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M. Casanellas

GLICCI VERSUS GLICOG

Abstract. We discuss the problem of whether arithmetically Gorenstein schemes are in the Gorenstein liaison class of a complete intersection. We present some axamples of arithmetically Gorenstein schenes that are indeed in the Gorenstein liaison class of a complete intersection.

In the recent research on Gorenstein liaison theory, the question whether any arithmetically Cohen-Macaulay scheme is in the Gorenstein liaison class of a complete intersection, has been of main importance. The intention of this question is to fully generalize Gaeta theorem to arbitrary codimension. Gaeta theorem says that in codimension 2 any arithmetically Cohen-Macaulay scheme is in the complete intersection liaison class of a complete intersection. As it has been shown in the recent papers on liaison theory, it is likely that in arbitrary codimension Gorenstein liaison behaves better than complete intersection liaison. Indeed, some of the theorems of complete intersection liaison in codimension 2, do not hold for complete intersection liaison in arbitrary codimension but hold when we link by means of arithmetically Gorenstein schemes instead of complete intersection schemes. So it is natural to ask if any arithmetically Cohen-Macaulay scheme is in the Gorenstein liaison class of a complete intersection or briefly *glicci*.

Since complete intersection schemes have been changed to arithmetically Gorenstein schemes in the study of liaison theory in arbitrary codimension, it is natural to ask why we formulate the question *"(1) Is any arithmetically Cohen-Macaulay scheme glicci?"* instead of formulating first the weaker question *"(2) Is any arithmetically Cohen-Macaulay scheme in the Gorenstein liaison class of an arithmetically Gorenstein scheme?"* If we use the acronym *glicog* for the schemes that are in the **G**orenstein **Li**aison **C**lass **O**f an arithmetically **G**orenstein scheme, then Question (2) asks if any arithmetically Cohen-Macaulay scheme is glicog.

If Question (1) could be answered affirmatively, then Question (2) would also have an affirmative answer. Moreover, if it could be proved that any arithmetically Gorenstein scheme is glicci, then both questions would be equivalent. So one of the first questions that Gorenstein liaison has to address is *"(3) Is any arithmetically Gorenstein scheme glicci?"* This is still an open question. The purpose of this note is to present some results that answer this question affirmatively in some cases.

First of all, the main result that gives a partial affirmative answer to Question (3) is that any arithmetically Gorenstein scheme of codimension 3 is *licci*, i.e. it is in the complete intersection liaison class of a complete intersection (in particular it is glicci). This result is attributed to Watanabe because it can be derived from the proof of his main theorem in [4]. It is known that in higher codimension there exist arithmetically Gorenstein schemes that are not licci, but it is not known if they are glicci or not.

To study Gorenstein liaison classes of schemes of arbitrary codimension, we have used the results of [3] where there are given useful methods to study Gorenstein liaison classes of divisors on arithmetically Cohen-Macaulay schemes. In this setting, we have studied the Gorenstein liaison classes of divisors on rational normal scrolls and we can apply our results to arithmetically Gorenstein schemes. First of all we have proved:

THEOREM 1 ([1], THEOREM 3.3.1). *Let X and X' be two effective divisors on a smooth rational normal scroll $S \subset \mathbb{P}^n$. Then, X and X' have isomorphic deficiency modules (up to shift and dual) if and only if they belong to the same G-liaison class. In particular, any arithmetically Cohen-Macaulay divisor on a rational normal scroll is glicci.*

It has been also proved in [1] that any arithmetically Cohen-Macaulay divisor on a rational normal scroll, not necessarily smooth, is glicci (this was first proved for divisors on rational normal scroll surfaces in [2] and then generalized in [1], Theorem 3.2.3). In particular, any arithmetically Gorenstein divisor on a rational normal scroll is glicci.

This result can be applied to arithmetically Cohen-Macaulay varieties that are known to be divisors on rational normal scrolls: varieties of maximum genus, varieties of maximum Castelnuovo-Mumford regularity, integral rational curves, elliptic linearly normal curves, hyperelliptic linearly normal curves ... (see [1]). In particular, this result can be applied to arithmetically Gorenstein schemes satisfying one of these conditions. As another consequence of this result we have that arithmetically Cohen-Macaulay K3 surfaces, which are arithmetically Gorenstein schemes, are glicci:

COROLLARY 1 ([1], COROLLARY 3.5.11). *Let $X \subset \mathbb{P}^n$ be a linearly normal smooth arithmetically Cohen-Macaulay K3 surface such that* $\deg X \geq 8$ *and the generic member of* $|H_X|$ *is a smooth non hyperelliptic curve. Assume that X contains an irreducible elliptic cubic curve E (or, equivalently, $I(X)$ is not generated by quadrics). Then X is arithmetically Gorenstein and it is glicci.*

As one of the main problems of Gorenstein liaison is the difficulty of constructing arithmetically Gorenstein schemes containing a given scheme such that it produces a useful Gorenstein link, it is thought that this is also the main problem to address Question (3) in full generality.

References

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Marta CASANELLAS
Dpt. Algebra i Geometria, Fac. Matemàtiques
Universitat de Barcelona
Gran Via 585
08007-Barcelona, SPAIN
e-mail: `casanell@mat.ub.es`

Present address
Departement of Mathematics
University of California
Berkeley, CA 94720-3840, USA