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COMPACTNESS FOR SUB-RIEMANNIAN LENGTH-MINIMIZERS AND SUBANALYTICITY

Abstract.
We establish compactness properties for sets of length-minimizing admissible paths of a prescribed small length. This implies subanayticity of small sub-Riemannian balls for a wide class of real-analytic sub-Riemannian structures: for any structure without abnormal minimizers and for many structures without strictly abnormal minimizers.

1. Introduction

Let M be a C^∞ Riemannian manifold, $\dim M = n$. A distribution on M is a smooth linear subbundle Δ of the tangent bundle TM. We denote by Δ_q the fiber of Δ at $q \in M$; $\Delta_q \subset T_q M$. A number $k = \dim \Delta_q$ is the *rank* of the distribution. We assume that $1 < k < n$. The restriction of the Riemannian structure to Δ is a *sub-Riemannian structure*.

Lipschitzian integral curves of the distribution Δ are called *admissible paths*; these are Lipschitzian curves $t \mapsto q(t)$, $t \in [0, 1]$, such that $\dot{q}(t) \in \Delta_{q(t)}$ for almost all t.

We fix a point $q_0 \in M$ and study only admissible paths started from this point, i.e. we impose the initial condition $q(0) = q_0$. Sections of the linear bundle Δ are smooth vector fields; iterated Lie brackets of these vector fields define a flag

$$\Delta_{q_0} \subset \Delta^2_{q_0} \subset \cdots \subset \Delta^m_{q_0} \cdots \subset T_q M$$

in the following way:

$$\Delta^m_{q_0} = \operatorname{span}\{[X_1, [X_2, [\ldots, X_m]\ldots](q_0) : X_i(q) \in \Delta_q,\ i = 1, \ldots, m,\ q \in M\}.$$

A distribution Δ is *bracket generating* at q_0 if $\Delta^m_{q_0} = T_{q_0} M$ for some $m > 0$. If Δ is bracket generating, then according to a classical Rashevski-Chow theorem (see [15, 22]) there exist admissible paths connecting q_0 with any point of an open neighborhood of q_0. Moreover, applying a general existence theorem for optimal controls [16] one obtains that for any q_1 from a small enough neighborhood of q_0 there exists a shortest admissible path connecting q_0 with q_1. The length of this shortest path is the *sub-Riemannian* or *Carnot-Caratheodory distance* between q_0 and q_1.

For the rest of the paper we assume that Δ is bracket generating at the given initial point q_0. We denote by $\rho(q)$ the sub-Riemannian distance between q_0 and q. It follows from the Rashevsky-Chow theorem that ρ is a continuous function defined on a neighborhood of q_0. Moreover, ρ is Hölder-continuous with the Hölder exponent $\frac{1}{m}$, where $\Delta^m_{q_0} = T_{q_0} M$. A *sub-Riemannian sphere* $S(r)$ is the set of all points at sub-Riemannian distance r from q_0, $S(r) = \rho^{-1}(r)$.

In contrast to the Riemannian distance, the sub-Riemannian distance ρ is never smooth in a punctured neighborhood of q_0 (see Theorem 1) and the main motivation for this research is to understand regularity properties of ρ. In the Riemannian case, where all paths are available, the set of shortest paths connecting q_0 with the sphere of a small radius r is parametrized by the points of the sphere. This is not true for the set of shortests admissible paths connecting q_0 with the sub-Riemannian sphere $S(r)$. The structure of the last set may be rather complicated; we show that this set is at least compact in H^1-topology (Theorem 2). The situation is much simpler if no one among so called abnormal geodesics of length r connect q_0 with $S(r)$. In the last case, the mentioned set of shortests admissible paths can be parametrized by a compact part of a cylinder $S^{k-1} \times \mathbb{R}^{n-k}$ (Theorem 3). In Theorem 4 we recall an efficient necessary condition for a length r admissible path to be a shortest one. In Theorem 5 we state a result, which is similar to that of Theorem 3 but more efficient and admitting nonstrictly abnormal geodesics as well.

We apply all mentioned results to the case of real-analytic M and Δ. The main problem here is to know whether the distance function ρ is subanalytic. Positive results for some special classes of distributions were obtained in [8, 17, 19, 20, 23] and the first counterexample was described in [10] (see [13, 14] for further examples and for study of the "transcendence" of ρ).

Both positive results and the counterexamples gave an indication that the problem is intimately related to the existence of abnormal length-minimizers. Corollaries 2, 3, 4 below make this statement a well-established fact: they show very clear that only abnormal length-minimizers may destroy subanalyticity of ρ out of q_0.

What remains? The situation with subanalyticity in a whole neighborhood including q_0 is not yet clarified. This subanalyticity is known only for a rather special type of distributions (the best result is stated in [20]). Another problem is to pass from examples to general statements for sub-Riemannian structures with abnormal length-minimizers. Such length-minimizers are exclusive for rank $k \geq 3$ distributions (see discussion at the end of the paper) and typical for rank 2 distributions (see [7, 21, 24]). A natural conjecture is:
If $k = 2$ and $\Delta^2_{q_0} \neq \Delta^3_{q_0}$, then ρ is not subanalytic.

2. Geodesics

We are working in a small neighborhood O_{q_0} of $q_0 \in M$, where we fix an orthonormal frame $X_1, \dots, X_k \in \operatorname{Vect} M$ of the sub-Riemannian structure under consideration. Admissible paths are thus solutions to the differential equations

$$\dot{q} = \sum_{i=1}^{k} u_i(t) X_i(q), \ q \in O_{q_0}, \ q(0) = q_0, \tag{1}$$

where $u = (u_1(\cdot), \dots, u_k(\cdot)) \in L_2^k[0, 1]$.

Below $\|u\| = \left(\int_0^1 \sum_{i=0}^k u_i^2(t)\, dt \right)^{1/2}$ is the norm in $L_2^k[0, 1]$. We also set $\|q(\cdot)\| = \|u\|$, where $q(\cdot)$ is the solution to (1). Let

$$U_r = \{u \in L_2^k[0, 1] : \|u\| = r\}$$

be the sphere of radius r in $L_2^k[0, 1]$. Solutions to (1) are defined for all $t \in [0, 1]$, if u belongs to the sphere of a small enough radius r. In this paper we take u only from such spheres without special mentioning. The length $l(q(\cdot)) = \int_0^1 \left(\sum_{i=1}^k u_i^2(t) \right)^{1/2} dt$ is well-defined and satisfies the inequality

$$l(q(\cdot)) \leq \|q(\cdot)\| = r\,. \tag{2}$$

The length doesn't depend on the parametrization of the curve while the norm $\|u\|$ depends. We say that u and $q(\cdot)$ are *normalized* if $\sum_{i=1}^{k} u_i^2(t)$ doesn't depend on t. For normalized u, and only for them, inequality (2) becoms equality.

REMARK 1. The notations $\|q(\cdot)\|$ and $l(q(\cdot))$ reflect the fact that these quantities do not depend on the choice of the orthonormal frame $X_1, \ldots, X_k$ and are characteristics of the *trajectory* $q(\cdot)$ rather than the *control* u. L_2-topology in the space of controls is H_1-topology in the space of trajectories.

We consider the endpoint mapping $f : u \mapsto q(1)$. This is a well-defined smooth mapping of a neighborhood of the origin of $L_2^k[0, 1]$ into M. We set $f_r = f\big|_{U_r}$. Critical points of the mapping $f_r : U_r \to M$ are called *extremal controls* and correspondent solutions to the equation (1) are called *extremal trajectories* or *geodesics*.

An extremal control u and the correspondent geodesic $q(\cdot)$ are *regular* if u is a regular point of f; otherwise they are *singular* or *abnormal*.

Let C_r be the set of normalized critical points of f_r; in other words, C_r is the set of normalized extremal controls of the length r. It is easy to check that $f_r^{-1}(S(r)) \subset C_r$. Indeed, among all admissible curves of the length no greater than r only geodesics of the length exactly r can reach the sub-Riemannian sphere $S(r)$. Controls $u \in f_r^{-1}(S(r))$ and correspondent geodesics are called *minimal*.

Let $D_u f : L_2^k[0, 1] \to T_{f(u)}M$ be the differential of f at u. Extremal controls (and only them) satisfy the equation

$$\lambda D_u f = \nu u \tag{3}$$

with some "Lagrange multipliers" $\lambda \in T^*_{f(u)}M \setminus 0$, $\nu \in \mathbb{R}$. Here $\lambda D_u f$ is the composition of the linear mapping $D_u f$ and the linear form $\lambda : T_{f(u)}M \to \mathbb{R}$, i.e. $(\lambda D_u f) \in L_2^k[0, 1]^* = L_2^k[0, 1]$. We have $\nu \neq 0$ for regular extremal controls, while for abnormal controls ν can be taken 0. In principle, abnormal controls may admit Lagrange multipliers with both zero and nonzero ν. If it is not the case, then the control and the geodesic are called *strictly abnormal*.

Pontryagin maximum principle gives an efficient way to solve equation (3), i.e. to find extremal controls and Lagrange multipliers. A coordinate free formulation of the maximum principle uses the canonical symplectic structure on the cotangent bundle T^*M. The symplectic structure associates a Hamiltonian vector field $\vec{a} \in \operatorname{Vect} T^*M$ to any smooth function $a : T^*M \to \mathbb{R}$ (see [11] for the introduction to symplectic methods).

We define the functions h_i, $i = 1, \ldots, k$, and h on T^*M by the formulas

$$h_i(\psi) = \langle \psi, X_i(q) \rangle\,, \quad h(\psi) = \frac{1}{2}\sum_{i=1}^{k} h_i^2(\psi)\,, \quad \forall q \in M,\ \psi \in T_q^*M\,.$$

Pontryagin maximum principle implies the following

PROPOSITION 1. *A triple (u, λ, ν) satisfies equation (3) if and only if there exists a solution*

$\psi(t),\ 0 \le t \le 1$, *to the system of differential and pointwise equations*

$$\dot{\psi} = \sum_{i=1}^{k} u_i(t)\vec{h}_i(\psi)\,, \quad h_i(\psi(t)) = \nu u_i(t) \tag{4}$$

with boundary conditions $\psi(0) \in T^*_{q_0}M,\ \psi(1) = \lambda$.

Here $(\psi(t), \nu)$ are Lagrange multipliers for the extremal control $u_t : \tau \mapsto tu(t\tau)$; in other words, $\psi(t) D_{u_t} f = \nu u_t$.

Note that abnormal geodesics remain to be geodesics after an arbitrary reparametrization, while regular geodesics are automatically normalized. We say that a geodesic is *quasi-regular* if it is normalized and is not strictly abnormal. Setting $\nu = 1$ we obtain a simple description of all quasi-regular geodesics.

COROLLARY 1. *Quasi-regular geodesics are exactly projections to* M *of the solutions to the differential equation* $\dot{\psi} = \vec{h}(\psi)$ *with initial conditions* $\psi(0) \in T^*_{q_0}M$. *If* $h(\psi(0))$ *is small enough, then such a solution exists (i.e. is defined on the whole segment* $[0, 1]$*). The length of the geodesic equals* $\sqrt{2h(\psi(0))}$ *and the Lagrange multiplier* $\lambda = \psi(1)$.

The next result demonstrates a sharp difference between Riemannian and sub-Riemannian distance functions.

THEOREM 1. *Any neighbourhood of* q_0 *in* M *contains a point* $q \neq q_0$, *where the distance function* ρ *is not continuously differentiable.*

This theorem is a kind of folklore; everybody agrees it is true but I have never seen the proof. What follows is a sketch of the proof.

Suppose ρ is continuously differentiable out of q_0. Take a minimal geodesic $q(\cdot)$ of the length r. Then $\tau \mapsto q(t\tau)$ is a minimal geodesic of the length tr for any $t \in [0, 1]$ and we have $\rho(q(t)) \equiv rt$; hence $\langle d_{q(t)}\rho, \dot{q}(t)\rangle = r$. Since any point of a neighborhood of q_0 belongs to some minimal geodesic, we obtain that ρ has no critical points in the punctured neighborhood. In particular, the spheres $S(r) = \rho^{-1}(r)$ are C^1-hypersurfaces in M. Moreover, $S(r) = \partial f(U_r)$; hence $\left(d_{q(1)}\rho\right) D_u f_r = 0$ and we obtain the equality $\left(d_{q(1)}\rho\right) D_u f = \frac{1}{r}u$, where u is the extremal control associated with $q(\cdot)$. Hence $q(\cdot)$ is the projection to M of the solution to the equation $\dot{\psi} = \vec{h}(\psi)$ with the boundary condition $\psi(1) = r d_{q(1)}\rho$. Moreover, we easily conclude that $\psi(t) = r d_{q(t)}\rho$ and come to the equation

$$\dot{q}(t) = r\sum_{i=1}^{k}\langle d_{q(t)}\rho, X_i(q(t))\rangle X_i(q(t))\,.$$

For the rest of the proof we fix local coordinates in a neighborhood of q_0. We are going to prove that the vector field $V(q) = r\sum_{i=1}^{k}\langle d_q\rho, X_i(q)\rangle X_i(q)$, $q \neq q_0$, has index 1 at its isolated singularity q_0. Let $B_\varepsilon = \{q \in \mathbb{R}^n : |q - q_0| \le \varepsilon\}$ be a so small ball that $\rho(q) < \frac{r}{2}$, $\forall q \in B_\varepsilon$. Let $s \mapsto q(s; q_\varepsilon)$ be the solution to the equation $\dot{q} = V(q)$ with the initial condition $q(0; q_\varepsilon) = q_\varepsilon \in B_\varepsilon$. Then $q(\frac{r}{2}; q_\varepsilon) \notin B_\varepsilon$. In particular, the vector field W_ε on B_ε defined by the formula $W(q_\varepsilon) = q(\frac{r}{2}; q_\varepsilon) - q_\varepsilon$ looks "outward" and has index 1. The family of the fields $V_s(q_\varepsilon) = \frac{1}{s}(q(s; q_\varepsilon) - q_\varepsilon)$, $0 \le s \le \frac{r}{2}$ provides a homotopy of $V\big|_{B_\varepsilon}$ and $\frac{r}{2}W$, hence V has index 1 at q_0 as well.

On the other hand, the field V is a linear combination of $X_1, \dots, X_k$ and takes its values near the k-dimensional subspace $\operatorname{span}\{X_1(q_0), \dots, X_k(q_0)\}$. Such a field must have index 0 at q_0. This contradiction completes the proof.

Corollary 1 gives us a parametrization of the space of quasi-regular geodesics by the poins of an open subset Ψ of $T^*_{q_0}M$. Namely, Ψ consists of $\psi_0 \in T^*_{q_0}M$ such that the solution $\psi(t)$ to the equation $\dot{\psi} = \vec{h}(\psi)$ with the initial condition $\psi(0) = \psi_0$ is defined for all $t \in [0, 1]$. The composition of this parametrization with the endpoint mapping f is the *exponential mapping* $\mathcal{E} : \Psi \to M$. Thus $\mathcal{E}(\psi(0)) = \pi(\psi(1))$, where $\pi : T^*M \to M$ is the canonical projection.

The space of quasi-regular geodesics of a small enough length r are parametrized by the points of the manifold $H(r) = h^{-1}(\frac{r^2}{2}) \cap T^*_{q_0}M \subset \Psi$. Clearly, $H(r)$ is diffeomorphic to $\mathbb{R}^{n-k} \times S^{k-1}$ and $H(sr) = sH(r)$ for any nonnegative s.

All results about subanalyticity of the distance function ρ are based on the following statement. As usually, the distances r are assumed to be small enough.

PROPOSITION 2. *Let M and the sub-Riemannian structure be real-analytic. Suppose that there exists a compact $K \subset h^{-1}(\frac{1}{2}) \cap T^*_{q_0}M$ such that $S(r) \subset \mathcal{E}(rK)$, $\forall r \in (r_0, r_1)$. Then ρ is subanalytic on $\rho^{-1}((r_0, r_1))$.*

Proof. It follows from our assumptions and Corollary 1 that

$$\rho(q) = \min\{r : \psi \in K,\ \mathcal{E}(r\psi) = q\}, \quad \forall q \in \rho^{-1}((r_0, r_1)) .$$

The mapping $\mathcal{E}$ is analytic thanks to the analyticity of the vector field $\vec{h}$. The compact K can obviously be chosen semi-analytic. The proposition follows now from [25, Prop. 1.3.7].

□

3. Compactness

Let $\mathcal{O} \subset L_2^k[0, 1]$ be the domain of the endpoint mapping f. Recall that $\mathcal{O}$ is a neighborhood of the origin of $L_2^k[0, 1]$ and $f : \mathcal{O} \to M$ is a smooth mapping. We are going to use not only defined by the norm "strong" topology in the Hilbert space $L_2^k[0, 1]$, but also weak topology. We denote by $\mathcal{O}_{\text{weak}}$ the topological space defined by weak topology restricted to $\mathcal{O}$.

PROPOSITION 3. *$f : \mathcal{O}_{weak} \to M$ is a continuous mapping.*

This proposition easily follows from some classical results on the continuous dependence of solutions to ordinary differential equations on the right-hand side. Nevertheless, I give an independent proof in terms of the chronological calculus (see [1, 5]) since it is very short. We have

$$\begin{aligned} f(u) &= q_0 \overrightarrow{\exp} \int_0^1 \sum_{i=1}^k u_i(t) X_i \, dt \\ &= q_0 + \sum_{i=1}^k q_0 \int_0^1 \left(u_i(t) \, \overrightarrow{\exp} \int_0^t \sum_{j=1}^k u_j(t) X_j \, d\tau \right) dt \circ X_i \, . \end{aligned}$$

The integration by parts gives:

$$\int_0^1 \left(u_i(t) \overrightarrow{\exp} \int_0^t \sum_{j=1}^k u_j(t) X_j \, d\tau \right) dt = \int_0^1 u_i(t)\, dt \, \overrightarrow{\exp} \int_0^1 \sum_{j=1}^k u_j(t) X_j \, dt$$

$$- \sum_{i=1}^k \int_0^1 \left(u_j(t) \int_0^t u_i(\tau)\, d\tau \, \overrightarrow{\exp} \int_0^t \sum_{j=1}^k u_j(t) X_j \, d\tau \right) dt \circ X_j \, .$$

It remains to mention that the mapping $u(\cdot) \mapsto \int_0^{\cdot} u(\tau)\, d\tau$ is a compact operator in $L_2^k[0,1]$. A detailed study of the continuity of $\overrightarrow{\exp}$ in various topologies see in [18].

THEOREM 2. *The set of minimal geodesics of a prescribed length r is compact in H_1-topology for any small enough r.*

Proof. We have to prove that $f_r^{-1}(S(r))$ is a compact subset of U_r. First of all, $f_r^{-1}(S(r)) = f^{-1}(S(r)) \cap \operatorname{conv} U_r$, where $\operatorname{conv} U_r$ is a ball in $L_2^k[0,1]$. This is just because $S(r)$ cannot be reached by trajectories of the length smaller than r. Then the continuity of ρ implies that $S(r) = \rho^{-1}(r)$ is a closed set and the continuity of f in weak topology implies that $f^{-1}(S(r))$ is weakly closed. Since $\operatorname{conv} U_r$ is weakly compact we obtain that $f_r^{-1}(S(r))$ is weakly compact. What remains is to note that weak topology resricted to the sphere U_r in the Hilbert space is equivalent to strong topology.

□

THEOREM 3. *Suppose that all minimal geodesics of the length r are regular. Then we have that $\mathcal{E}^{-1}(S(r)) \cap H(r)$ is compact.*

Proof. Denote by $u_{\psi(0)}$ the extremal control associated with $\psi(0) \in H(r)$ so that $\mathcal{E}(\psi(0)) = f(u_{\psi(0)})$. We have $u_{\psi(0)} = (h_1(\psi(\cdot)), \ldots, h_k(\psi(\cdot)))$ (see Proposition 1 and its Corollary). In particular, $u_{\psi(0)}$ continuously depends on $\psi(0)$.

Take a sequence $\psi_m(0) \in \mathcal{E}^{-1}(S(r)) \cap H(r)$, $m = 1, 2, \ldots$; the controls $u_{\psi_m(0)}$ are minimal, the set of minimal controls of the length r is compact, hence there exists a convergent subsequence of this sequence of controls and the limit is again a minimal control. To simplify notations, we suppose without losing generality that the sequence $u_{\psi_m(0)}$, $m = 1, 2, \ldots$, is already convergent, $\exists \lim_{m\to\infty} u_{\psi_m(0)} = \bar{u}$.

It follows from Proposition 1 that $\psi_m(1) D_{u_{\psi_m(0)}} f = u_{\psi_m(0)}$. Suppose that M is endowed with some Riemannian structure so that the length $|\psi_m(1)|$ of the cotangent vector $\psi_m(1)$ has a sense. There are two possibilities: either $|\psi_m(1)| \to \infty$ ($m \to \infty$) or $\psi_m(1)$, $m = 1, 2, \ldots$, contains a convergent subsequence.

In the first case we come to the equation $\lambda D_{\bar{u}} f = 0$, where λ is a limiting point of the sequence $\frac{1}{|\psi_m(1)|}\psi_m(1)$, $|\lambda| = 1$. Hence $\bar{u}$ is an abnormal minimal control that contradicts the assumption of the theorem.

In the second case let $\psi_{m_l}(1)$, $l = 1, 2, \ldots$, be a convergent subsequence. Then $\psi_{m_l}(0)$, $l = 1, 2, \ldots$, is also convergent, $\exists \lim_{l\to\infty} \psi_{m_l}(0) = \bar{\psi}(0) \in H(r)$. Then $\bar{u} = u_{\bar{\psi}(0)}$ and we are done.

□

COROLLARY 2. *Let M and the sub-Riemannian structure be real-analytic. Suppose that all minimal geodesics of the length r_0 are regular for some $r_0 < r$. Then ρ is subanalytic on*

$\rho^{-1}((r_0, r])$.

Proof. According to Theorem 3, $K_0 = \mathcal{E}^{-1}(S(r_0)) \cap H(r_0)$ is a compact set and $\{u_{\psi(0)} : \psi(0) \in K_0\}$ is the set of all minimal extremal controls of the length r_0. The minimality of an extremal control $u_{\psi(0)}$ implies the minimality of the control $u_{s\psi(0)}$ for $s < 1$, since $u_{s\psi(0)}(\tau) = su_{\psi(0)}(\tau)$ and a reparametrized piece of a minimal geodesic is automatically minimal. Hence $S(r_1) \subset \mathcal{E}\left(\frac{r_1}{r_0} K_0\right)$ for $r_1 \geq r_0$ and the required subanalyticity follows from Proposition 2.

□

Corollary 2 gives a rather strong sufficient condition for subanalyticity of the distance function ρ out of q_0. In particular, the absence of abnormal minimal geodesics implies subanalyticity of ρ in a punctured neighborhood of q_0. This condition is not however quite satisfactory because it doesn't admit abnormal quasi-regular geodesics. Though being non generic, abnormal quasi-regular geodesics appear naturally in problems with symmetries. Moreover, they are common in so called nilpotent approximations of sub-Riemannian structures at (see [5, 12]). The nilpotent approximation (or nilpotenization) of a generic sub-Riemannian structure q_0 leads to a simplified quasi-homogeneous approximation of the original distance function. It is very unlikely that ρ loses subanalyticity under the nilpotent approximation, although the above sufficient condition loses its validity. In the next section we give chekable sufficient conditions for subanalyticity, wich are free of the above mentioned defect.

4. Second Variation

Let $u \in U_r$ be an extremal control, i.e. a critical point of f_r. Recall that the Hessian of f_r at u is a quadratic mapping

$$\mathrm{Hes}_u f_r : \ker D_u f_r \to \operatorname{coker} D_u f_r \,,$$

an independent on the choice of local coordinates part of the second derivative of f_r at u. Let (λ, ν) be Lagrange multipliers associated with u so that equation (3) is satisfied. Then the covector $\lambda : T_{f(u)}M \to \mathbb{R}$ annihilates $\operatorname{im} D_u f_r$ and the composition

$$\lambda \mathrm{Hes}_u f_r : \ker D_u f_r \to \mathbb{R} \tag{5}$$

is well-defined.

Quadratic form (5) is the *second variation* of the sub-Riemannian problem at (u, λ, ν). We have

$$\lambda \mathrm{Hes}_u f_r(v) = \lambda D_u^2 f(v, v) - \nu |v|^2 \,, \quad v \in \ker D_u f_r \,.$$

Let $q(\cdot)$ be the geodesic associated with the control u. We set

$$\operatorname{ind}(q(\cdot), \lambda, \nu) = \operatorname{ind}_+(\lambda \mathrm{Hes}_u f_r) - \dim \operatorname{coker} D_u f_r \,, \tag{6}$$

where $\operatorname{ind}_+(\lambda \mathrm{Hes}_u f_r)$ is the positive inertia index of the quadratic form $\lambda \mathrm{Hes}_u f_r$. Decoding some of the symbols we can re-write:

$$\begin{aligned} \operatorname{ind}(q(\cdot), \lambda, \nu) \;&=\; \sup\{\dim V : V \subset \ker D_u f_r,\ \lambda D_u^2 f(v, v) > \nu |v|^2,\ \forall v \in V \setminus 0\} \\ &\quad - \dim\{\lambda' \in T^*_{f(u)} M : \lambda' D_u f_r = 0\} \,. \end{aligned}$$

The value of $\operatorname{ind}(q(\cdot), \lambda, \nu)$ may be an integer or $+\infty$.

REMARK 2. Index (5) doesn't depend on the choice of the orthonormal frame $X_1, \dots, X_k$ and is actually a characteristic of the geodesic $q(\cdot)$ and the Lagrange multipliers (λ, ν). Indeed, a change of the frame leads to a smooth transformation of the Hilbert manifold U_r and to a linear transformation of variables in the quadratic form $\lambda \mathrm{Hes}_u f_r$ and linear mapping $D_u f_r$. Both terms in the right-hand side of (5) remain unchanged.

PROPOSITION 4. $(u, \lambda, \nu) \mapsto \mathrm{ind}(q(\cdot), \lambda, \nu)$ *is a lower semicontinuous function on the space of solutions of (3).*

Proof. We have $\dim \operatorname{coker} D_u f_r = \operatorname{codim} \ker D_u f_r$. Here $\ker D_u f_r = \ker D_u f \cap \{u\}^\perp \subset L_2^k[0,1]$ is a subspace of finite codimension in $L_2^k[0,1]$. The multivalued mapping $u \longmapsto (\ker D_u f_r) \cap U_r$ is upper semicontinuous in the Hausdorff topology, just because $u \mapsto D_u f$ is continuous.

Take (u, λ, ν) satisfying (3). If u' is close enough to u, then $\ker D_{u'} f_r$ is arbitraryly close to a subspace of codimension

$$\dim \operatorname{coker} D_u f_r - \dim \operatorname{coker} D_{u'} f_r$$

in $D_u f_r$. Suppose $V \subset \ker D_u f_r$ is a finite-dimensional subspace such that $\lambda \mathrm{Hes}_u f_r \big|_V$ is a positive definite quadratic form. If u' is sufficiently close to u, then $\ker D_{u'} f_r$ contains a subspace V' of dimension

$$\dim V - (\dim \operatorname{coker} D_u f_r - \dim \operatorname{coker} D_{u'} f_r)$$

that is arbitrarily close to a subspace of V. If λ' is sufficiently close to λ, then the quadratic form $\lambda' \mathrm{Hes}_{u'} f_r \big|_{V'}$ is positive definite.

We come to the inequality $\mathrm{ind}(q'(\cdot), \lambda', \nu') \geq \mathrm{ind}(q(\cdot), \lambda, \nu)$ for any solution (u', λ', ν') of (3) close enough to (u, λ, ν); here $q'(\cdot)$ is the geodesic associated to the control u'.

□

THEOREM 4. *If $q(\cdot)$ is minimal geodesic, then there exist associated with $q(\cdot)$ Lagrange multipliers λ, ν such that* $\mathrm{ind}(q(\cdot), \lambda, \nu) < 0$.

This theorem is a direct corollary of a general result announced in [2] and proved in [3]; see also [8] for the updated proof of exactly this corollary.

THEOREM 5. *Suppose that* $\mathrm{ind}(q(\cdot), \lambda, 0) \geq 0$ *for any abnormal geodesic $q(\cdot)$ of the length r and associated Lagrange multipliers $(\lambda, 0)$. Then there exists a compact $K_r \subset H(r)$ such that $S(r) = \mathcal{E}(K_r)$.*

Proof. We use notations introduced in the first paragraph of the proof of Theorem 3. Let $q_{\psi(0)}$ be the geodesic associated to the control $u_{\psi(0)}$. We set

$$K_r = \{\psi(0) \in H(r) \cap \mathcal{E}^{-1}(S(r)) : \mathrm{ind}(q_{\psi(0)}, \psi(1), 1) < 0\}. \tag{7}$$

It follows from Theorem 4 and the assumption of Theorem 5 that $\mathcal{E}(K_r) = S(r)$. What remains is to prove that K_r is compact.

Take a sequence $\psi_m(0) \in K_r$, $m = 1, 2, \dots$; the controls $u_{\psi_m(0)}$ are minimal, the set of minimal controls of the length r is compact, hence there exists a convergent subsequence of this sequence of controls and the limit is again a minimal control. To simplify notations, we suppose without losing generality that the sequence $u_{\psi_m(0)}$, $m = 1, 2, \ldots$, is already convergent, $\exists \lim_{m\to\infty} u_{\psi_m(0)} = \bar{u}$.

It follows from Proposition 1 that $\psi_m(1) D_{u_{\psi_m(0)}} f = u_{\psi_m(0)}$. There are two possibilities: either $|\psi_m(1)| \to \infty$ $(m \to \infty)$ or $\psi_m(1)$, $m = 1, 2, \ldots$, contains a convergent subsequence.

In the first case we come to the equation $\bar{\lambda} D_{\bar{u}} f = 0$, where $\bar{\lambda}$ is a limiting point of the sequence $\frac{1}{|\psi_m(1)|}\psi_m(1)$, $|\bar{\lambda}| = 1$. Lower semicontinuity of $\operatorname{ind}(q(\cdot), \lambda, \nu)$ implies the inequality $\operatorname{ind}(\bar{q}(\cdot), \bar{\lambda}, 0) < 0$, where $\bar{q}(\cdot)$ is the geodesic associated with the control $\bar{u}$. We come to a contradiction with the assumption of the theorem.

In the second case let $\psi_{m_l}(1)$, $l = 1, 2, \ldots$, be a convergent subsequence. Then $\psi_{m_l}(0)$, $l = 1, 2, \ldots$, is also convergent, $\exists \lim_{l\to\infty} \psi_{m_l}(0) = \bar{\psi}(0) \in H(r)$. Then $\bar{u} = u_{\bar{\psi}(0)}$ and $\operatorname{ind}(\bar{q}(\cdot), \bar{\psi}(1), 1) < 0$ because of lower semicontinuity of $\operatorname{ind}(q(\cdot), \lambda, \nu)$. Hence $\bar{\psi}(0) \in K_r$ and we are done.

□

COROLLARY 3. *Let M and the sub-Riemannian structure be real-analytic. Suppose $r_0 < r$ is such that* $\operatorname{ind}(q(\cdot), \lambda, 0) \geq 0$ *for any abnormal geodesic $q(\cdot)$ of the length r_0 and associated Lagrange multipliers* $(\lambda, 0)$. *Then ρ is subanalytic on* $\rho^{-1}((r_0, r])$.

Proof. Let K_{r_0} be defined as in (7). Then K_{r_0} is compact and $\{u_{\psi(0)} : \psi(0) \in K_{r_0}\}$ is the set of all minimal extremal controls of the length r_0. The minimality of an extremal control $u_{\psi(0)}$ implies the minimality of the control $u_{s\psi(0)}$ for $s < 1$, since $u_{s\psi(0)}(\tau) = s u_{\psi(0)}(\tau)$ and a reparametrized piece of a minimal geodesic is automatically minimal. Hence $S(r_1) \subset \mathcal{E}\left(\frac{r_1}{r_0} K_{r_0}\right)$ for $r_1 \geq r_0$ and the required subanalyticity follows from Proposition 2.

□

Among 2 terms in expression (6) for $\operatorname{ind}(q(\cdot), \lambda, \nu)$ only the first one, the inertia index of the second variation, is nontrivial to evaluate. Fortunately, there is an efficient way to compute this index for both regular and singular (abnormal) geodesics, as well as a good supply of conditions that garantee the finiteness or infinity of the index (see [2, 4, 6, 9]). The simplest one is the *Goh condition* (see [6]):

If $\operatorname{ind}(q(\cdot), \psi(1), 0) < +\infty$, then $\psi(t)$ annihilates $\Delta^2_{q(t)}$, $\forall t \in [0, 1]$.

Recall that $\psi(t)$ annihilates $\Delta_{q(t)}$, $0 \leq t \leq 1$, for any Lagrange multiplier $(\psi(1), 0)$ associated with $q(\cdot)$. We say that $q(\cdot)$ is a *Goh geodesic* if there exist Lagrange multipliers $(\psi(1), 0)$ such that $\psi(t)$ annihilates $\Delta^2_{q(t)}$, $\forall t \in [0, 1]$. In particuar, strictly abnormal minimal geodesics must be Goh geodesics. Besides that, the Goh condition and Corollary 3 imply

COROLLARY 4. *Let M and the sub-Riemannian structure be real-analytic and $r_0 < r$. If there are no Goh geodesics of the length r_0, then ρ is subanalytic on* $\rho^{-1}((r_0, r])$.

I'll finish the paper with a brief analysis of the Goh condition. Suppose that $q(\cdot)$ is an abnormal geodesic with Lagrange multipliers $(\psi(1), 0)$, and $k = 2$. Differentiating the identities $h_1(\psi(t)) = h_2(\psi(t)) = 0$ with respect to t, we obtain $u_2(t)\{h_2, h_1\}(\psi(t)) = u_1(t)\{h_1, h_2\}(\psi(t)) = 0$, where $\{h_1, h_2\}(\psi(t)) = \langle \psi(t), [X_1, X_2](q(t)) \rangle$ is the Poisson bracket. In other words, the Goh condition is automatically satisfied by any abnormal geodesic.

The situation changes dramatically if $k > 2$. In order to understand why, we need some notation. Take $\lambda \in T^*M$ and set

$$b_0(\lambda) = \big(\{h_1, h_2\}(\lambda), \{h_1, h_3\}(\lambda), \dots, \{h_{k-1}, h_k\}(\lambda)\big),$$

a vector in $\mathbb{R}^{\frac{k(k-1)}{2}}$ whose coordinates are numbers $\{h_i, h_j\}(\lambda)$, $1 \le i < j \le k$, with lexicographically ordered indeces (i, j). Set also $\beta_0 = \frac{k(k-1)}{2}$. The Goh condition for $q(\cdot), \psi(1)$ implies the identity $b_0(\psi(t)) = 0$, $\forall t \in [0, 1]$. The differentiation of this identity with respect to t in virtue of (4) gives the equality

$$\sum_{i=1}^{k} u_i(t)\{h_i, b_0\}(\psi(t)) = 0, \quad 0 \le t \le 1. \tag{8}$$

Consider the space $\bigwedge^k \mathbb{R}^{\beta_0}$, the k-th exterior power of $\mathbb{R}^{\beta_0}$. The standard lexicographic basis in $\bigwedge^k \mathbb{R}^{\beta_0}$ gives the identification $\bigwedge^k \mathbb{R}^{\beta_0} \cong \mathbb{R}^{\binom{\beta_0}{k}}$. We set $\beta_1 = \beta + \binom{\beta_0}{k}$ and

$$b_1(\lambda) = (b_0(\lambda), \{h_1, b_0\}(\lambda) \wedge \cdots \wedge \{h_k, b_0\}(\lambda)) \in \mathbb{R}^{\beta_1}.$$

Equality (8) implies: $b_1(\psi(t)) = 0$, $0 \le t \le 1$.

Now we set by induction $\beta_{i+1} = \beta_i + \binom{\beta_i}{k}$, $i = 0, 1, 2, \dots$, and fix identifications $\mathbb{R}^{\beta_i} \times \mathbb{R}^{\binom{\beta_i}{k}} \cong \mathbb{R}^{\beta_{i+1}}$. Finaly, we define

$$b_{i+1}(\lambda) = (b_i(\lambda), \{h_1, b_i\}(\lambda) \wedge \cdots \wedge \{h_k, b_i\}(\lambda)) \in \mathbb{R}^{\beta_{i+1}}, \ i = 1, 2, \dots.$$

Successive differentiations of the Goh condition give the equations $b_i(\psi(t)) = 0$, $i = 1, 2, \dots$. It is easy to check that the equation $b_{i+1}(\lambda) = 0$ is not, in general, a consequence of the equation $b_i(\lambda) = 0$ and we indeed impose more and more restrictive conditions on the locus of Goh geodesics.

A natural conjecture is that admitting Goh geodesics distributions of rank $k > 2$ form a set of infinite codimension in the space of all rank k distributions, i.e. they do not appear in generic smooth families of distributions parametrized by finite-dimensional manifolds. It may be not technically easy, however, to turn this conjecture into the theorem.

Anyway, Goh geodesics are very exclusive for the distributions of rank greater than 2. Yet they may become typical under a priori restictions on the growth vector of the distribution (see [6]).

Note in proof. An essential progress was made while the paper was waiting for the publication. In particular, the conjecture on Goh geodesics has been proved as well as the conjecture stated at the end of the Introduction. These and other results will be included in our joined paper with Jean Paul Gauthier, now in preparation.

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