in $[0, t_2]$ prescribed by the couple (n_1, δ_1), as in Lemma 4, if s belongs to I_1, and, moreover, the curve $\gamma_1(s) = z(t_2; 0, \gamma_0(s))$ is defined for $s \in I_1$, is contained either in the first or the third quadrant, it is unbounded when s tends to one of the endpoints of I_1, while it lies inside a circle of radius R_1^* for s belonging to a neighborhood of the other endpoint. Therefore we can apply Lemma 4 on the successive interval $[t_2, t_4]$ and to the curve γ_1 with the choice $R = R_1^*$.

After j successive applications of Lemma 4 to the intervals $[t_{2i-2}, t_{2i}]$, for $i = 1, \ldots, j$, we get $R_j^* > 0$ and j positive integers $n_1^*, \ldots, n_j^*$ such that, for every j-tuple $(n_1, \ldots, n_j) \in \mathbb{N}^j$, with $n_i > n_i^*$, and for every j-tuple $(\delta_1, \ldots, \delta_j) \in \{0, 1\}^j$, there is a final interval $I_j \subset [0, +\infty[$ with the following properties:

- the curve $\gamma(s) = z(t_{2j}; 0, (0, s))$ is defined for $s \in I_j$, lies in the first or in the third quadrant (it depends on the parity of $n_1 + \delta_1 + \cdots + n_j + \delta_j$), it is unbounded when s tends to one of the endpoints of I_j, while it is inside the circle of radius R_j^* if s belongs to a neighborhood of the other endpoint;

- if $s \in I_j$ and $i = 1, \dots, j$, the solution $x(t; 0, (0, s))$ has a nodal behavior in $[t_{2i-2}, t_{2i}]$ which is described by the couple (n_i, δ_i) as in Lemma 4.

It remains to find some s in the interval I_j such that the solution starting at $t = t_{2j}$ from $\gamma(s)$ reaches the y-axis exactly at $t = T$ and this can be done by a result of Struwe [83], since the weight q is nonnegative in the interval $[t_{2j}, T]$ (see Example 2 for an idea of the argument).

Clearly another set of solutions can be found starting from the negative y-axis and it is not difficult to obtain the same kind of result if q is negative either near $t = 0$ or near $t = T$ or both. However, a more important fact is perhaps that we can adjust the technique explained above in order to find multiple solutions of more general boundary value problems for (1), namely all those problems whose boundary conditions can be expressed by:

$$(x(0), \dot{x}(0)) \in \Gamma_0 \qquad \text{and} \qquad (x(T), \dot{x}(T)) \in \Gamma_T,$$

where Γ_0 and Γ_T are suitable subsets of the phase plane. They are called "generalized" Sturm–Liouville boundary conditions (see [83]) since they coincide with the usual Sturm–Liouville conditions when Γ_0 and Γ_T are two straight lines. In particular, when q is positive near 0 and T, it is possible to adapt the technique to cover all the cases in which Γ_0 and Γ_T are two unbounded continua (i.e. connected, closed and unbounded sets) contained, for instance, in some half-planes: in fact, by approximating bounded portions of continua by means of supports of continuous curves, it is possible to prove a generalization of Lemma 4 which holds also when the path γ is substituted by an unbounded continuum Γ contained either in the right half plane or in the left one.

4.1. Application to homoclinic solutions

Assume that:

$$q(t) \le 0 \qquad \forall\, t \in \,]-\infty, a] \cup [b, +\infty[$$

and that:

$$\int_{-\infty} q = \int^{+\infty} q = -\infty.$$

Then, using an argument similar to that employed by Conley in [29], it is possible to show that there are four unbounded continua $\Gamma_a^+ \subset Q_1$, $\Gamma_a^- \subset Q_3$, $\Gamma_b^+ \subset Q_4$ and $\Gamma_b^- \subset Q_2$ such that:

- $\lim_{t \to -\infty} z(t; a, p) = (0, 0)$ for every $p \in \Gamma_a^{\pm}$;
- $\lim_{t \to +\infty} z(t; b, p) = (0, 0)$ for every $p \in \Gamma_b^{\pm}$

(see Lemmas 5 and 7 in [72] for precise statements and proof). Therefore the problem of finding homoclinics solutions of (1) is reduced to that of determining solutions of (1) in $[a, b]$ which satisfy the generalized Sturm–Liouville boundary condition:

$$(x(a), \dot{x}(a)) \in \Gamma_a^{\pm} \qquad\qquad (x(b), \dot{x}(b)) \in \Gamma_b^{\pm}$$

and this can be done in the superlinear case by the technique already explained in this section (Lemma 4 plus Struwe's result [83]).

4.2. Application to blow-up solutions

In [61] (see also [62]) the problem of finding solutions of (1) which blow up at a precise time was considered when g has a superlinear growth at infinity and $q :]0, 1[\to \mathbb{R}$ is a continuous weight such that q is nonpositive in some neighborhood of 0 and of 1 and both 0 and 1 are accumulation points of the set in which q is strictly negative. See the paper [27] for recent results about the analogous problem for partial differential equations.

To be precise, let us assume that q is nonpositive in $]0, a]$ and in $[b, 1[$; then there are two unbounded continua Γ_0 and Γ_1 which are contained in the right half plane $x \geq 0$ and moreover:

1. there are $R > r > 0$ and $\epsilon > 0$ such that:

$$\begin{aligned}
&\Gamma_0 \cap [0, r] \times \mathbb{R} \subset [0, r] \times]-\infty, -\epsilon] \\
&\Gamma_0 \cap [R, +\infty[\times \mathbb{R} \subset [R, +\infty[\times [\epsilon, +\infty[\\
&\Gamma_1 \cap [0, r] \times \mathbb{R} \subset [0, r] \times [\epsilon, +\infty[\\
&\Gamma_1 \cap [R, +\infty[\times \mathbb{R} \subset [R, +\infty[\times]-\infty, -\epsilon]
\end{aligned}$$

2. $\lim_{t\to 0} x(t; a, p) = \lim_{t\to 1} x(t; b, q) = +\infty$ if $p \in \Gamma_0$ and $q \in \Gamma_1$.

If $q \leq 0$ in the whole $]0, 1[$, then we can choose $a = b = 1/2$ and the localization properties in statement 1 imply that $\Gamma_0 \cap \Gamma_1 \neq \emptyset$ and this proves that there is a positive solution which blows up at 0 and 1.

On the other hand, if q changes sign a finite number of times inside $]0, 1[$, we can consider the generalized Sturm–Liouville boundary value conditions:

$$(x(a), \dot{x}(a)) \in \Gamma_0 \qquad\qquad (x(b), \dot{x}(b)) \in \Gamma_1$$

and apply the procedure previously explained in order to find solutions of (1) in $]0, 1[$ which blows up at 0 and 1 and have a prescribed nodal behavior inside the interval.

5. Chaotic-like dynamics

The chaotic features of (1) were studied in the papers [85] and [26] when g is superlinear at infinity. Here we would like to give an interpretation of chaos in the sense of "coin-tossing", as it is defined in [53] for the discrete dynamical system generated by the iterations of a continuous planar map ψ which is not required to be defined in the whole plane (like the Poincaré map associated to our equation (1) when g is superlinear at infinity and q is somewhere negative). To be more precise, consider the set X which is the union of two disjoint, nonempty and compact sets K_0 and K_1. We say that the discrete dynamical system generated by the iterates of a continuous mapping ψ is *chaotic in the sense of coin-tossing* if, for every doubly infinite sequence of binary digits $(\delta_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, there is a doubly infinite sequence $(p_i)_{i\in\mathbb{Z}}$ of points of X such that:

1. $\psi(p_i) = p_{i+1}$
2. $p_i \in K_{\delta_i}$

for every $i \in \mathbb{Z}$. The first condition states that the sequence $(p_i)_{i\in\mathbb{Z}}$ is an *orbit* of the dynamical system generated by ψ; the second one guarantees the possibility of finding orbits which touch at each time the prescribed component of X.

We remark that in this definition ψ is not necessarily defined in the whole X and it is not required to be 1-to-1. Actually we are interested in the case of planar maps, since we wish to study the Poincaré map associated to (1), and, in particular, we will consider compact sets K_i with a particular structure: we call an *oriented cell* a couple $(\mathcal{A}, \mathcal{A}^-)$ where $\mathcal{A} \subset \mathbb{R}^2$ is a two-dimensional cell (i.e., a subset of the plane homeomorphic to the unit square $Q = [-1,1]^2$) and $\mathcal{A}^- \subset \partial\mathcal{A}$ is the union of two disjoint compact arcs. The two components of $\mathcal{A}^-$ will be denoted by $\mathcal{A}_l^-$ and $\mathcal{A}_r^-$ and conventionally called the left and the right sides of $\mathcal{A}$. The order in which we make the choice of naming $\mathcal{A}_l^-$ and $\mathcal{A}_r^-$ is immaterial in what follows.

If ψ is a continuous map $\mathbb{R}^2 \supset \mathrm{Dom}(\psi) \to \mathbb{R}^2$ and $(\mathcal{A}, \mathcal{A}^-)$, $(\mathcal{B}, \mathcal{B}^-)$ are two oriented cells, we say that ψ *stretches* $(\mathcal{A}, \mathcal{A}^-)$ *to* $(\mathcal{B}, \mathcal{B}^-)$ and write:

$$\psi : (\mathcal{A}, \mathcal{A}^-) \Diamond\!\!\rightsquigarrow (\mathcal{B}, \mathcal{B}^-),$$

if:

- ψ is *proper* on $\mathcal{A}$, which means that $|\psi(p)| \to +\infty$ whenever $\mathrm{Dom}(\psi) \cap \mathcal{A} \ni p \to p_0 \in \partial\,\mathrm{Dom}(\psi) \cap \mathcal{A}$;
- for any path $\Gamma \subset \mathcal{A}$ such that $\Gamma \cap \mathcal{A}_l^- \neq \emptyset$ and $\Gamma \cap \mathcal{A}_r^- \neq \emptyset$, there is a path $\Gamma' \subset \Gamma \cap \mathrm{Dom}(\psi)$ such that:

$$\psi(\Gamma') \subset \mathcal{B}, \quad \psi(\Gamma') \cap \mathcal{B}_l^- \neq \emptyset, \quad \psi(\Gamma') \cap \mathcal{B}_r^- \neq \emptyset.$$

THEOREM 2. *If* $\psi : (\mathcal{A}, \mathcal{A}^-) \Diamond\!\!\rightsquigarrow (\mathcal{A}, \mathcal{A}^-)$, *then* ψ *has at least one fixed point in* $\mathcal{A}$.

Sketch of the proof. Let us consider just the case of $\mathcal{A} = [0,1] \times [0,1]$, with $\mathcal{A}_l^- = \{0\} \times [0,1]$ and $\mathcal{A}_r^- = \{1\} \times [0,1]$, and let $\psi(x_1,x_2) = (\psi_1(x_1,x_2), \psi_2(x_1,x_2))$. If $\Gamma \subset \mathcal{A}$ is a path joining the vertical sides of $\mathcal{A}$, let $\Gamma' \subset \Gamma$ be the subpath such that $\psi(\Gamma')$ is again a path in $\mathcal{A}$ which joins its vertical sides and, in particular, let $p = (p_1,p_2)$ and $q = (q_1,q_2)$ two points in Γ' such that $\psi(p) \in \{0\} \times [0,1]$ and $\psi(q) \in \{1\} \times [0,1]$. Therefore we have:

$$\psi_1(p_1,p_2) - p_1 = -p_1 \le 0 \qquad \text{and} \qquad \psi_1(q_1,q_2) - q_1 = 1 - q_1 \ge 0.$$

Hence, every path in $\mathcal{A}$ joining the vertical sides meets the closed set in which the function $\psi_1(x_1, x_2) - x_1$ vanishes and this implies that actually there is a whole continuum $\Gamma_1 \subset \mathcal{A}$ joining the *horizontal* sides of $\mathcal{A}$ such that $\psi_1(x_1, x_2) - x_1$ vanishes in Γ_1 and $\psi(\Gamma_1) \subset \mathcal{A}$ (see the argument to find Γ_0 in (6)). Again, this implies that the function $\psi_2(x_1, x_2) - x_2$ changes sign on Γ_1 : there is a point in Γ_1 where also $\psi_2(x_1, x_2) - x_2$ vanishes, and such a point is clearly a fixed point of ψ.

$\square$

THEOREM 3. *Let $(\mathcal{A}_0, \mathcal{A}_0^-)$ and $(\mathcal{A}_1, \mathcal{A}_1^-)$ be two oriented cells. If ψ stretches each of them to itself and to the other one:*

$$\psi : (\mathcal{A}_i, \mathcal{A}_i^-) \multimap\!\!\!\!\leftarrow (\mathcal{A}_j, \mathcal{A}_j^-), \qquad \textit{for } (i, j) \in \{0, 1\}^2,$$

then ψ shows a chaotic dynamics of coin-tossing type.

These results can be applied, for instance, to the following situation:

$$\ddot{x} + [\alpha q^+(t) - \beta q^-(t)]g(x) = 0, \tag{8}$$

where α and β are positive constants, $q^+(t) = \max\{q(t), 0\}$ and $q^-(t) = \max\{-q(t), 0\}$ are respectively the positive and the negative part of a continuous and periodic function q which changes sign, and g is a nonlinear function such that:

$$0 < g'(0) \ll g'(\infty).$$

The parameter α regulates the twisting effect of the Poincaré map along the intervals of positivity of q, while β controls the stretching of the arcs along the intervals of negativity of q. Assume, for simplicity, that q is T-periodic with exactly one change of sign in $\tau \in]0, T[$ in such a way that:

$$q > 0 \quad \text{in }]0, \tau[\qquad \text{and} \qquad q < 0 \quad \text{in }]\tau, T[\,.$$

For every fixed $n \in \mathbb{N}$, using the theorems stated above, it is possible to find $\alpha_n > 0$ such that, for every $\alpha > \alpha_n$, there is $\beta_\alpha > 0$ such that for each $\beta > \beta_\alpha$ we have the following results (see Theorem 2.1 in [30]):

1. for any $m \in \mathbb{N}$ and any m-tuple of binary digits $(\delta_1, \ldots, \delta_m) \in \{0, 1\}^m$ such that $mn + \delta_1 + \cdots + \delta_m$ is an even number, there are at least two mT-periodic solutions x^+ and x^- of (8) which have exactly n zeros in $[(i-1)T, (i-1)T + \tau]$ and δ_i zeros in $[(i-1)T + \tau, iT]$, for each $i = 1, \ldots, m$; moreover $x^+(0) > 0$ and $x^-(0) < 0$;

2. for any doubly infinite sequence of binary digits $(\delta_i)_{i\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$, there is at least a globally defined solution x of (8) which has exactly n zeros in $[iT, iT + \tau]$ and δ_i zeros in $[iT + \tau, (i+1)T]$, for all $i \in \mathbb{Z}$.

6. Subharmonic solutions

Our aim here is to consider large solutions of equations like (1) in which the nonlinearity g is sublinear at infinity, as in Hill's equation (2) when $0 < \gamma < 1$. The results we are going to present are contained in a joint work with B. Liu [59] and are valid also for the forced version of (1):

$$\ddot{x} + q(t)g(x) = e(t).$$

Throughout this section we assume that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that:

- $g(0) = 0$;
- there is $R_0 \geq 0$ such that $g(s) \cdot s > 0$ and $g'(s) \geq 0$ if $|s| > R_0$ ($\Longrightarrow g(-\infty) < 0 < g(+\infty)$);
- $\displaystyle\lim_{s \to \pm\infty} \frac{g(s)}{s} = 0.$

The third condition is the so-called condition of sublinearity at infinity. Moreover we will suppose that q is a continuous and T-periodic function, even if continuity is not necessary: local integrability would be enough.

THEOREM 4. *Besides the assumptions stated above, suppose that:*

$$\overline{q} = \frac{1}{T}\int_0^T q(t)dt > 0. \tag{9}$$

Then for each integer $j \geq 1$ there is $m_j^* \in \mathbb{N}$ such that, for every $m \geq m_j^*$ equation (1) *has at least one mT-periodic solution $x_{j,m}$ which has exactly $2j$-zeros in* $[0, mT[$ *. Moreover, for each $m \geq 1$ there is $M_m > 0$ such that any mT-periodic solution x of* (1) *satisfies:*

$$\|x\|_{C^1} \leq M_m;$$

on the other hand, for every fixed $j \geq 1$ we have:

$$\lim_{m \to +\infty} (|x_{j,m}(t)| + |\dot{x}_{j,m}(t)|) = +\infty,$$

uniformly with respect to $t \in \mathbb{R}$.

EXAMPLE 5. Theorem 4 holds, for instance, for the following Hill's equation:

$$\ddot{x} + [k + \cos(t + \theta)]|x|^{\gamma - 1}x = 0,$$

where $0 < \gamma < 1$, $k > 0$ and $\theta \in \mathbb{R}$. The same is true if we substitute $|x|^{\gamma-1}x$ with another sublinear function like $x/(1 + |x|)$, for instance.

We remark that condition (9) was already considered by other authors dealing with the superlinear case (see for instance [76]). A partial converse, with respect to this assumption, holds in the case of Hill's equation (2) for $0 < \gamma < 1$; in this case, if $\overline{q} < 0$, there is a constant $B > 0$, such that every solution of (2) satisfying:

$$|x(0)| + |\dot{x}(0)| > B$$

is unbounded, that is:

$$\sup_{t \in \mathbb{R}} (|x(t)| + |\dot{x}(t)|) = +\infty.$$

REMARK 2. In the book [34, p. 129] it is pointed out that "the question is whether we can find *for each* $k \geq 2$ a subharmonic x_k [that is a kT-periodic solution] such that the x_k are pairwise distinct. No result is known in the subquadratic case". The same question was pointed out by Các and Lazer in [24]. Of course we deal with a scalar model, which is a very simple case of a Hamiltonian system.

The trick to study (1) is the introduction of the so-called "Riccati integral equation" associated to (1):

$$\frac{\dot{x}(t)}{g(x(t))} = \frac{\dot{x}(s)}{g(x(s))} - \int_s^t \left[\frac{\dot{x}(\xi)}{g(x(\xi))} \right]^2 g'(x(\xi)) d\xi - \int_s^t q(\xi) d\xi,$$

which is easily deduced recalling that:

$$\frac{\ddot{x}(t)}{g(x(t))} = -q(t),$$

by equation (1). This integral equation was already used by people working in oscillation theory.

We use here a small variant of a notation already introduced. If z is a solution of (3), we denote by $\mathrm{rot}(z; t_1, t_2)$ the amplitude of the angle spanned by the vector $z(t)$ as t varies from t_1 to t_2, measured in clockwise sense. Thus we do not normalize any more by dividing by π, as we did in the previous sections.

Sketch of the proof. For simplicity we assume the uniqueness property for the Cauchy problems associated to (1) and divide the proof into several lemmas.

1. *The continuability of the solutions:* the sublinear growth of g at infinity implies that every maximal solution of (1) is defined on $\mathbb{R}$.

2. *There is* $\nu > 1/2$ *such that for every* $R_1 > R_0$ *there exists* $R_2 > R_1$ *such that, if* $z(t) = (x(t), y(t))$ *is any solution of* (3) *satisfying* $\|z(t_1)\| = R_1$, $\|z(t_2)\| = R_2$ *(or* $\|z(t_2)\| = R_1$, $\|z(t_1)\| = R_2$*) and* $R_1 \leq \|z(t)\| \leq R_2$, *for all* $t \in [t_1, t_2]$, *it follows that:*

$$rot(z; t_1, t_2) > \nu 2\pi.$$

This lemma can be proved by arguments similar to those used in [44, 35, 32].

3. *Iteration of Step 2:* let us write $\nu = \delta + 1/2$, so that $\delta > 0$; we fix $R_1 > R_0$ and apply Step 2 obtaining $R_2 > R_1$; then we apply again Step 2 with R_2 in place of R_1 obtaining $R_3 > R_2$. Let z be a solution of (3) such that $\|z(t_1)\| = R_3$, $\|z(t_2)\| = R_1$ and $R_1 \leq \|z(t)\| \leq R_3$ for $t \in [t_1, t_2]$, and consider the first instant s_1 and the last instant s_2 in $[t_1, t_2]$ such that $\|z(s)\| = R_2$. Since the trajectories of (3) cross the positive y-axis from the left to the right hand side and the negative y-axis from the right to the left one, it is easy to see that actually $\mathrm{rot}(z; s_1, s_2) > -\pi$, therefore we obtain:

$$\begin{aligned}\mathrm{rot}(z; t_1, t_2) &= \mathrm{rot}(z; t_1, s_1) + \mathrm{rot}(z; s_1, s_2) + \mathrm{rot}(z; s_2, t_2) \\ &> \nu 2\pi - \pi + \nu 2\pi = \left(\frac{1}{2} + 2\delta\right) 2\pi.\end{aligned}$$

   Therefore, for every $j > 0$, it is possible to find sufficiently large annuli such that every solution which crosses them must rotate around the origin at least j times.

4. *If $\mathcal{A}$ is a sufficiently large annulus and z is a solution such that $z(t) \in \mathcal{A}$ for all $t \geq t_0$, then:*
$$\mathit{rot}(z; t_0, t) \to +\infty \qquad \text{as } t \to +\infty,$$
   uniformly with respect to $t_0 \in [0, T]$.

5. *Large solutions rotate little:* using the sublinear condition at infinity it is possible to show that for every $L > 0$ there is $\widehat{R}_L > R_0$ such that, if $0 < t_1 - t_2 \leq L$ and z is any solution satisfying $\|z(t)\| \geq \widehat{R}_L$ for all $t \in [t_1, t_2]$, then:
$$\mathrm{rot}(z; t_1, t_2) < 2\pi.$$

Now, let us fix j and, by Step 3, consider $R_0 < R_1 < R_2 < R_3$ such that each solution crossing either $B[R_2] \setminus B(R_1)$ or $B[R_3] \setminus B(R_2)$ turns at least $j + 1$ times around the origin. Let $\mathcal{A} = B[R_3] \setminus B(R_1)$. By Step 4, there is m_j^* such that:

$$m \geq m_j^* \qquad \Longrightarrow \qquad \mathrm{rot}(z; 0, mT) > j2\pi \quad \text{if } R_1 \leq \|z(t)\| \leq R_3 \quad \forall t \in [0, mT].$$

Consider any solution with $\|z(0)\| = R_2$: either $z(t)$ remains in $\mathcal{A}$ for all $t \in [0, mT]$ or there is a first instant $\hat{t}$ in which the solution z exits the annulus $\mathcal{A}$. In the former case we already know that $\mathrm{rot}(z; 0, mT) > j2\pi$; in the latter one we can select an interval $[t_1, t_2] \subset [0, mT]$ such that:

- either $\|z(t_1)\| = R_2$, $\|z(t_2)\| = R_1$ and $R_1 \leq \|z(t)\| \leq R_2$ for all $t \in [t_1, t_2]$
- or $\|z(t_1)\| = R_2$, $\|z(t_2)\| = R_3$ and $R_2 \leq \|z(t)\| \leq R_3$ for all $t \in [t_1, t_2]$.

In both these situations we can conclude that $\mathrm{rot}(z; t_1, t_2) > (j+1)2\pi$ by the choice of R_1, R_2 and R_3. Therefore, arguing as in Step 3, we conclude again that $\mathrm{rot}(z; 0, mT) > j2\pi$. We can summarize this by the following implication:

$$\|z(0)\| = R_2 \quad \Longrightarrow \quad \mathrm{rot}(z; 0, mT) > j2\pi.$$

Let us fix now $m \geq m_j^*$ and apply Step 5 with $L = mT$: we get $S_1 \geq R_2$ such that the conclusion of Step 5 holds if $\|z(t)\| \geq S_1$ for all $[0, mT]$. By the continuability of all the solutions of (3) (Step 1), it is possible to find $S_2 \geq S_1$ such that $\|z(t)\| \geq S_1$ for every $t \in [0, mT]$, if $\|z(0)\| = S_2$. Hence:

$$\|z(0)\| = S_2 \quad \Longrightarrow \quad \mathrm{rot}(z; 0, mT) < 2\pi.$$

Finally, consider the mT-Poincaré map:

$$B(S_2) \setminus B[R_2] \ni p \mapsto z(mT; 0, p)$$

whose fixed points are the mT-periodic solutions of (3). It turns out that the mT-Poincaré map satisfies the Poincaré–Birkhoff fixed point theorem by the discussion carried above, and, therefore, it has a fixed point such that the corresponding mT-periodic solution rotate exactly j times around the origin in $[0, mT]$ and, hence, has exactly $2j$ zeros in $[0, mT[$.

□

We remark that if, j and m are coprime numbers and x is the mT-periodic solution of (1) given by Theorem 4 with these choices, then it turns out that mT is actually the minimal period of x in the class of the integral multiples of T.

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