Rend. Sem. Mat. Univ. Pol. Torino
Vol. 59, 3 (2001)

J. S. Aujla - M. S. Rawla *

SOME RESULTS ON OPERATOR MEANS AND SHORTED OPERATORS

Abstract. We prove some results on operator equalities and inequalities involving positive maps, operator means and shorted operators. Inequalities for shorted operators involving convex operator functions and tensor product have also been proved.

1. Introduction

With a view to studying electrical network connections, Anderson and Duffin [2] introduced the concept of parallel sum of two positive semidefinite matrices. Subsequently, Anderson [1] defined a matrix operation, called shorted operation to a subspace, for each positive semidefinite matrix. If A and B are impedance matrices of two resistive n-port networks, then their parallel sum $A : B$ is the impedance matrix of the parallel connection. If ports are partitioned to a group of s ports and to the remaining group of $n - s$ ports, then the shorted matrix $A_{\mathcal{S}}$ to the subspace $\mathcal{S}$ spanned by the former group is the impedance matrix of the network obtained by shorting the last $n - s$ ports.

Anderson and Trapp [3] have extended the notions of parallel addition and shorted operation to bounded linear positive operators on a Hilbert space $\mathcal{H}$ and demonstrated its importance in operator theory. They have studied fundamental properties of these operations and their interconnetions.

The axiomatic theory for connections and means for pairs of positive operators have been developed by Nishio and Ando [12] and Kubo and Ando [11]. This theory has found a number of applications in operator theory.

In Section 2, we shall study when the equalities of the type $\phi(A\sigma B) = \phi(A)\sigma\phi(B)$ hold for a connection σ, positive operators A, B on a Hilbert space $\mathcal{H}$, and positive map ϕ. In these results ϕ is not assumed to be linear. In Section 3, we shall obtain some operator inequalities involving shorted operators and convex operator functions. An inequality for shorted operation of tensor product of two positive operators has also been proved in this section.

*The authors would like to thank a referee for pointing out a mistake in the earlier version of this paper and for giving useful suggestions.

2. Positive Maps And Operator Means

In what follows, $\mathcal{S}(\mathcal{H})$ shall denote the set of positive linear operators on a Hilbert space $\mathcal{H}$, whereas $\mathcal{P}(\mathcal{H})$ shall denote the set of positive linear invertible operators on $\mathcal{H}$. An operator connection σ according to Kubo and Ando [11] is defined as a binary operation among positive operators satisfying the following axioms:

monotonicity:

$$A \leq C,\ B \leq D \text{ imply } A\sigma B \leq C\sigma D,$$

upper continuity:

$$A_n \downarrow A \text{ and } B_n \downarrow B \text{ imply } (A_n\sigma B_n) \downarrow (A\sigma B),$$

transformer inequality:

$$T^*(A\sigma B)T \leq (T^*AT)\sigma(T^*BT).$$

A mean is a connection with normalization condition

$$A\sigma A = A.$$

The main result of Kubo-Ando theory is the order isomorphism between the class of connections and the class of positive operator monotone functions on $\mathbb{R}_+$. This isomorphism $\sigma \leftrightarrow f$ is characterized by the relation

$$A\sigma B = A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2}$$

for all $A, B \in \mathcal{P}(\mathcal{H})$. The operator monotone function f is called the representing function of σ.

The operator connection corresponding to operator monotone function $f(x) = s + tx$, $s, t > 0$, is denoted by $\triangledown_{s,t}$. $\triangledown_{1/2,1/2}$ is called the arithmetic mean and is denoted by $\triangledown$. The operator mean corresponding to the operator monotone function $x \to x^{1/2}$ is called the geometric mean and is denoted by #. The operator connection corresponding to the operator monotone function $x \to \frac{x}{s+tx}$, $s, t > 0$, is denoted by $!_{s,t}$. $!_{1/2,1/2}$ is called the harmonic mean and is denoted by !.

The transpose σ' of a connection σ is defined by $A\sigma' B = B\sigma A$. For a connection σ, the adjoint σ^* and the dual $\sigma^\perp$ are respectively defined by $A\sigma^* B = (A^{-1}\sigma B^{-1})^{-1}$ and $A\sigma^\perp B = (B^{-1}\sigma A^{-1})^{-1}$ for all $A, B \in \mathcal{P}(\mathcal{H})$. These definitions extend to $\mathcal{S}(\mathcal{H})$ by continuity. A connection σ is called symmetric if $\sigma' = \sigma$, selfadjoint if $\sigma^* = \sigma$ and selfdual if $\sigma^\perp = \sigma$. It follows that if f is the representing function of σ then $xf(x^{-1})$ is the representing function of σ', $(f(x^{-1}))^{-1}$ is the representing function of σ^* and $x(f(x))^{-1}$ is the representing function of $\sigma^\perp$. $\triangledown$, # and ! are examples of symmetric means. $\triangledown$ and ! are adjoints of each other while # is selfadjoint. Moreover it follows that # is the only operator mean which is the dual of itself.

By a positive map, we mean a mapping from the set of bounded linear operators on a Hilbert space $\mathcal{H}$ to the set of bounded linear operators on a Hilbert space $\mathcal{K}$ which maps positive intertible operators into positive invertible operators. A map ϕ is called unital if $\phi(I) = I$.

In [10], it is proved that if ϕ is a C*-homomorphism, then

$$\phi(A\sigma B) = \phi(A)\sigma\phi(B)$$

for any operator mean σ, here we shall obtain similar type of results for a positive map ϕ.

THEOREM 1. *Let ϕ be a positive map such that*

$$\phi(A\#B) = \phi(A)\#\phi(B)$$

for all $A, B \in \mathcal{P}(\mathcal{H})$. Then

$$\phi(A\sigma B) = \phi(A)\sigma\phi(B)$$

implies

$$\phi(A\sigma^{\perp}B) = \phi(A)\sigma^{\perp}\phi(B)$$

for all connections σ.

Proof. The equality

$$(A\sigma B)\#(A\sigma^{\perp}B) = A\#B$$

implies

$$\begin{aligned}
\phi(A\sigma B)\#\phi(A\sigma^{\perp}B) &= \phi(A\#B)\\
&= \phi(A)\#\phi(B)\\
&= (\phi(A)\sigma\phi(B))\#(\phi(A)\sigma^{\perp}\phi(B))\\
&= \phi(A\sigma B)\#(\phi(A)\sigma^{\perp}\phi(B))
\end{aligned}$$

which further implies

$$\phi(A\sigma^{\perp}B) = \phi(A)\sigma^{\perp}\phi(B),$$

since $A\#B = A\#C$ implies $B = C$.

□

REMARK 1. It is not always true that the inequality $A\#B \leq A\#C$ implies $B \leq C$. Indeed, let $A = I$, $B = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$, $C = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$. Then the inequality $A\#B \leq A\#C$ is satisfied. However, $B \leq C$ is not true.

THEOREM 2. *Let ϕ be a unital positive map. Then any two of the following conditions imply the third:*

(i) $\phi(A^{-1}) = (\phi(A))^{-1}$ for all $A \in \mathcal{P}(\mathcal{H})$.

(ii) $\phi(A \triangledown_{s,t} B) = \phi(A) \triangledown_{s,t} \phi(B)$ for all $A, B \in \mathcal{P}(\mathcal{H})$ and $s, t > 0$.

(iii) $\phi(A!_{s,t}B) = \phi(A)!_{s,t}\phi(B)$ for all $A, B \in \mathcal{P}(\mathcal{H})$ and $s, t > 0$.

Proof. (i) and (ii) imply (iii): Observe that

$$\begin{aligned}\phi(A^{-1} \bigtriangledown_{s,t} B^{-1}) &= \phi(A^{-1}) \bigtriangledown_{s,t} \phi(B^{-1}) \\ &= (\phi(A))^{-1} \bigtriangledown_{s,t} (\phi(B))^{-1}.\end{aligned}$$

The above equality implies

$$\begin{aligned}\phi(A!_{s,t}B) &= \phi((A^{-1} \bigtriangledown_{s,t} B^{-1})^{-1}) \\ &= (\phi(A^{-1} \bigtriangledown_{s,t} B^{-1}))^{-1} \\ &= \phi(A)!_{s,t}\phi(B).\end{aligned}$$

(i) and (iii) imply (ii):

$$\begin{aligned}\phi(A^{-1}!_{s,t}B^{-1}) &= \phi(A^{-1})!_{s,t}\phi(B^{-1}) \\ &= (\phi(A))^{-1}!_{s,t}(\phi(B))^{-1}.\end{aligned}$$

Consequently,

$$\begin{aligned}(\phi(A \bigtriangledown_{s,t} B))^{-1} &= \phi((A \bigtriangledown_{s,t} B)^{-1}) \\ &= \phi(A^{-1}!_{s,t}B^{-1}) \\ &= (\phi(A))^{-1}!_{s,t}(\phi(B))^{-1} \\ &= (\phi(A) \bigtriangledown_{s,t} \phi(B))^{-1}.\end{aligned}$$

Thus

$$\phi(A \bigtriangledown_{s,t} B) = \phi(A) \bigtriangledown_{s,t} \phi(B).$$

(ii) and (iii) imply (i):

The equality

$$I!A + I!A^{-1} = 2I$$

implies

$$I!\phi(A) + I!\phi(A^{-1}) = 2I,$$

i.e.,

$$(I + (\phi(A))^{-1})^{-1} + (I + (\phi(A^{-1}))^{-1})^{-1} = I,$$

which implies

$$(I + (\phi(A))^{-1}) + (I + (\phi(A^{-1}))^{-1}) = (I + (\phi(A))^{-1})(I + (\phi(A^{-1}))^{-1}).$$

Consequently,

$$\phi(A)\phi(A^{-1}) = I.$$

Hence

$$\phi(A^{-1}) = (\phi(A))^{-1}.$$

$\square$

REMARK 2. Note that in Theorem 2 to prove (ii) and (iii) imply (i) we use (ii) and (iii) for particular choice of s, t when $s = t = \frac{1}{2}$.

COROLLARY 1. *Let ϕ be a unital positive map such that*
(i) $\phi(A \bigtriangledown B) = \phi(A) \bigtriangledown \phi(B)$ for all $A, B \in \mathcal{P}(\mathcal{H})$,
(ii) $\phi(A!B) = \phi(A)!\phi(B)$ for all $A, B \in \mathcal{P}(\mathcal{H})$.
Then

$$\phi(A^2) = (\phi(A))^2$$

for all $A \in \mathcal{P}(\mathcal{H})$.

Proof. For a fixed $A \in \mathcal{P}(\mathcal{H})$, consider the map ψ defined on $\mathcal{P}(\mathcal{H})$ by

$$\psi(X) = (\phi(A))^{-1/2}\phi(A^{1/2}XA^{1/2})(\phi(A))^{-1/2}.$$

Then

$$\begin{aligned} \psi(I) &= I, \\ \psi(X \bigtriangledown Y) &= \psi(X) \bigtriangledown \psi(Y), \\ \psi(X!Y) &= \psi(X)!\psi(Y), \end{aligned}$$

since ϕ satisfies these. Therefore by Theorem 2 and Remark 2,

$$\psi(A^{-1}) = (\psi(A))^{-1},$$

i.e.,

$$(\phi(A))^{-1} = (\phi(A))^{1/2}(\phi(A^2))^{-1}(\phi(A))^{1/2},$$

which gives the desired equality

$$\phi(A^2) = (\phi(A))^2.$$

□

COROLLARY 2. *Let ϕ be a unital positive map. Then any two of the following conditions imply the third:*
(i) $\phi(A\#B) = \phi(A)\#\phi(B)$ for all $A, B \in \mathcal{P}(\mathcal{H})$.
(ii) $\phi(A \bigtriangledown_{s,t} B) = \phi(A) \bigtriangledown_{s,t} \phi(B)$ for all $A, B \in \mathcal{P}(\mathcal{H})$ and $s, t > 0$.
(iii) $\phi(A!_{s,t}B) = \phi(A)!_{s,t}\phi(B)$ for all $A, B \in \mathcal{P}(\mathcal{H})$ and $s, t > 0$.

Proof. The implications (i) and (ii) imply (iii), and (i) and (iii) imply (ii) follows from Theorem 1.

(ii) and (iii) imply (i):

If ψ is the map considered in Corollary 1, then

$$\begin{aligned}\psi(I) &= I,\\ \psi(X \bigtriangledown Y) &= \psi(X) \bigtriangledown \psi(Y),\\ \psi(X!Y) &= \psi(X)!\psi(Y).\end{aligned}$$

Therefore, by Corollary 1,

$$\psi(X^2) = (\psi(X))^2.$$

Using that $x \to x^{1/2}$ is operator monotone on $(0, \infty)$, we obtain

$$\psi(X^{1/2}) = (\psi(X))^{1/2},$$

i.e.,

$$\psi(I\#X) = I\#\psi(X)$$

for all $X \in \mathcal{P}(\mathcal{H})$. Now

$$\begin{aligned}\phi(A\#B) &= \phi(A^{1/2}(I\#(A^{-1/2}BA^{-1/2}))A^{1/2})\\ &= (\phi(A))^{1/2}\psi(I\#(A^{-1/2}BA^{-1/2}))(\phi(A))^{1/2}\\ &= (\phi(A))^{1/2}(I\#\psi(A^{-1/2}BA^{-1/2}))(\phi(A))^{1/2}\\ &= \phi(A)\#\phi(B).\end{aligned}$$

$\square$

3. Shorted Operators and Operator Means

Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, the shorted operator $A_{\mathcal{S}}$ of a positive operator A to $\mathcal{S}$ is defined as:

$$A_{\mathcal{S}} = \max\{D : 0 \leq D \leq A,\ Ran(D) \subseteq \mathcal{S}\}.$$

The existence of such a maximum is guaranteed by Anderson and Trapp [3]. The operation $A \to A_{\mathcal{S}}$ is called the shorted operation. The shorted operation has the following properties [3]:

(i) $A_{\mathcal{S}} \leq A$,

(ii) $(\alpha A)_{\mathcal{S}} = \alpha A_{\mathcal{S}}$ for $\alpha \geq 0$,

(iii) $(A_{\mathcal{S}})_{\mathcal{S}} = A_{\mathcal{S}}$,

(iv) $A_{\mathcal{S}} + B_{\mathcal{S}} \leq (A + B)_{\mathcal{S}}$.

The parallel addition $A : B = (A^{-1} + B^{-1})^{-1}$ for $A, B \in \mathcal{P}(\mathcal{H})$ is the operator connection corresponding to the operator monotone function $x \to \frac{x}{1+x}$, $x > 0$. Thus $A : B = \frac{1}{2}(A!B)$. The important interconnections between parallel addition and shorted operator were established by Anderson and Trapp [3]:

$$\lim_{\alpha\to\infty}(A : \alpha P) = A_{\mathcal{S}} \tag{1}$$

where P is the projection to the subspace $\mathcal{S}$. An important consequence of (1) is the commutativity of parallel addition and shorted operation:

$$(A : B)_{\mathcal{S}} = A_{\mathcal{S}} : B = A : B_{\mathcal{S}}.$$

Our first result of this section is an inequality involving operator convex function and shorted operator.

THEOREM 3. *Let f be a strictly increasing operator convex function on* $[0, \infty)$ *with* $f(0) = 0$ *and* $f(x^{-1}) = (f(x))^{-1}$ *for all* $x > 0$. *Then*

$$(f(A))_{\mathcal{S}} \leq f(A_{\mathcal{S}}),$$

for all $A \in \mathcal{P}(\mathcal{H})$.

Proof. Let P be a projection onto $\mathcal{S}$ and $\alpha > 1$. Then for all $\epsilon > 0$, we have

$$\begin{aligned}
f((1-\alpha^{-1})^{-1}A : \alpha(P+\epsilon)) &= f([(1-\alpha^{-1})A^{-1} + \alpha^{-1}(P+\epsilon)^{-1}]^{-1}) \\
&= [f((1-\alpha^{-1})A^{-1} + \alpha^{-1}(P+\epsilon)^{-1})]^{-1} \\
&\geq [((1-\alpha^{-1})f(A^{-1}) + \alpha^{-1}f((P+\epsilon)^{-1})]^{-1} \\
&= (1-\alpha^{-1})^{-1}f(A) : \alpha f(P+\epsilon).
\end{aligned}$$

On taking the limit when $\epsilon \to 0$, we get

$$f((1-\alpha^{-1})^{-1}A : \alpha P) \geq (1-\alpha^{-1})^{-1}f(A) : \alpha f(P). \tag{2}$$

Since $f(0) = 0$ and $f(1) = 1$, we have, $f(P) = P$. Also note that for any $X \in \mathcal{P}(\mathcal{H})$ and for any projection P

$$(1-\alpha^{-1})^{-1}X : \alpha P = (1-\alpha^{-1})^{-1}[X : (\alpha-1)P] = (1+\gamma^{-1})[X : \gamma P] \tag{3}$$

where $\gamma = \alpha - 1$. Now on taking limit when $\alpha \to \infty$ in inequality (2) and using the identities (3) and (1), we obtain

$$f(A_{\mathcal{S}}) \geq (f(A))_{\mathcal{S}}.$$

This completes the proof. □

Since the function $x \to x^r,\ 1 \leq r \leq 2$ is operator convex on $[0, \infty)$, we have the following corollary:

COROLLARY 3. *Let* $A \in \mathcal{P}(\mathcal{H})$. *Then*

$$(A^r)_{\mathcal{S}} \leq (A_{\mathcal{S}})^r$$

for all $1 \leq r \leq 2$.

Since a positive operator concave function on $[0, \infty)$ is operator monotone and hence is strictly increasing, one can prove the following theorem by an argument similar to that used in Theorem 3.

THEOREM 4. *Let f be a positive operator concave function on $[0, \infty)$ with $f(0) = 0$ and $f(x^{-1}) = (f(x))^{-1}$ for all $x > 0$. Then*

$$(f(A))_{\mathcal{S}} \geq f(A_{\mathcal{S}}),$$

for all $A \in \mathcal{P}(\mathcal{H})$.

COROLLARY 4. *Let $A \in \mathcal{P}(\mathcal{H})$. Then*

$$(A^r)_{\mathcal{S}} \geq (A_{\mathcal{S}})^r$$

for all $0 \leq r \leq 1$.

Proof. Since the function $x \to x^r,\ 0 \leq r \leq 1$ is operator concave on $[0, \infty)$, one have the desired inequality by Theorem 4.

□

Let e_i be a complete orthonormal system for $\mathcal{H}$. Then for operators A, B on $\mathcal{H}$, their tensor product $A \otimes B$ is determined by

$$\langle (A \otimes B)(e_i \otimes e_j), e_k \otimes e_l \rangle = \langle Ae_i, e_k \rangle \langle Be_j, e_l \rangle.$$

We have the following theorem:

THEOREM 5. *Let $A, B \in \mathcal{S}(\mathcal{H})$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then*

$$(A \otimes B)_{\mathcal{S} \otimes \mathcal{S}} \geq A_{\mathcal{S}} \otimes B_{\mathcal{S}}.$$

Proof. Indeed, by definition

$$A_{\mathcal{S}} = \max\{D : 0 \leq D \leq A,\ Ran(D) \subseteq \mathcal{S}\} = \max \sum\nolimits_1$$

and

$$B_{\mathcal{S}} = \max\{D : 0 \leq D \leq B,\ Ran(D) \subseteq \mathcal{S}\} = \max \sum\nolimits_2 .$$

Let $D_1 \in \sum_1$ and $D_2 \in \sum_2$. Then it is clear that

$$D_1 \otimes D_2 \in \sum\nolimits_1 \otimes \sum\nolimits_2 \subseteq \max\{D : 0 \leq D \leq A \otimes B\ Ran(D) \subseteq \mathcal{S} \otimes \mathcal{S}\},$$

since $Ran(D_1 \otimes D_2) \subseteq \mathcal{S} \otimes \mathcal{S}$ and $0 \leq D_1 \otimes D_2 \leq A \otimes B$. Consequently,

$$(A \otimes B)_{\mathcal{S} \otimes \mathcal{S}} \geq A_{\mathcal{S}} \otimes B_{\mathcal{S}}.$$

This completes the proof of the theorem.

□

References

[1] ANDERSON W. N. JR., *Shorted Operators*, Siam J. Appl. Math. **20** (1971), 520–525.

[2] ANDERSON W. N. JR. AND DUFFIN R. J., *Series and parallel addition of matrices*, J. Math. Anal. Appl. **26** (1969), 576–594.

[3] ANDERSON W. N. JR. AND TRAPP G. E., *Shorted Operators II*, Siam J. Appl. Math. **28** (1975), 60–71.

[4] ANDERSON W. N. JR. AND TRAPP G. E., *A class of monotone operator functions related to electrical network theory*, Linear Alg. Appl. **15** (1976), 53–67.

[5] ANDO T., *Topics on operator inequalities*, Lecture notes, Hokkaido University, Sapporro 1978.

[6] AUJLA J. S., *Some operator equalities involving connections and means*, Linear Alg. Appl. **258** (1997), 223–228.

[7] AUJLA J. S. AND VASUDEVA H. L., *Operator inequalities related to means of operators*, Math. Japon. **41** (1995), 383–388.

[8] FILLMORE P. A. AND WILLIAMS J. P., *On operator ranges*, Advances in Math. **7** (1971), 254–281.

[9] FUJII J. I., *The Markus-Khan Theorem for Hilbert space operators*, Math. Japon. **41** (1995), no. 3, 531–535.

[10] KAINUMA D. AND KAMEI E., *C^*-homomorphisms and operator means*, Math. Japon. **28** (1983), 627–631.

[11] KUBO F. AND ANDO T., *Means of positive linear operators*, Math. Ann. **246** (1980), 205–224.

[12] NISHIO K. AND ANDO T., *Characterizations of operations derived from network connections*, J. Math. Anal. Appl. **53** (1976), 539–549.

AMS Subject Classification: 47A64, 47A63.

Jaspal Singh AUJLA
Department of Mathematics
National Institute of Technology
144011 Jalandhar
Punjab, INDIA

Mandeep Singh RAWLA
Department of Mathematics
Panjab University
Chandigarh, INDIA

Lavoro pervenuto in redazione il 02.11.1999 e, in forma definitiva, il 10.02.2001.