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INTEGRAL PROPERTIES OF RAPIDLY AND REGULARLY VARYING FUNCTIONS

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Abstract. Regularly and rapidly varying functions are studied as well as the asymptotic properties related to several classical inequalities and integral sums.

1. Introduction

Regular and rapid variation of functions was initiated by Karamata [3]. It is sometimes called Karamata theory. Nowadays, it is a well developed theory used in asymptotic analysis of functions, Tauberian theorems, probability and analytic number theory.

Recall that a measurable function $f : [a, \infty) \to (0, \infty)$, $a > 0$ is called *regularly varying* in the sense of Karamata if for some $\alpha \in R$ it satisfies

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha$$

for every $\lambda > 0$, and we denote $f \in R_\alpha$. The classes R_α, $\alpha \in R$ were introduced in [3], where it is proved (see also [1]) that if a function $f \in R_\alpha$, $\alpha > 0$, is locally bounded, then

$$\int_a^x f(t)\,dt \sim \frac{x}{\alpha+1} f(x), \quad x \to \infty.$$

We will use Potter's theorem (see e.g., [1]): If $f \in R_\alpha$, $\alpha > 0$, then for every $\mu > 1$ and $\varepsilon > 0$ there exists $x_0 > 0$ such that

$$\frac{f(y)}{f(x)} < \mu\Big(\frac{y}{x}\Big)^{\alpha+\varepsilon}, \quad x_0 \leqslant x < y$$

Recall [1], a measurable function $f : [a, \infty) \to (0, \infty)$, $a > 0$, is called *rapidly varying* in the sense of de Haan, with the index of variability ∞, if it satisfies

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \infty$$

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for every $\lambda > 1$. This functional class is denoted by R_∞ (see e.g., [1]). Example: If $f(x) = x^{r(x)}$, $r(x) \to \infty$, and r is a nondecreasing function, then $f \in R_\infty$.

Here we consider functions $f \in R_\infty$ defined on the interval $[0, \infty)$. Analogous results can be obtained if the domain of a function f is $[a, \infty)$, $a > 0$. Let $f \in R_\infty$ if $f(x) = O(\underline{f}(x))$, $x \to \infty$, where $\underline{f}(x) = \inf\{f(t) \mid x \leqslant t\}$, then we say $f \in MR_\infty$. In this case we also know that $\underline{f}$ is a nondecreasing function and $\underline{f} \leqslant f$. We will use the following properties of rapidly varying functions:

(1) If $f \in R_\infty$, then $\lim_{x\to\infty} \frac{\underline{f}(\lambda x)}{f(x)} = \infty$ and $\underline{f} \in R_\infty$.

(2) Using (1) we have: for $f \in R_\infty$ $\lim_{x\to\infty} \frac{f(\varphi(x))}{f(\psi(x))} = \infty$, if $\psi(x) \to \infty$ and $\liminf_{x\to\infty} \frac{\varphi(x)}{\psi(x)} > 1$.

(3) If $\lambda > 1$ and $f \in R_\infty$ is locally bounded on $[0, \infty)$, then
$$\int_0^x f(t)dt \sim \int_{\frac{x}{\lambda}}^x f(t)dt, \quad x \to \infty.$$

(4) If $f \in R_\infty$, $\varphi \in R_\alpha$ and f, φ are locally bounded on $[0, \infty)$, then
$$\int_0^x f(t)\varphi(t)dt \sim \varphi(x) \int_0^x f(t)\, dt, \quad x \to \infty.$$

(5) If $f \in MR_\infty$ is locally bounded on $[0, \infty)$, then
$$\frac{1}{x} \int_0^x f(t)\, dt = o(f(x)), \quad x \to \infty.$$

We use notation $(f(x) \gg g(x)$, $x \to a$ for $g(x) = o(f(x))$, $x \to a$.

2. Results

Our first theorem is connected with *Chebyshev's inequality*: if $f, g : [a, b] \to R$ are monotonic functions of the same monotonicity, then
$$\int_a^b f(t)\, g(t)\, dt \geqslant \frac{1}{b-a} \int_a^b f(t)\, dt \int_a^b g(t)\, dt.$$
If f, g are of different monotonicity, then the above inequality holds in the opposite direction.

Theorem 2.1. *Let $f, g \in MR_\infty$ be locally bounded on $[0, \infty)$. Then*
$$\int_0^x f(t)\, g(t)\, dt \gg \frac{1}{x} \int_0^x f(t)\, dt \int_0^x g(t)\, dt \gg \int_0^x f(t)\, g(x-t)\, dt, \quad x \to \infty.$$

The following theorem is connected with *Jensen's inequality*: let $f : R \to R$ be a convex function and $\varphi : [a, b] \to (0, +\infty)$ a nondecreasing function. Then
$$f\left(\frac{1}{b-a} \int_a^b \varphi(x)\, dx\right) \leqslant \frac{1}{b-a} \int_a^b f(\varphi(x))\, dx.$$

Theorem 2.2. *Let $f \in R_\infty$, $\varphi \in R_\alpha$, $\alpha > 0$ be locally bounded on $[0, \infty)$. Then*
$$f\left(\frac{1}{x} \int_0^x \varphi(t)\, dt\right) \ll \frac{1}{x} \int_0^x f(\varphi(t))\, dt, \quad x \to \infty.$$

THEOREM 2.3. *Let $f \in R_\infty$ and $\varepsilon > 0$. Then*

$$f\Big(\frac{n}{1+\varepsilon}\Big) \ll \frac{f(1)+f(2)+\cdots+f(n)}{n}, \quad n \to \infty.$$

THEOREM 2.4. *Let $f \in R_\infty$ be a locally bounded on $[0,\infty)$. Then for every $p > 1$,*

$$\frac{1}{x}\int_0^x f(t)\,dt \ll \left(\frac{1}{x}\int_0^x f(t)^p dt\right)^{\frac{1}{p}}, \quad x \to \infty.$$

THEOREM 2.5. *If $f \in R_\infty, \varphi \in R_\alpha$ are locally bounded functions on $[0,\infty)$, then*

$$\int_0^x f(t)\varphi(t)\,dt \ll \left(\int_0^x f(t)^p dt\right)^{\frac{1}{p}} \left(\int_0^x \varphi(t)^q dt\right)^{\frac{1}{q}}, \quad x \to \infty$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p>0$, $q>0$.

Using Theorem 2.4 for the function $f([x]+1) \in R_\infty$ on $(0,n)$ we get:

COROLLARY 2.1. *If $f \in R_\infty$ and $p > 1$, then*

$$\frac{f(1)+f(2)+\cdots+f(n)}{n} \ll \left(\frac{f(1)^p+f(2)^p+\cdots+f(n)^p}{n}\right)^{\frac{1}{p}}, \quad n \to \infty.$$

THEOREM 2.6. *Let $f \in R_\infty$ be a locally bounded function on $[0,\infty)$ and $n \in N$. Then*

$$\frac{f(\frac{x}{n})+f(\frac{2x}{n})+\cdots+f(\frac{(n-1)x}{n})}{n} \ll \frac{1}{x}\int_0^x f(t)\,dt, \quad x \to \infty.$$

THEOREM 2.7. *Let $f \in MR_\infty$ be a locally bounded function on $[0,\infty)$ and let $n \in N$. Then*

$$\frac{f(\frac{x}{n})+f(\frac{2x}{n})+\cdots+f(x)}{n} \gg \frac{1}{x}\int_0^x f(t)\,dt, \quad x \to \infty.$$

3. Proofs

PROOF OF THEOREM 2.1. It is enough to prove that for every $M > 1$

$$\frac{1}{M}\int_0^x f(t)\,g(t)\,dt > \frac{1}{x}\int_0^x f(t)\,dt\int_0^x g(t)\,dt > M\int_0^x f(t)\,g(x-t)\,dt$$

for sufficiently large x. There exists $m \in (0,1)$ and $x_1 > 0$ such that

$$mf(x) \leqslant \underline{f}(x), mg(x) \leqslant \underline{g}(x), \quad x > x_1.$$

Let $1 < \lambda < \frac{4M}{4M-m^2}$; then $\frac{m^2}{4M(1-\frac{1}{\lambda})} > 1$. Further on, there exists $x_2 > 0$, such that if $x > x_2$, then from (1) and (3) we obtain

$$\frac{1}{2}\int_0^x \underline{f}(t)\,dt \leqslant \int_{\frac{x}{\lambda}}^x \underline{f}(t)\,dt, \frac{1}{2}\int_0^x \underline{g}(t)\,dt \leqslant \int_{\frac{x}{\lambda}}^x \underline{g}(t)\,dt.$$

Now for $x > x_0 = \max\{x_1, x_2\}$, by Chebyshev's inequality for the nondecreasing functions $\underline{f}, \underline{g}$ on the interval $\big(\frac{x}{\lambda}, x\big)$, we have

$$\begin{aligned}\frac{1}{M}\int_0^x f(t)\,g(t)\,dt &\geqslant \frac{1}{M}\int_0^x \underline{f}(t)\,\underline{g}(t)\,dt \geqslant \frac{1}{M}\int_{\frac{x}{\lambda}}^x \underline{f}(t)\,\underline{g}(t)\,dt\\ &\geqslant \frac{1}{M}\frac{1}{x-\frac{x}{\lambda}}\int_{\frac{x}{\lambda}}^x \underline{f}(t)\,dt\int_{\frac{x}{\lambda}}^x \underline{g}(t)\,dt \geqslant \frac{1}{x}\frac{1}{M(1-\frac{1}{\lambda})}\frac{1}{4}\int_0^x \underline{f}(t)\,dt\int_0^x \underline{g}(t)\,dt.\\ &\geqslant \frac{1}{4xM(1-\frac{1}{\lambda})}m^2\int_0^x f(t)\,dt\int_0^x g(t)\,dt > \frac{1}{x}\int_0^x f(t)\,dt\int_0^x g(t)\,dt\end{aligned}$$

In a similar way we can show $\frac{1}{x}\int_0^x f(t)\,dt\int_0^x g(t)\,dt > M\int_a^x f(t)g(x-t)\,dt$ for sufficiently large x. $\square$

Proof of Theorem 2.2. If $\varphi \in R_\alpha$, then for every $\lambda > 0$

$$\varphi(\lambda x) \sim \lambda^\alpha \varphi(x), \ \ x\to\infty; \qquad \int_0^x \varphi(t)\,dt \sim \frac{x}{\alpha+1}\varphi(x), \ \ x\to\infty.$$

Let $i : [0,+\infty) \to (0,\infty)$ be a locally integrable function such that $\lim_{x\to\infty} i(x) = 1$ and $\frac{1}{x}\int_0^x \varphi(t)\,dt = \frac{1}{1+\alpha}\varphi(x)i(x)$, for every $x > 0$. Let $1 < \lambda < (1+\alpha)^{\frac{1}{\alpha}}$; then $\frac{\alpha+1}{\lambda^\alpha} > 1$. For sufficiently large x and $t \geqslant \frac{x}{\sqrt{\lambda}}$, using $f\circ\varphi \in R_\infty$, we have

$$\frac{1}{x}\int_0^x f(\varphi(t))\,dt > \frac{1}{x}\int_{\frac{x}{\sqrt{\lambda}}}^x f(\varphi(t))\,dt \geqslant \frac{1}{x}\left(x-\frac{x}{\sqrt{\lambda}}\right)\underline{f\circ\varphi}\left(\frac{x}{\sqrt{\lambda}}\right).$$

Now by (1), we obtain

$$\frac{\underline{f\circ\varphi}(\frac{x}{\sqrt{\lambda}}))}{f(\varphi(\frac{x}{\lambda}))} \to \infty, \quad x\to\infty.$$

There is a function $j : (0,+\infty) \to R$, $\lim_{x\to\infty} j(x) = 1$ such that, for every $x > 0$

$$\varphi\left(\frac{x}{\lambda}\right) = \frac{1}{\lambda^\alpha}\varphi(x)j(x).$$

Using

$$\liminf_{x\to\infty}\frac{\frac{1}{\lambda^\alpha}\varphi(x)j(x)}{\frac{1}{1+\alpha}\varphi(x)i(x)} = \frac{1+\alpha}{\lambda^\alpha} > 1,$$

and (2), we have

$$\lim_{x\to\infty}\frac{f(\varphi\left(\frac{x}{\lambda}\right))}{f\left(\frac{1}{x}\int_0^x\varphi(t)\,dt\right)} = \lim_{x\to\infty}\frac{f\left(\frac{1}{\lambda^\alpha}\varphi(x)j(x)\right)}{f\left(\frac{1}{1+\alpha}\varphi(x)i(x)\right)} = \infty.$$

Finally

$$f\left(\frac{1}{x}\int_0^x \varphi(t)\,dt\right) \ll \frac{1}{x}\int_0^x f(\varphi(t))\,dt, \quad x\to\infty. \qquad \square$$

Proof of Theorem 2.3. Let $k \in N$ so that $(1+2k)^{\frac{1}{k}} < 1+\epsilon$, $g(x) = f(\sqrt[k]{x})$ and $\varphi(x) = [x+1]^k$ (where $[\cdot]$ denotes the integer part). Obviously $\varphi \in R_k$ and $g \in R_\infty$. Using Theorem 2.2 we get

$$\frac{1}{n}\int_0^n g(\varphi(x))\,dx \gg g\left(\frac{1}{n}\int_0^n \varphi(x)\,dx\right), \quad n\to\infty.$$

This leads to

$$\frac{g(1)+g(2^k)+\cdots+g(n^k)}{n} \gg g\left(\frac{1+2^k+\cdots+n^k}{n}\right), \quad n\to\infty.$$

Since $\lim_{n\to\infty}\frac{(1+2^k+\cdots+n^k)(1+2k)}{n^{k+1}}=\frac{2k+1}{k+1}>1$, by (2) we have

$$\lim_{n\to\infty}\frac{g\left(\frac{1+2^k+\cdots+n^k}{n}\right)}{g\left(\frac{n^k}{1+2k}\right)}=\infty.$$

Finally

$$\frac{f(1)+\cdots+f(n)}{n}=\frac{g(1)+g(2^k)+\cdots+g(n^k)}{n}\gg g\left(\frac{n^k}{1+2k}\right), \quad n\to\infty,$$

and now by (2) and $(1+2k)^{\frac{1}{k}}<1+\varepsilon$ we obtain

$$g\left(\frac{n^k}{1+2k}\right)=f\left(\frac{n}{(1+2k)^{\frac{1}{k}}}\right)\gg f\left(\frac{n}{1+\varepsilon}\right), \quad n\to\infty \qquad \square$$

Proof of Theorem 2.4. We will use the inequality

$$(*) \qquad \frac{1}{b-a}\int_a^b f(t)\,dt \leqslant \left(\frac{1}{b-a}\int_a^b f(t)^p dt\right)^{\frac{1}{p}}$$

where $p>1$ and f is a nonnegative function. Let $M>1$ be an arbitrary number and $p>1$. Let $1<\lambda\leqslant\frac{1}{1-\left(\frac{1}{2M}\right)^{\frac{p}{p-1}}}$; then $2M\left(1-\frac{1}{\lambda}\right)^{1-\frac{1}{p}}\leqslant 1$. For sufficiently large x, using (3) and (*), we get

$$\begin{aligned}\frac{M}{x}\int_0^x f(t)\,dt &\leqslant \frac{2M}{x}\int_{\frac{x}{\lambda}}^x f(t)\,dt = 2M\frac{\lambda-1}{\lambda}\frac{1}{x-\frac{x}{\lambda}}\int_{\frac{x}{\lambda}}^x f(t)\,dt\\ &\leqslant 2M\frac{\lambda-1}{\lambda}\left(\frac{1}{x-\frac{x}{\lambda}}\int_{\frac{x}{\lambda}}^x f(t)^p dt\right)^{\frac{1}{p}}\\ &= 2M\left(\frac{\lambda-1}{\lambda}\right)^{1-\frac{1}{p}}\left(\frac{1}{x}\int_{\frac{x}{\lambda}}^x f(t)^p dt\right)^{\frac{1}{p}} \leqslant \left(\frac{1}{x}\int_0^x f(t)^p dt\right)^{\frac{1}{p}}. \qquad \square\end{aligned}$$

Proof of Theorem 2.5. Note that $\varphi(x)^q\in R_{\alpha q}$, by (4) we have

$$\int_0^x \varphi(t)^q dt \sim \frac{x\varphi(t)^q}{\alpha q+1}, \quad x\to\infty$$

$$\int_0^x f(t)\varphi(t)\,dt \sim \varphi(x)\int_0^x f(t)\,dt, \quad x\to\infty.$$

Now we apply Theorem 2.4 and obtain

$$\varphi(x)\int_0^x f(t)\,dt = x\varphi(x)\frac{1}{x}\int_0^x f(t)\,dt \ll x\varphi(x)\left(\frac{1}{x}\int_0^x f(t)^p dt\right)^{\frac{1}{p}}, \quad x\to\infty.$$

Finally

$$x\varphi(x)\left(\frac{1}{x}\int_0^x f(t)^p dt\right)^{\frac{1}{p}} = x^{\frac{1}{q}}\varphi(x)\left(\int_0^x f(t)^p dt\right)^{\frac{1}{p}}$$
$$\sim \left(\int_0^x f(t)^p\right)^{\frac{1}{p}} x^{\frac{1}{q}}\left(\frac{\alpha q+1}{x}\right)^{\frac{1}{q}}\left(\int_0^x \varphi(t)^q\right)^{\frac{1}{q}}, \quad x\to\infty$$
$$\int_0^x f(t)\varphi(t)\,dt \ll (\alpha q+1)^{\frac{1}{q}}\left(\int_0^x f(t)^p\right)^{\frac{1}{p}}\left(\int_0^x \varphi(t)^q dt\right)^{\frac{1}{q}}, \quad x\to\infty. \qquad \square$$

Proof of Theorem 2.6. Let $M > 0$, $n \in N$ and $\lambda = \frac{n-\frac{1}{2}}{n-1} > 1$. For a sufficiently large x we have
$$\underline{f}\left(\frac{k-\frac{1}{2}}{n}x\right) \geqslant \underline{f}\left(\lambda\frac{k-1}{n}x\right) > 2Mf\left(\frac{k-1}{n}x\right)$$
for every $k \in \{2,3,\dots,n\}$. Now it follows
$$\frac{1}{x}\int_0^x f(t)\,dt = \frac{1}{x}\sum_{k=1}^n \int_{\frac{k-1}{n}x}^{\frac{k}{n}x} f(t)\,dt > \frac{1}{x}\sum_{k=2}^n \int_{\frac{k-\frac{1}{2}}{n}x}^{\frac{k}{n}x} f(t)\,dt$$
$$\geqslant \frac{1}{x}\sum_{k=2}^n \frac{x}{2n}\underline{f}\Big(\frac{k-\frac{1}{2}}{n}x\Big) > \frac{1}{2n}\sum_{k=2}^n 2Mf\Big(\frac{k-1}{n}x\Big) = M\frac{f(\frac{x}{n})+\cdots+f(\frac{n-1}{n}x)}{n}. \quad \square$$

Proof of Theorem 2.7. Let $n \in N$ and $M > 1$. Then for a sufficiently large x and every $k \in \{1,2,\dots,n\}$ we have by (5),
$$f\Big(\frac{k}{n}x\Big) \geqslant Mn^2\frac{k}{nx}\int_0^{\frac{k}{n}x} f(t)\,dt \geqslant \frac{Mn}{x}\int_{\frac{k-1}{n}x}^{\frac{k}{n}x} f(t)\,dt.$$
Summing by $k \in 1,2,\dots,n$ we have
$$\frac{f\left(\frac{x}{n}\right)+f\left(\frac{2x}{n}\right)+\cdots+f(x)}{n} \geqslant M\frac{1}{x}\int_0^x f(t)\,dt. \qquad \square$$

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