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THE STRUCTURE OF MATERIAL FORCES IN ELECTROMAGNETIC MATERIALS

Abstract. Material forces govern the behaviour and the evolution of a defect or of an inhomogeneity in a solid material. In elastic materials these forces are associated with the Eshelby tensor, as is known. In structured or micro-structured materials, an Eshelby–like stress can be assembled by following a simple rule. By appealing to this rule, one is able to propose an expression for the Eshelby tensor in electromagnetism.

A variational procedure is hereby expounded, from which an expression for the classical electromagnetic stress tensor, whose form is otherwise controversial, stems straightforwardly. The electromagnetic Eshelby–like tensor is derived on this base.

1. Introduction

Material forces and configurational forces are customarily understood as two synonymous which label the same notion. In the continuum framework, the configurational forces are usually associated with the energy–stress tensor and they acquire a special importance in structured materials. Most people, who are concerned with materials with an internal structure or with microstructures, are familiar with the notion of energy–stress. Such an energy–stress naturally appears in the theory whenever the material response depends on the gradient of the fields or of the microfields of interest. We shall stress out that there are two kinds (at least) of configurational forces and only one of the two is related to the notion of material force. The latter governs the behaviour of material defects or inhomogeneities [1, 2].

The Maxwell electromagnetism can be viewed as one of the first theories of a material endowed with a structure (i.e.the electromagnetic fields), although the Maxwell–Faraday's electromagnetic fields are defined also out of a body of finite extent. As the electromagnetic fields pervade the whole physical space, the mathematical problem for the electromagnetic materials has to be formulated not only in the domain occupied by the body, but also in the exterior domain, accordingly. The electromagnetic fields obey a set of differential equations that are in general coupled with the mechanical equations [3, 4]. Should the Maxwell equations possibly decouple from the mechanical ones, one could think to solve them first and afterward to look for the associated stress tensor, in order to enquire about the mechanical behaviour of the material.

Some people who are interested in liquid crystals share this attitude with those who deal with electromagnetic materials or with other sort of structured materials. However, in most of the cases, the form of the mechanical stress to associate with a structured material may be a controversial matter [5, 6, 7]. This is the case for electromagnetic materials [4]. It is worth to recalling that the quarrel, about the proper form the stress (and the momentum) should have in electromagnetic materials, is still unsettled [8]. As there is not a general agreement on the form of the electromagnetic stress, a challenge for new proposals is open.

Here, we discuss this point basing on a variational approach. In this approach, the Maxwell electromagnetic stress tensor will naturally stem from a Hamilton-like principle for a Lagrangian that we are going also to introduce. Along with the electromagnetic stress, other electromagnetic quantities of interest are derived. These quantities are useful for recovering the final form of the energy–stress tensor which is related to the *material forces* [9, 10]. This energy-stress definitely differs from the Maxwell energy–stress. Maxwell introduced, in his treatise, the electromagnetic stress tensor in order to evaluate the electromagnetic force acting over a body [11]. Thus, the Maxwell tensor is an energy-stress tensor that it is related to the classical notion of force, *not* to the *material forces*, as we will try to show in evidence.

Nonetheless, the procedure suggested by Maxwell in establishing the stress and the force acting on a material body is appealing, as it can be re–proposed in other fields of Continuum Mechanics or Continuum Physics. Eshelby [1] was the first who proposed to apply the Maxwell's procedure to elasticity, in order to evaluate the force acting upon a point-wise defect. In this respect, Eshelby introduced the notion of material force.

2. Maxwell equations in material form

Hereafter, we consider a solid body of infinite extent, which fills the whole physical space. This space is here represented by the Euclidean space E_3. The classical Maxwellian fields **E, D, B, H, P** and **M** (the electric field, the electric displacement, the magnetic induction, the magnetic field, the polarisation and the magnetisation, respectively) are *transformed* in a suitable chosen reference configuration of the deformable body, in the following fashion:

$$
\begin{aligned}
\mathfrak{E} &= \mathbf{F}^{\mathbf{T}}(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) = \mathbf{F}^{\mathbf{T}}\mathcal{E};\\
\mathfrak{D} &= J\mathbf{F}^{-1}\mathbf{D};\\
\mathfrak{B} &= J\mathbf{F}^{-1}\mathbf{B};\\
\mathfrak{H} &= \mathbf{F}^{\mathbf{T}}\mathcal{H} \equiv \mathbf{F}^{\mathbf{T}}\mathbf{H} + \mathbf{V} \wedge \mathbf{D};\\
\mathfrak{P} &= J\mathbf{F}^{-1}\mathbf{P};\\
\mathfrak{M} &= \mathbf{F}^{\mathbf{T}}\mathbf{M}
\end{aligned}
\tag{1}
$$

The introduction of the following auxiliary fields will be also useful:

$$
\begin{aligned}
\mathfrak{E}^* &= \mathbf{F}^{\mathbf{T}}\mathbf{E} \equiv \mathfrak{E} + \mathbf{V} \wedge \mathfrak{B};\\
\mathfrak{B}^* &= J\mathbf{F}^{-1}\mathcal{B} = J\mathbf{F}^{-1}[\mathbf{B} - (1/c^2)\mathbf{v} \wedge \mathbf{E}]\\
&\equiv \mathfrak{B} + (1/c^2)\mathbf{C}^{-1}(\mathbf{V} \wedge \varepsilon_0 J\mathbf{C}^{-1}\mathfrak{E}^*);\\
\mathfrak{M}^* &= \mathbf{F}^{\mathbf{T}}\mathcal{M} \equiv \mathfrak{M} - (\mathbf{V} \wedge \mathfrak{P}).
\end{aligned}
\tag{2}
$$

The transformation has to be understood as through the mapping $\chi : (\mathbf{X}, t) \rightarrow \mathbf{x}$, which is assumed here to be regular enough for our purposes.

X belongs to the reference configuration and **x** to the actual configuration of a body. $t \in \mathbf{R}$ represents the time. $\mathbf{F} = \nabla_R \chi$, where ∇_R stands for the spatial gradient in the referential frame. $J = det\,\mathbf{F}$. $\mathbf{F}^{\mathbf{T}}$ denotes the transpose of **F**. $\mathbf{v} = \dot{\mathbf{x}}$ and $\mathbf{V} = -\mathbf{F}^{-1}\mathbf{v}$. We also assume, as usual, that $J > 0$.

The fields introduced in (1) satisfy the Maxwell equations in the following form:

$$
\begin{aligned}
div_R\mathfrak{D} &= J\rho_e\\
div_R\mathfrak{B} &= 0\\
rot_R\mathfrak{E} + (\partial\mathfrak{B}/\partial t)|_X &= 0\\
rot_R\mathfrak{H} - (\partial\mathfrak{D}/\partial t)|_X &= \mathbf{g}.
\end{aligned}
\tag{3}
$$

div_R represents the divergence operator and rot_R the curl operator in the reference configuration; $\mathbf{g}$ is defined as follows:

$$\mathbf{g} = J\mathbf{F}^{-1}\mathbf{j} + J\rho_e\mathbf{V}. \tag{4}$$

ρ_e and $\mathbf{j}$ are the free charge and the free current densities, respectively, per unit volume of the current configuration.

The transformations of the electromagnetic fields, such as given by the relationships (1) and (2), provide the Maxwell equations (3) in the referential frame. These equations are clearly form–invariant.
The reader is referred to [10] for further details on this point.
For future use, we will introduce the following relationships:

$$\mathfrak{P} = \mathfrak{D} - \varepsilon_0 J\mathbf{C}^{-1}\mathfrak{E}^* \tag{5}$$

and

$$\mathfrak{M}^* = \mu_0 J^{-1}\mathbf{C}\mathfrak{B} - \mathfrak{H}. \tag{6}$$

where $\mathbf{C} \equiv \mathbf{F}^{\mathbf{T}}\mathbf{F}$. ϵ_0 and μ_0 are the electric permittivity and the magnetic permeability of a vacuum, respectively.

2.1. The material electromagnetic potentials

The classical electrodynamical potentials, namely the scalar potential Φ and the vector potential $\mathbf{A}$ are also transformed in two analogous fields ϕ and $\mathcal{A}$, respectively, in such a way that they are consistent with the equations (3) [10, 11, 12, 13]. More specifically, one introduces the vector field $\mathcal{A}$, (the vector potential in the material form) as follows:

$$rot_R\mathcal{A} = \mathfrak{B} \tag{7}$$

so that the equation $(3)_2$ is identically satisfied. It is worth to mentioning that $\mathcal{A}$ is uniquely defined, provided that the quantity $div_R\mathcal{A}$ is specified.
Basing on the equations $(3)_3$ and (7), one also introduces the material scalar potential ϕ, so that

$$\mathfrak{E} = -\nabla_R\phi - \dot{\mathcal{A}} \tag{8}$$

The superposed dot on $\mathcal{A}$ denotes the total time derivative of $\mathcal{A}$.

The equations (3) can be now written in terms of ϕ, $\mathcal{A}$, $\mathfrak{P}$ and $\mathfrak{M}^*$, by taking into account the equations (1), (4), (5), (6), (7) and (8). Hereafter, we will be concerned with this form of the Maxwell equations, which is known as the Lorentz form [4, 12].

It is worth to recalling that, had we dealt with bounded domains, the fields $\mathfrak{P}$ and $\mathfrak{M}^*$ would have been identically vanishing out of these domains.

3. A Lagrangian approach

Motivations for introducing the Lagrangian density in the material form, such as written below, will not be reported here as they are illustrated in [10]. Here, we only remark that such a Lagrangian provides the equations (3), in the Lorentz form. This Lagrangian reads:

$$\begin{aligned} L \;=\;& \frac{1}{2}\{\varepsilon_0 J\mathfrak{E}^*\cdot\mathbf{C}^{-1}\mathfrak{E}^* - (\mu_0 J)^{-1}\mathfrak{B}\cdot\mathbf{C}\mathfrak{B}\} + \mathfrak{P}\cdot\mathfrak{E} + \mathfrak{M}^*\cdot\mathfrak{B} + \frac{\rho_0 v^2}{2} \\ -\;& W(\mathbf{F}, \mathbf{F}\mathfrak{P}, J\mathbf{F}^{-1\mathbf{T}}\mathfrak{M}^*, \mathbf{X}). \end{aligned} \tag{9}$$

L is a Lagrangian density per unit volume of the reference configuration and possibly is of the following form:

$$L = \hat{L}(\phi, \dot{\phi}, \nabla_R \phi, \mathcal{A}, \dot{\mathcal{A}}, \nabla_R \mathcal{A}, \mathfrak{P}, \dot{\mathfrak{P}}, \nabla_R \mathfrak{P}, \mathfrak{M}^*, \dot{\mathfrak{M}}^*, \nabla_R \mathfrak{M}^*, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{F}, \mathbf{X})\ . \tag{10}$$

The corresponding Lagrange equations read:

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} + \mathrm{div}_R \frac{\partial L}{\partial \nabla_R \phi} = 0,\\
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathcal{A}}} - \frac{\partial L}{\partial \mathcal{A}} + \mathrm{div}_R \frac{\partial L}{\partial \nabla_R \mathcal{A}} = 0,\\
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} - \frac{\partial L}{\partial \mathbf{x}} + \mathrm{div}_R \frac{\partial L}{\partial \mathbf{F}} = 0,\\
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathfrak{P}}} - \frac{\partial L}{\partial \mathfrak{P}} + \mathrm{div}_R \frac{\partial L}{\partial \nabla_R \mathfrak{P}} = 0,\\
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathfrak{M}}^*} - \frac{\partial L}{\partial \mathfrak{M}^*} + \mathrm{div}_R \frac{\partial L}{\partial \nabla_R \mathfrak{M}^*} = 0.
\end{aligned} \tag{11}$$

With reference to the expression (9), one notes that:

$$\begin{aligned}
&\frac{\partial L}{\partial \phi} = 0,\\
&\frac{\partial L}{\partial \mathcal{A}} = 0,\\
&\frac{\partial L}{\partial \dot{\phi}} = 0.
\end{aligned} \tag{12}$$

The equations $(11)_1$ and $(11)_2$ simplify accordingly and provide, as a final result, two of the Maxwell equations of interest, in the Lorentz form.

For sake of simplicity, we also assume that

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\mathfrak{P}}} &= 0,\\
\frac{\partial L}{\partial \dot{\mathfrak{M}}^*} &= 0,\\
\frac{\partial L}{\partial \nabla_R \mathfrak{P}} &= 0,\\
\frac{\partial L}{\partial \nabla_R \mathfrak{M}^*} &= 0.
\end{aligned} \tag{13}$$

In accordance with this assumption, the equations $(11)_4$ and $(11)_5$ reduce to the following algebraic equations:

$$\begin{aligned}
&-\frac{\partial W}{\partial \mathfrak{P}} = \mathfrak{E},\\
&\frac{\partial W}{\partial \mathfrak{M}^*} = \mathfrak{B},
\end{aligned} \tag{14}$$

which happen to correspond to the classical constitutive equations. The equation $(11)_3$ has a natural mechanical interpretation, according to which the quantity $\partial L/\partial \dot{\mathbf{x}}$ represents the *momentum*.

4. The electromagnetic stress tensor and momentum

With reference to the expression (9) the momentum density explicitly reads:

$$\frac{\partial L}{\partial \dot{\mathbf{x}}} = \rho_0 \mathbf{v} - (\mathbf{F}^{-1\mathbf{T}})\frac{\partial L}{\partial \mathbf{V}} \equiv \rho_0 \mathbf{v} + J(\mathbf{D}_0 \wedge \mathbf{B}), \tag{15}$$

where $\mathbf{D}_0 \equiv \varepsilon_0 \mathbf{E}$. The electromagnetic stress tensor, in the Piola form, has the following explicit expression

$$\begin{aligned} -\frac{\partial L}{\partial \mathbf{F}} &= \mathbf{E} \otimes \mathfrak{D}_0 + \mu_0^{-1}\mathbf{B} \otimes \mathfrak{B} - \frac{1}{2}J[\varepsilon_0 \mathrm{E}^2 + \mu_0^{-1}\mathrm{B}^2]\mathbf{F}^{-1\mathbf{T}} - J[\mathbf{D}_0 \wedge \mathbf{B} \otimes \mathbf{V}] + \\ &+ \frac{\partial W}{\partial \mathbf{F}} + \frac{\partial W}{\partial(\mathbf{F}\mathfrak{P})} \otimes \mathfrak{P} - J\mathcal{M} \otimes \mathbf{F}^{-1}\left(\frac{\partial W}{\partial J\mathbf{F}^{-1\mathbf{T}}\mathfrak{M}^*}\right) + \\ &+ \left[\frac{\partial W}{\partial J\mathbf{F}^{-1\mathbf{T}}\mathfrak{M}^*} \cdot J\mathbf{F}^{-1\mathbf{T}}\mathfrak{M}^*\right]\mathbf{F}^{-1\mathbf{T}}, \end{aligned} \tag{16}$$

having noted that the dependence of L on $\mathbf{F}$ is through $\mathbf{V} \equiv -\mathbf{F}^{-1}\dot{\mathbf{x}}$ and through W. W, in turn, depends on $\mathbf{F}$ explicitly and through $J\mathfrak{P}$ and $J\mathfrak{M}^*$. The tensor $\partial L/\partial \mathbf{F}$ can be transformed in the *Cauchy–form* and, if we take into account the equation $(11)_4$, $(11)_5$ and (14), we eventually write:

$$\begin{aligned} -J^{-1}\frac{\partial L}{\partial \mathbf{F}}\mathbf{F}^T &= [\varepsilon_0 \mathbf{E} \otimes \mathbf{E} + \mu_0^{-1}\mathbf{B} \otimes \mathbf{B}] - \frac{1}{2}[\varepsilon_0 \mathrm{E}^2 + \mu_0^{-1}\mathrm{B}^2]\mathbf{I} + \\ &- [\varepsilon_0(\mathbf{E} \wedge \mathbf{B}) \otimes \mathbf{v}] + \\ &+ J^{-1}\left(\frac{\partial W}{\partial \mathbf{F}}\right)\mathbf{F}^{\mathbf{T}} + \mathcal{E} \otimes \mathbf{P} - \mathcal{M} \otimes \mathbf{B} + (\mathcal{M} \cdot \mathbf{B})\mathbf{I}. \end{aligned} \tag{17}$$

The expression (17) is consistent with the classical expression of the Maxwell stress tensor in a vacuum, which reads:

$$\mathbf{t}_M = \varepsilon_0 \mathbf{E} \otimes \mathbf{E} + \mu_0^{-1}\mathbf{B} \otimes \mathbf{B} - \frac{1}{2}(\varepsilon_0 \mathrm{E}^2 + \mu_0^{-1}\mathrm{B}^2)\mathbf{I}, \tag{18}$$

having disregarded the velocity $\mathbf{v}$ of the material points.

5. The electromagnetic material tensor

The variational procedure which is based on the Lagrangian density L not only provides the Maxwell equations and the balance of momentum [10, 13]. In fact, along with the *momentum* $\partial L/\partial \dot{\mathbf{x}}$, *two additional canonical momenta*, $\partial L/\partial \dot{\phi}$ and $\partial L/\partial \dot{\mathcal{A}}$, are also introduced.

In the specific case of electromagnetism, one of these momenta vanishes:

$$\frac{\partial L}{\partial \dot{\phi}} \equiv 0, \tag{19}$$

as remarked previously in $(12)_3$.

The following result holds true for the second canonical momentum:

$$\frac{\partial L}{\partial \dot{\mathcal{A}}} = -\mathfrak{D}, \tag{20}$$

taking into account the relationships $(2)_1$, (8), (9) and (10).

Having this remarked, an *additional mechanical quantity*, which is a combination of the two non–vanishing canonical momenta, can be introduced. This mechanical quantity is defined as follows [10]:

$$\mathbf{p}_0 = -\mathbf{F}^{\mathbf{T}} \frac{\partial L}{\partial \dot{\mathbf{x}}} - (\nabla_R \mathcal{A})^T \frac{\partial L}{\partial \dot{\mathcal{A}}}. \tag{21}$$

The expression (21) is a density per unit volume of the reference configuration and leads to the definition of the *material momentum* or *pseudomomentum* [8, 14, 15], which writes as follows:

$$\mathbf{p}_R = \rho_0 \mathbf{C}\mathbf{V} + \mathfrak{P} \wedge \mathfrak{B} \equiv \rho_0 \mathbf{C}\mathbf{V} + J(\mathbf{P} \wedge \mathbf{B}) \tag{22}$$

or, per unit volume of the current configuration,

$$\mathbf{p} = \rho \mathbf{C}\mathbf{V} + \mathbf{P} \wedge \mathbf{B}. \tag{23}$$

This procedure for defining the novel mechanical quantity may not be unfamiliar to people who work on materials with microstructures, from the viewpoint of continuum mechanics. An analogous procedure can be employed for a combination of the quantities $\partial L/\partial \nabla_R \phi$, $\partial L/\partial \nabla_R \mathcal{A}$ and $\partial L/\partial \mathbf{F}$.

This combination defines the *material energy–stress* (an Eshelby–like stress) as follows

$$\mathbf{b} = -L\mathbf{I} + \mathbf{F}^{\mathbf{T}} \frac{\partial L}{\partial \mathbf{F}} + (\nabla_R \phi) \otimes \frac{\partial L}{\partial \nabla_R \phi} + (\nabla_R \mathcal{A})^T \frac{\partial L}{\partial \nabla_R \mathcal{A}}. \tag{24}$$

The expression (24) can be explicitly evaluated by taking into account the equations $(11)_4$, $(11)_5$ and (14). The computations will not reported here as they can be found in [10].

One of the result of interest is the expression of **b** that specialises in the following form, for the electrostatics of a dielectric material:

$$\mathbf{b}^{diel} = (W - \mathfrak{P} \cdot \mathfrak{E})\mathbf{I} - \mathbf{F}^{\mathbf{T}} \frac{\partial W}{\partial \mathbf{F}} - \mathfrak{E} \otimes \mathfrak{P}. \tag{25}$$

The corresponding Cauchy–like stress is reported here below for comparison. With reference to the formula (17), it reads:

$$\mathbf{T}^{diel} = -\frac{1}{2}\varepsilon_0 \mathrm{E}^2 \mathbf{I} + \mathbf{E} \otimes \mathbf{D} + J^{-1} \frac{\partial W}{\partial \mathbf{F}} \mathbf{F}^{\mathbf{T}}. \tag{26}$$

where

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}. \tag{27}$$

6. Comments

By comparing (26) with (25) one can notice the following. First, although the two mentioned expressions are in the form of energy–stress tensors they completely differ from one another. It is not possible to transform one into the other by means of a simple rule, like in pure elasticity.

Second, the Cauchy form of the electromagnetic stress tensor reduces to the Maxwell stress tensor, not only in a vacuum but also in all simple cases that are dealt with in the classical literature. Third, the electrostatic stress tensor survives also out of the domain occupied by the material, whereas the corresponding *electrostatic material stress tensor* $\mathbf{b}^{diel}$ identically vanishes in a vacuum.

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