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CONSTITUTIVE THEORY IN GENERAL RELATIVITY: SPIN-MATERIAL IN SPACES WITH TORSION

Abstract. Some of the problems arising in general relativistic constitutive theory can be solved by using the Riemann-Cartan geometry, a generalization of the Riemann geometry containing torsion. As an example the ideal spinning fluid (Weyssenhoff fluid) is discussed and different results for Einstein and Einstein-Cartan theories are compared.

1. Introduction

It is possible to formulate a relativistic constitutive theory in the framework of Einstein's theory of gravitation [1], but there are several unsatisfying points. One problem is that only symmetric energy-momentum tensors are compatible with the field equations, another problem is that the energy-momentum tensor has to have a vanishing divergence (this is also a consequence of the field equations). Other problems arise from the principle of minimal coupling. One can expect, that at least some problems can be solved by using a generalized theory of gravitation that includes spin (angular momentum) as source of gravitation. The Einstein-Cartan theory of gravitation is such a generalized theory, it is based on a spacetime with curvature and torsion, the Riemann-Cartan geometry.

2. Einstein-Cartan theory

2.1. Geometry

There is a general connection Γ, which is different from the Christoffel symbol. This connection is not symmetric, the antisymmetric part defines the torsion $\mathcal{T}$, which is a tensor of degree 3. The torsion vector is defined by a contraction of the torsion with respect to the first and third indices:

$$\begin{aligned} \mathcal{T}_{\mu\lambda\cdot}^{\cdot\cdot\kappa} &:= \Gamma^{\kappa}_{[\mu\lambda]} \\ \mathcal{T}_{\lambda} &:= \frac{3}{2}\mathcal{T}_{\mu\lambda\cdot}^{\cdot\cdot\mu} \end{aligned}$$

It is possible to represent the connection as a combination of the Christoffel symbol and the so-called contorsion:

$$\Gamma^{\kappa}_{\mu\lambda} = \underbrace{\left\{{}^{\kappa}_{\mu\lambda}\right\}}_{\text{Christoffel symbols}} + \underbrace{\mathcal{T}_{\mu\lambda\cdot}^{\cdot\cdot\kappa} - \mathcal{T}_{\lambda\cdot\mu}^{\cdot\kappa\cdot} + \mathcal{T}^{\kappa\cdot\cdot}_{\cdot\mu\lambda}}_{\text{Contorsion}}$$

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The tensor of curvature and the Ricci tensor are defined as usual:

$$
\begin{aligned}
R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} &= 2\partial_{[\nu}\Gamma^{\kappa}_{\mu]\lambda} + 2\Gamma^{\kappa}_{[\nu|\rho|}\Gamma^{\rho}_{\mu]\lambda} \\
R_{\mu\lambda} &:= R_{\kappa\mu\lambda}^{\cdot\cdot\cdot\kappa}
\end{aligned}
$$

The covariant derivatives are defined in the same way as in Riemann geometry, but the symmetric Christoffel symbols are replaced by the non-symmetric connection Γ:

$$
\begin{aligned}
u^{\nu}_{\cdot;\mu} &= u^{\nu}_{\cdot,\mu} + \Gamma^{\nu}_{\lambda\mu}u^{\lambda} \\
u_{\lambda;\mu} &= u_{\lambda,\mu} - \Gamma^{\nu}_{\lambda\mu}u_{\nu}
\end{aligned}
$$

2.2. Field equations

It is possible to derive field equations by a variation principle [2]. The variation of the special Lagrange density $\mathcal{L}(g_{\mu\nu}, \Gamma^{\alpha}_{\mu\nu}, \phi, \partial_m u\phi)$ given in [2] with respect to tetrades and connection results in two sets of field equations with curvature and torsion:

(1)
$$
\begin{aligned}
R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R &= \kappa T^{\mu\nu} \\
\mathcal{T}_{\alpha\beta}^{\cdot\cdot\mu} + 3\delta^{\mu}_{[\alpha}\mathcal{T}_{\beta]} &= \kappa S_{\alpha\beta}^{\cdot\cdot\mu} \\
\text{geometry} &\Leftrightarrow \text{material}
\end{aligned}
$$

The first set of field equations reads the same as in Einstein theory, but neither the Ricci tensor nor the energy-momentum tensor are symmetric. Both sides of this equation are not divergence-free, in contrast to Einstein's theory. The second set of the field equations connects the torsion with the spin-tensor, which is a constitutive function.

Differentiating the Einstein-Cartan tensor, i.e. the left side of the first set of field-equations (1) and contracting over the second index results in the following equations (the contracted Bianchi identity):

$$
\nabla_{\kappa}(R_{\nu}^{\cdot\kappa} - \frac{1}{2}\delta^{\kappa}_{\nu}R) = 2\mathcal{T}_{\nu\kappa}^{\cdot\cdot\rho}R_{\rho}^{\cdot\kappa} - \mathcal{T}_{\kappa\mu}^{\cdot\cdot\rho}R_{\nu\rho}^{\cdot\cdot\mu\kappa}
$$

Using the field equation one finds that the divergence of the energy-momentum tensor is given by:

$$
\Longrightarrow \kappa\nabla_{\kappa}T_{\nu}^{\cdot\kappa} = 2\mathcal{T}_{\nu\kappa}^{\cdot\cdot\rho}R_{\rho}^{\cdot\kappa} - \mathcal{T}_{\kappa\mu}^{\cdot\cdot\rho}R_{\nu\rho}^{\cdot\cdot\mu\kappa}
$$

In contrast to the Einstein theory the divergence of the energy-momentum tensor does not vanish anymore, but is geometrically determined.

2.3. Balances

It is possible to derive balances for the energy-momentum and for the spin from the field equations. This can be done by splitting the first set of field equations into a symmetric and an antisymmetric part:

$$
\begin{aligned}
R_{(\mu\nu)} - \frac{1}{2}g_{(\mu\nu)}R &= \kappa T_{(\mu\nu)} \\
R_{[\mu\nu]} &= \kappa T_{[\mu\nu]}
\end{aligned}
$$

By taking the divergence of the symmetric equation and using the contracted Bianchi identity one can derive the balance of energy-momentum:

$$(\nabla_\nu - 3\mathcal{T}_\nu) T^{\nu}_{\cdot\mu} + 2\mathcal{T}_{\mu\beta}^{\cdot\,\cdot\,\alpha} T^{\beta}_{\cdot\alpha} + S_{\alpha\beta}^{\cdot\,\cdot\,\nu} R^{\alpha\beta}_{\cdot\,\cdot\,\mu\nu} = 0 \tag{2}$$

The balance of angular momentum can be directly derived from the antisymmetric part by using a geometrical identity for the antisymmetric part of the Ricci tensor and the second set of field equations:

$$(\nabla_\alpha - 3\mathcal{T}_\alpha) \underbrace{(\mathcal{T}_{\mu\lambda}^{\cdot\,\cdot\,\alpha} + 3\delta^{\alpha}_{[\mu} \mathcal{T}_{\lambda]})}_{\kappa S_{\mu\lambda}^{\cdot\,\cdot\,\alpha}} \quad = \quad \kappa T_{[\mu\nu]} \tag{3}$$

The balance of angular momentum connects the change of the spin tensor to the antisymmetric part of the energy-momentum-tensor.

3. Weyssenhoff fluid

3.1. Heuristic description

Now the Weyssenhoff fluid [3] will be discussed as it is done by Obukhov and Korotky [4].

The Weyssenhoff fluid is defined as an ideal spinning fluid. A spin density is now introduced as a skew-symmetric tensor:

$$S^{\mu\nu} = -S^{\nu\mu}$$

The spin density is spacelike, what is ensured by the Frenkel condition:

$$S^{\mu\nu} u_\nu = 0$$

The constitutive assumptions (postulates) for a Weyssenhoff fluid are as follows:

- The spin tensor is a function of the spin density and the following constitutive equation is assumed:

$$S_{\alpha\beta}^{\cdot\,\cdot\,\mu} \quad = \quad u^\mu S_{\alpha\beta}$$

- The energy-momentum tensor should be a function of the energy-momentum density, and is defined as follows:

$$T^{\mu}_{\cdot\alpha} \quad = \quad u^\mu P_\alpha$$

Next one calculate the explicit form of the energy-momentum-density P_α. This can be done by starting out with the antisymmetric part of the energy-momentum tensor (3) and (4):

$$2T_{[\mu\nu]} = u_\mu P_\nu - u_\nu P_\mu \quad = \quad 2(\nabla_\alpha - 3\mathcal{T}_\alpha) S_{\mu\nu}^{\cdot\,\cdot\,\alpha}$$

- and with the usual definition of the internal-energy

$$u^\mu P_\mu \quad \overset{!}{=} \quad \epsilon$$

  one obtains:

$$\begin{aligned} -c^2 P_\nu &= \epsilon u_\nu + 2u^\mu (\nabla_\alpha - 3\mathcal{T}_\alpha)(u^\alpha S_{\mu\nu}) \\ T^{\mu}_{\cdot\nu} &= -\frac{1}{c^2}\epsilon u^\mu u_\nu - \frac{1}{c^2} 2u^\mu u^\alpha \nabla_\beta S_{\alpha\nu}^{\cdot\,\cdot\,\beta} \end{aligned}$$

If it is now assumed, that the interaction between the elements of the fluid is given in such a way that

- Pascal law is valid, one has to modify the equations by an isotropic pressure:

$$\hat{T}^{\mu}_{\cdot\nu} = +\frac{1}{c^2} p\delta^\mu_\nu - \frac{1}{c^2} u^\mu (u_\nu(\epsilon + p) + 2u^\alpha \nabla_\beta S_{\alpha\nu}^{\cdot\,\cdot\,\beta})$$

3.2. Exploiting the 2nd law

Balances

From the thermodynamical point of view the correct way would be to write down the balances and the constraints and for deriving restrictions to the constitutive equations by use of the Liu procedure.

First there is the balance of particle number density which is given in the same way as in Einstein's theory:

$$\nabla_\mu N^\mu = 0 = (nu^\mu)_{;\mu}$$

Next there are the balances of energy-momentum and angular momentum, which are given by the geometrical identities (2) and (3):

$$(\nabla_\nu - 3\mathcal{T}_\nu) T^{\nu}_{\cdot\mu} + 2\mathcal{T}_{\mu\beta}^{\cdot\,\cdot\,\alpha} T^{\beta}_{\cdot\alpha} + S_{\alpha\beta}^{\cdot\,\cdot\,\nu} R^{\alpha\beta}_{\cdot\,\cdot\,\mu\nu} = 0$$

$$(\nabla_\alpha - 3\mathcal{T}_\alpha) S_{\mu\lambda}^{\cdot\,\cdot\,\alpha} = T_{[\mu\nu]}$$

The next equation one needs is the balance of entropy, representing the second law of thermodynamics

$$\nabla_\mu \Sigma^\mu = (su^\mu)_{;\mu} + s^{\mu}_{\cdot;\mu} \geq 0$$

and the field equations are

$$\begin{aligned} R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R &= \kappa T^{\mu\nu} \\ \mathcal{T}_{\alpha\beta}^{\cdot\,\cdot\,\mu} + 3\delta^\mu_{[\alpha}\mathcal{T}_{\beta]} &= \kappa S_{\alpha\beta}^{\cdot\,\cdot\,\mu} \end{aligned}$$

Other constaints, as there are the normalization of the 4-velocity and the form of the covariant derivative have also to be taken into account.

We now choose the state space for an ideal fluid with spin. This state space has to contain the wanted fields, the metric and the connection:

$$\mathcal{Z} = \{n, u_\alpha, \epsilon, S_{\alpha\beta}, g_{\alpha\beta}, \Gamma^{\gamma}_{\alpha\beta}\}$$

Liu procedure

In order to apply Liu's procedure [5, 6, 7] one has to insert the explicit form of the covariant derivative into the balances and then use the chain rule for differentiating the constitutive quantities.

Next the balances and constraints have to be rewritten in matrix formulation:

$$\begin{aligned} \underline{\underline{A}} \cdot \underline{y} + \underline{B} &= 0, \\ \underline{\alpha} \cdot \underline{y} + \beta &\geq 0 \end{aligned}$$

PROPOSITION 1 (COLEMAN-MIZEL-FORMULATION OF THE 2$^{\text{ND}}$ LAW [8]).
If Z is no trap, the following inclusion is valid for all $\underline{y}$*:*

$$\underline{\underline{A}} \cdot \underline{y} = -\underline{B} \quad \Longrightarrow \quad \underline{\alpha} \cdot \underline{y} \geq -\beta$$

that means, all $\underline{y}$ *which are solutions of the balances satisfy the dissipation inequality.*

Then one can apply Liu's proposition, which runs as follows:

PROPOSITION 2 (LIU [5]). Starting with proposition 1 one can show:
In large state spaces exist state space functions $\underline{\Lambda}$ *so that the constitutive equations satisfy the Liu relations*

$$\underline{\Lambda} \cdot \underline{\underline{A}} = \underline{\alpha}, \tag{4}$$

and the residual inequality

$$-\underline{\Lambda} \cdot \underline{B} \geq -\beta. \tag{5}$$

From (4) and (5) we obtain the restrictions to the constitutive equations we are looking for. Taking these restrictions into account we obtain constitutive equations which are in accordance with the second law of thermodynamics.

4. Comparison of Einstein and Einstein-Cartan theory

We now discuss differences and similarities of Einstein and Einstein-Cartan theories with respect to coupling of constitutive properties to geometry.

In Einstein-Cartan theory with non-vanishing torsion and curvature the spin couples to torsion, and the energy-momentum tensor which is spin-dependent, non-symmetric, and not divergence-free couples to curvature. If the torsion vanishes, also the spin tensor and the skew-symme-tric part of the energy-momentum tensor vanish.

In Einstein theory with vanishing torsion and non-vanishing curvature the spin appears as in Einstein-Cartan theory in the non-symmetric and not divergence-free energy-momentum tensor which is split into its symmetric and skew-symme-tric part. The divergence-free symmetric part couples by the Einstein equations to curvature, whereas the skew-symmetric part does not couple to any geometric quantity. It represents the source in spin balance.

In Minkowski theory being flat and torsion-free there are no geometric objects to which spin and energy-momentum tensor can couple. If we regard Minkowski and Einstein theory as special cases of the Einstein-Cartan theory all having the same type of coupling, then Einstein theory has to be spin-free and Minkowski theory is only valid in vacuum. Of course, this is not the case by experience and therefore we have to regard these three theories as having different types of coupling to constitutive properties.

5. Conclusion

As dicussed above the energy-momentum tensor of the Weyssenhoff fluid was obtained by use of a variational problem without taking into account the second law of thermodynamics. This variational problem generates the balance equations of energy-momentum and spin which now are supplemented by the dissipation inequality. The Liu procedure of exploiting this dissipation inequality generates restrictions to the constitutive quantities energy-momentum and spin.

References

[1] HERRMANN H., RÜCKNER G. AND MUSCHIK W., *Constitutive theory in general relativity: spin-material in spaces with torsion*; in: “Proceedings of the 4th international seminar on geometry, continua and microstructures 2000”, Rend. Sem. Mat. Univ. e Polit. Torino, **58** (2) 2000, 135–141.

[2] HEHL F.W., MCCREA J.D., MIELKE E.W. AND NE'EMAN Y., *Metric-affine Gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance*, Physics Reports **258** (1995), 1–171.

[3] WEYSSENHOFF J. AND RAABE A., *Relativistic dynamics of spin-fluids and spin-particles*, Acta Phys. Polon. **9** (1947), 7–53.

[4] OBUKHOV YU.N. AND KOROTKY V.A., *The Weyssenhoff fluid in Einstein-Cartan theory*, Class. Quantum Grav. **4** (1987), 1633–1657.

[5] I-SHIH AND LIU, *Method of Lagrange multipliers for exploitation of the entropy principle*, Arch. Rat. Mech. Anal. **46** (1972), 131.

[6] MUSCHIK W., PAPENFUSS C. AND EHRENTRAUT H., *Concepts of continuum thermodynamics*, Technische Universität Berlin und Kielce University of Technology 1996.

[7] MUSCHIK W., *Aspects of non-equilibrium thermodynamics*, World Scientific, Singapore 1990.

[8] COLEMAN B.D. AND MIZEL V.J., *Existence of caloric equations of state in thermodynamics*, J. Chem. Phys. **40** (1964), 1116–1125.

[9] MUSCHIK W. AND EHRENTRAUT H., *An amendment to the second law*, J. Non-Equilib. Thermodyn. **21** (1996), 175.

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