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ON THE SECOND ORDER INFINITESIMAL BENDINGS OF A CLASS OF TOROIDS

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Abstract. This paper is devoted to a study of the second order infinitesimal bendings of a class of toroids generated by a simple polygonal meridian. A necessary and sufficient condition for the existence of the second order infinitesimal bendings is determined.

1. Introduction

Let us suppose that the meridian of a rotational surface T is a simple polygon with apeces $A_i(u_i,\rho_i)$, $\rho_i>0$, $i=1,\dots,n$, $A_{n+1}\equiv A_1$ at the orthogonal coordinate system $uO\rho$, with orths $\bar e$, $\bar a(v)$, $\bar a'(v)$, where $\bar e$ is unit vector of the axis of rotation, $\bar a(v)$ unit vector of the ρ-axis where v is the angle between the plane of initial position of the meridian and $\bar a(v)$, then $\bar a'(v)\perp\bar a(v)$ and $\bar a'(v)\perp\bar e$ (see [1], page 90). The equations of the sides are

$$\rho_{(i)}=\rho_i+\frac{\rho_{i+1}-\rho_i}{u_{i+1}-u_i}(u-u_i)=k_iu+n_i, \tag{1.1}$$

where $\rho_{(i)}$ is the value of ρ on A_iA_{i+1}, and

$$k_i=\frac{\rho_{i+1}-\rho_i}{u_{i+1}-u_i},\ n_i=\rho_i-\frac{\rho_{i+1}-\rho_i}{u_{i+1}-u_i}. \tag{1.2}$$

By rotation of the mentioned polygon round the u-axis one gets the toroid T. Let us consider the second order infinitesimal bendings of such a toroid.

We shall use here Cohn-Vossen's method [1], [2] and [3]. Infinitesimal bendings of the first order for the toroid rotational surfaces with polygonal meridian were considered at [4] and [5].

The correspondence between the volume variation and the flow of the infinitesimal bending for piecewise smooth surfaces was studied at remarkable paper [8].

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References for later investigations on infinitesimal bendings can also be found at [6] and [7].

The radius vector of surface T is represented in the known way [1]

$$\bar{r}(u,v)=u\bar{e}+\rho(u)\bar{a}(v), \tag{1.3}$$

where $\rho=\rho(u)$ is the equation of the meridian. Let

$$\overset{(1)}{\bar{z}}(u,v)=\overset{(1)}{\alpha}(u,v)\bar{e}+\overset{(1)}{\beta}(u,v)\bar{a}+\overset{(1)}{\gamma}(u,v)\bar{a}', \tag{1.4}$$

$$\overset{(2)}{\bar{z}}(u,v)=\overset{(2)}{\alpha}(u,v)\bar{e}+\overset{(2)}{\beta}(u,v)\bar{a}+\overset{(2)}{\gamma}(u,v)\bar{a}' \tag{1.5}$$

be continuous fields of infinitesimal bendings of the first and the second order respectively, where $\overset{(2)}{\bar{z}}(u,v)$ is an extension of the field $\overset{(1)}{\bar{z}}(u,v)$. The fields $\overset{(1)}{\bar{z}}(u,v)$ and $\overset{(2)}{\bar{z}}(u,v)$ satisfy the systems of differential equations [1]:

$$\overset{(1)}{\alpha}_u+\rho'\overset{(1)}{\beta}_u=0,\ \overset{(1)}{\beta}+\overset{(1)}{\gamma}_v=0,\ \overset{(1)}{\alpha}_v+\rho'(\overset{(1)}{\beta}_v-\overset{(1)}{\gamma})+\rho\overset{(1)}{\gamma}_u=0, \tag{1.6}$$

$$\begin{aligned}\overset{(2)}{\alpha}_u+\rho'\overset{(2)}{\beta}_u&=-\frac{1}{2}\left[\overset{(1)}{\alpha}{}^2_u+\overset{(1)}{\beta}{}^2_u+\overset{(1)}{\gamma}{}^2_u\right],\\ \overset{(2)}{\beta}+\overset{(2)}{\gamma}_v&=-\frac{1}{2\rho}\left[\overset{(1)}{\alpha}{}^2_v+(\overset{(1)}{\beta}_v-\overset{(1)}{\gamma})^2\right],\\ \overset{(2)}{\alpha}_v+\rho'(\overset{(2)}{\beta}_v-\overset{(2)}{\gamma})+\rho'\overset{(2)}{\gamma}_u&=-\left[\overset{(1)}{\alpha}_u\overset{(1)}{\alpha}_v+\overset{(1)}{\beta}_u(\overset{(1)}{\beta}_v-\overset{(1)}{\gamma})\right].\end{aligned} \tag{1.7}$$

According to [1] we have the first order fundamental field

$$\overset{(1)}{\bar{z}}_k(u,v)=\overset{(1)}{\alpha}_k(u,v)\bar{e}+\overset{(1)}{\beta}_k(u,v)\bar{a}+\overset{(1)}{\gamma}_k(u,v)\bar{a}',\quad k\geq 2,\quad k\in N \tag{1.8}$$

where

$$\begin{aligned}\overset{(1)}{\alpha}_k(u,v)&=\overset{(1)}{\varphi}_k(u)e^{ikv}+\overset{(1)}{\varphi}_{-k}(u)e^{-ikv},\\ \overset{(1)}{\beta}_k(u,v)&=\overset{(1)}{\psi}_k(u)e^{ikv}+\overset{(1)}{\psi}_{-k}(u)e^{-ikv},\\ \overset{(1)}{\gamma}_k(u,v)&=\overset{(1)}{\chi}_k(u)e^{ikv}+\overset{(1)}{\chi}_{-k}(u)e^{-ikv}.\end{aligned} \tag{1.9}$$

The field of infinitesimal bendings $\overset{(2)}{\bar{z}}_{2k}(u,v)$ of the second order is

$$\overset{(2)}{\bar{z}}_{2k}(u,v)=\overset{(2)}{\alpha}_{2k}(u,v)\bar{e}+\overset{(2)}{\beta}_{2k}(u,v)\bar{a}+\overset{(2)}{\gamma}_{2k}(u,v)\bar{a}',\ k\geq 2,\ k\in N, \tag{1.10}$$

where

$$\begin{aligned}\overset{(2)}{\alpha}_{2k}(u,v)&=\overset{(2)}{\varphi}_{2k}(u)e^{2ikv}+\overset{(2)}{\varphi}_{-2k}(u)e^{-2ikv}+\overset{(2)}{\varphi}_0(u),\\ \overset{(2)}{\beta}_{2k}(u,v)&=\overset{(2)}{\psi}_{2k}(u)e^{2ikv}+\overset{(2)}{\psi}_{-2k}(u)e^{-2ikv}+\overset{(2)}{\psi}_0(u),\\ \overset{(2)}{\gamma}_{2k}(u,v)&=\overset{(2)}{\chi}_{2k}(u)e^{2ikv}+\overset{(2)}{\chi}_{-2k}(u)e^{-2ikv}+\overset{(2)}{\chi}_0(u).\end{aligned} \tag{1.11}$$

The functions $\overset{(1)}{\varphi}_k(u)$, $\overset{(1)}{\psi}_k(u)$, $\overset{(1)}{\chi}_k(u)$, $\overset{(2)}{\varphi}_{2k}(u)$, $\overset{(2)}{\psi}_{2k}(u)$, $\overset{(2)}{\chi}_{2k}(u)$, $\overset{(2)}{\varphi}_0(u)$, $\overset{(2)}{\psi}_0(u)$, $\overset{(2)}{\chi}_0(u)$ satisfy the following systems of differential equations

$$\overset{(1)}{\varphi}{}'_k + \rho'\overset{(1)}{\psi}{}'_k = 0, \quad ik\overset{(1)}{\chi}_k + \overset{(1)}{\psi}_k = 0, \quad ik\overset{(1)}{\varphi}_k + \rho'(\overset{(1)}{\psi}_k - \overset{(1)}{\chi}_k) + \rho\overset{(1)}{\chi}_k = 0, \tag{1.12}$$

$$\begin{aligned}
\overset{(2)}{\varphi}{}'_{2k} + \rho'\overset{(2)}{\psi}_{2k} &= -\frac{1}{2}\left[\overset{(1)}{\varphi}{}'^2_k + \overset{(1)}{\psi}{}'^2_k + \overset{(1)}{\chi}{}'^2_k\right],\\
2ik\overset{(2)}{\chi}_{2k} + \overset{(2)}{\psi}_{2k} &= -\frac{1}{2\rho}\left[-k^2\overset{(1)}{\varphi}{}^2_k + (ik\overset{(1)}{\psi}_k - \overset{(1)}{\chi}_k)^2\right],\\
2ik\overset{(2)}{\varphi}_{2k} + \rho'(2ik\overset{(2)}{\psi}_{2k} - \overset{(2)}{\chi}_{2k}) + \rho\overset{(2)}{\chi}{}'_{2k} &= -ik\overset{(1)}{\varphi}_k\overset{(1)}{\varphi}{}'_k - \overset{(1)}{\psi}{}'_k(ik\overset{(1)}{\psi}_k - \overset{(1)}{\chi}_k),
\end{aligned} \tag{1.13}$$

$$\begin{aligned}
\overset{(2)}{\varphi}{}'_0 + \rho'\overset{(2)}{\psi}{}'_0 &= -\overset{(1)}{\varphi}{}'_k\overset{(1)}{\varphi}{}'_{-k} - \overset{(1)}{\psi}{}'_k\overset{(1)}{\psi}{}'_{-k} - \overset{(1)}{\chi}{}'_k\overset{(1)}{\chi}{}'_{-k},\\
\overset{(2)}{\psi}_0 &= -\frac{1}{\rho}\left[k^2\overset{(1)}{\varphi}_k\overset{(1)}{\varphi}_{-k} - (ik\overset{(1)}{\psi}_k - \overset{(1)}{\chi}_k)(ik\overset{(1)}{\psi}_{-k} - \overset{(1)}{\chi}_{-k})\right]\\
-\rho'\overset{(2)}{\chi}_0 + \rho\overset{(2)}{\chi}{}'_0 &= ik\overset{(1)}{\varphi}{}'_k\overset{(1)}{\varphi}_{-k} - ik\overset{(1)}{\varphi}_k\overset{(1)}{\varphi}{}'_{-k} - \overset{(1)}{\psi}{}'_k(-ik\overset{(1)}{\psi}_{-k} - \overset{(1)}{\chi}_{-k})\\
&\quad - \overset{(1)}{\psi}{}'_{-k}(ik\overset{(1)}{\psi}_k - \overset{(1)}{\chi}_k).
\end{aligned} \tag{1.14}$$

The system (1.13) can be written in the form

$$\begin{aligned}
\overset{(2)}{\varphi}{}'_{2k} + \rho'\overset{(2)}{\psi}{}'_{2k} &= \overset{(2)}{A}_{2k}(u),\\
2ik\overset{(2)}{\chi}_{2k} + \overset{(2)}{\psi}_{2k} &= \overset{(2)}{B}_{2k}(u),\\
2ik\overset{(2)}{\varphi}_{2k} + \rho'(2ik\overset{(2)}{\psi}_{2k} - \overset{(2)}{\chi}_{2k}) + \rho\overset{(2)}{\chi}{}'_{2k} &= \overset{(2)}{C}_{2k}(u),
\end{aligned} \tag{1.13$'$}$$

where $\overset{(2)}{A}_{2k}(u)$, $\overset{(2)}{B}_{2k}(u)$, $\overset{(2)}{C}_{2k}(u)$, are the right sides of (1.13). If we express the right sides of (1.13) by $\overset{(1)}{\psi}_k$ we'll have according to [2]

$$\begin{aligned}
\overset{(2)}{\varphi}{}'_{2k} + \rho'\overset{(2)}{\psi}_{2k} &= \frac{1}{2k^2}\left[1 - (1+\rho'^2)k^2\right]\overset{(1)}{\psi}{}'^2_k,\\
2ik\overset{(2)}{\chi}_{2k} + \overset{(2)}{\psi}_{2k} &= \left[(k^2-1)^2(1+\rho'^2)\overset{(1)}{\psi}{}^2_k + 2\rho\rho'(k^2-1)\overset{(1)}{\psi}_k\overset{(1)}{\psi}{}'_k + \rho^2\overset{(1)}{\psi}{}'^2_k\right]/(2\rho k^2),\\
2ik\overset{(2)}{\varphi}_{2k} + \rho'(2ik\overset{(2)}{\psi}_{2k} - \overset{(2)}{\chi}_{2k}) + \rho\overset{(2)}{\chi}{}'_{2k} &= -i\left[(1+\rho'^2)(k^2-1)\overset{(1)}{\psi}_k\overset{(1)}{\psi}{}'_k + \rho\rho'\overset{(1)}{\psi}{}'^2_k\right]/k.
\end{aligned} \tag{1.13$''$}$$

In the same way, the equations (1.14) are

$$\begin{aligned} \overset{(2)}{\varphi}{}'_0 + \rho' \overset{(2)}{\psi}_0 &= \overset{(2)}{A}_0(u), \\ \overset{(2)}{\psi}_0 &= \overset{(2)}{B}_0(u), \\ \rho \overset{(2)}{\chi}{}'_0 - \rho' \overset{(2)}{\chi}_0 &= \overset{(2)}{C}_0(u), \end{aligned} \tag{1.14'}$$

or

$$\begin{aligned} \overset{(2)}{\varphi}{}'_0 + \rho' \overset{(2)}{\psi}{}'_0 &= -\frac{1}{k^2}\left[k^2(1+\rho'^2)+1\right]\overset{(1)}{\psi}{}'_k \overset{(1)}{\psi}{}'_{-k}, \\ \overset{(2)}{\psi}_0 &= -\Big[(k^2-1)^2(1+\rho'^2)\overset{(1)}{\psi}_k \overset{(1)}{\psi}_{-k} \\ &\quad + \rho\rho'(k^2-1)(\overset{(1)}{\psi}_{-k}\overset{(1)}{\psi}{}'_k + \overset{(1)}{\psi}{}'_{-k}\overset{(1)}{\psi}_k) + \rho^2 \overset{(1)}{\psi}{}'_{-k}\overset{(1)}{\psi}{}'_k\Big]/(\rho k^2), \\ \rho \overset{(2)}{\chi}{}'_0 - \rho' \overset{(2)}{\chi}_0 &= i(k^2-1)(1+\rho'^2)(\overset{(1)}{\psi}{}'_k \overset{(1)}{\psi}_{-k} - \overset{(1)}{\psi}_k \overset{(1)}{\psi}{}'_{-k})/k. \end{aligned} \tag{1.14''}$$

The system (1.12) is equivalent to the differential equation of the second order with respect to $\overset{(1)}{\psi}_k$ (see[1]):

$$\rho \overset{(1)}{\psi}{}''_k + (k^2-1)\rho'' \overset{(1)}{\psi}_k = 0, \tag{1.15}$$

and the system (1.13) is equivalent to the differential equation (see [2]):

$$\rho \overset{(2)}{\psi}{}''_{2k} + (4k^2-1)\rho'' \overset{(2)}{\psi}_{2k} = \overset{(2)}{R}_{2k}(u) \tag{1.16}$$

where

$$\overset{(2)}{R}_{2k}(u) = -\rho'' \overset{(2)}{B}_{2k}(u) + \rho \overset{(2)}{B}{}''_{2k}(u) - 4k^2 \overset{(2)}{A}_{2k}(u) - 2ik \overset{(2)}{C}{}'_{2k}(u) \tag{1.17}$$

i.e.

$$\begin{aligned} \overset{(2)}{R}_{2k}(u) = \frac{(k^2-1)^2\rho'^2}{k^2\rho^2}&\left[1+\rho'^2+\rho\rho''(k^2-1)+\frac{k^2\rho\rho''}{\rho'^2}\right]\overset{(1)}{\psi}{}^2_k \\ &- \frac{2\rho'(k^2-1)}{k^2\rho}\left[\rho\rho''+(k^2-1)(1+\rho'^2)\right]\overset{(1)}{\psi}_k \overset{(1)}{\psi}{}'_k \\ &+ \frac{1}{k^2}\left[(1+\rho'^2)(1-k^2)^2-\rho\rho''(1+k^2)\right]\overset{(1)}{\psi}{}'^2_k \end{aligned} \tag{1.17'}$$

2. Infinitesimal bendings of the second order of a toroid with poligonal meridian

The meridian of our toroid consists of line segments $\rho_{(i)}(u)=k_iu+n_i$. From (1.15)

$$\overset{(1)}{\psi}_{k,i}(u)=\overset{(1)}{M}_{k,i}u+\overset{(1)}{N}_{k,i},\quad i=1,\dots,n,\quad (\overset{(1)}{M}_{k,i}=M_i,\overset{(1)}{N}_{k,i}=N_i) \tag{2.1}$$

i.e. we have the linearity of the field of infinitesimal bendings of the first order, where $M_i,\ N_i$ are the constants defined in [4]. For the determination of the second order field of infinitesimal bendings, according to (1.16) we have the equation

$$\rho_{(i)}(u)\overset{(2)}{\psi}{}''_{2k,i}(u)=\overset{(2)}{R}_{2k,i}(u) \tag{2.2}$$

where

$$\begin{aligned}\overset{(2)}{R}_{2k,i}(u)=&\frac{(k^2-1)^2\rho'^2_{(i)}}{k^2\rho^2_{(i)}}\left[1+\rho'^2_{(i)}+\rho_{(i)}\rho''_{(i)}(k^2-1)+\frac{k^2\rho_{(i)}\rho''_{(i)}}{\rho'^2_{(i)}}\right]\overset{(1)}{\psi}{}^2_{k,i}\\&-\frac{2\rho'_{(i)}(k^2-1)}{k^2\rho_{(i)}}\left[\rho_{(i)}\rho''_{(i)}+(k^2-1)(1+\rho'^2_{(i)})\right]\overset{(1)}{\psi}_{k,i}\overset{(1)}{\psi}{}'_{k,i}\\&+\frac{1}{k^2}\left[(1+\rho'^2_{(i)})(1-k^2)^2-\rho_{(i)}\rho''_{(i)}(1+k^2)\right]\overset{(1)}{\psi}{}'^2_{k,i}.\end{aligned} \tag{2.3}$$

Since $\rho'_{(i)}=k_i,\quad \rho''_{(i)}=0$, we have

$$\overset{(2)}{R}_{2k,i}(u)=\frac{(k^2-1)^2(1+k_i^2)}{k^2}\left[\frac{k_i^2(M_iu+N_i)^2}{(k_iu+n_i)^2}-2k_iM_i\frac{M_iu+N_i}{k_iu+n_i}+M_i^2\right]. \tag{2.4}$$

In order to find $\overset{(2)}{\psi}_{2k,i}(u)$, we have to solve the differential equation of the second order (2.2), which from (2.3,4) becomes:

$$\overset{(2)}{\psi}{}''_{2k,i}(u)=\frac{(k^2-1)^2(1+k_i^2)}{k^2}\left[\frac{k_i^2(M_iu+N_i)^2}{(k_iu+n_i)^3}-2k_iM_i\frac{M_iu+N_i}{(k_iu+n_i)^2}+\frac{M_i^2}{(k_iu+n_i)}\right]. \tag{2.5}$$

The solution of this equation is

$$\overset{(2)}{\psi}_{2k,i}(u)=\frac{(k^2-1)^2(1+k_i^2)(k_iN_i-M_in_i)^2}{2k^2k_i^2(k_iu+n_i)}+\overset{(2)}{M}_{2k,i}u+\overset{(2)}{N}_{2k,i}, \tag{2.6}$$

where $\overset{(2)}{M}_{2k,i},\ \overset{(2)}{N}_{2k,i},\ i=1,2,\dots,n$ are the constants that are to be found. Rotational toroid generated by a polygonal meridian consists of n regular parts. Let us denote them by $T_i,\ i=1,2,\dots,n$. Let $c_i,\ i=1,2,\dots,n$, be regular parts of meridian, $c_i:\rho_{(i)}=\rho_{(i)}(u)$. We shall denote by $\overset{(1)}{z}_{k,i}(u,v)$ and $\overset{(2)}{z}_{2k,i}(u,v)$ the fields of infinitesimal bendings at T_i. The fields $\overset{(1)}{z}_{k,i}(u,v)$, $\overset{(2)}{z}_{2k,i}(u,v)$ are to be continuous on the whole surface. At the circles circuscribed by the apeces of the meridian, the fields of infinitesimal bendings of the first and the second order are to be continuous [1]. From that fact, at the mentioned points $u = \sigma$ of the meridian we have for the field of infinitesimal bendings of the first order:

$$\overset{(1)}{\varphi}_{k,i}(\sigma) = \overset{(1)}{\varphi}_{k,i+1}(\sigma),\ \overset{(1)}{\psi}_{k,i}(\sigma) = \overset{(1)}{\psi}_{k,i+1}(\sigma),\ \overset{(1)}{\chi}_{k,i}(\sigma) = \overset{(1)}{\chi}_{k,i+1}(\sigma) \tag{2.7}$$

and in the same manner for the field of the second order:

$$\overset{(2)}{\varphi}_{2k,i}(\sigma) = \overset{(2)}{\varphi}_{2k,i+1}(\sigma),\ \overset{(2)}{\psi}_{2k,i}(\sigma) = \overset{(2)}{\psi}_{2k,i+1}(\sigma),\ \overset{(2)}{\chi}_{2k,i}(\sigma) = \overset{(2)}{\chi}_{2k,i+1}(\sigma), \tag{2.8}$$

$$\overset{(2)}{\varphi}_{0,i}(\sigma) = \overset{(2)}{\varphi}_{0,i+1}(\sigma),\ \overset{(2)}{\psi}_{0,i}(\sigma) = \overset{(2)}{\psi}_{0,i+1}(\sigma),\ \overset{(2)}{\chi}_{0,i}(\sigma) = \overset{(2)}{\chi}_{0,i+1}(\sigma). \tag{2.9}$$

At the points $u = \sigma$ of the meridian $\rho = \rho(u)$ we also have the following relations

$$\rho_{(i)}(\sigma) = \rho_{(i+1)}(\sigma),\ \rho'_{(i)}(\sigma) \neq \rho'_{(i+1)}(\sigma),\ i = 1, 2, \ldots, n. \tag{2.10}$$

According to (1.15,1.16) on the parts T_i of the surface T it is

$$\rho_{(i)}\overset{(1)}{\psi}{}''_{k,i} + (k^2-1)\rho''_{(i)}\overset{(1)}{\psi}_{k,i} = 0, \tag{2.11}$$

$$\rho_{(i)}\overset{(2)}{\psi}{}''_{2k,i} + (4k^2-1)\rho''_{(i)}\overset{(2)}{\psi}_{2k,i} = \overset{(2)}{R}_{2k,i},\ i = 1, 2, \ldots, n. \tag{2.12}$$

Based on [1], page 112, for the field of infinitesimal bendings $\overset{(1)}{\psi}_k(u)$ at the points $u = \sigma$ we have

$$\begin{aligned} &\rho_{(i)}(\sigma)\overset{(1)}{\psi}{}'_{k,i}(\sigma) + (k^2-1)\rho'_{(i)}(\sigma)\overset{(1)}{\psi}_{k,i}(\sigma) \\ &\qquad = \rho_{(i+1)}(\sigma)\overset{(1)}{\psi}{}'_{k,i+1}(\sigma) + (k^2-1)\rho'_{(i+1)}(\sigma)\overset{(1)}{\psi}_{k,i+1}(\sigma) \end{aligned} \tag{2.13}$$

For the field of the second order infinitesimal bendings we have according to [2]

$$\begin{aligned} &\rho_{(i+1)}(\sigma)\overset{(2)}{\psi}{}'_{2k,i+1}(\sigma) + (4k^2-1)\rho'_{(i+1)}(\sigma)\overset{(2)}{\psi}_{2k,i+1}(\sigma) \\ &\qquad = \rho_{(i)}(\sigma)\overset{(2)}{\psi}{}'_{2k,i}(\sigma) + (4k^2-1)\rho'_{(i)}(\sigma)\overset{(2)}{\psi}_{2k,i}(\sigma) + Q_k(\sigma), \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} Q_k(\sigma) = \frac{\rho'_{(i)} - \rho'_{(i+1)}}{\rho_{(i)}k^2}\Big[&(k^2-1)^2(\rho'^2_{(i)} - k^2 - \rho'_{(i)}\rho'_{(i+1)}k^2)\overset{(1)}{\psi}{}^2_{k,i} \\ &+ (1+k^2)\rho^2_{(i)}\overset{(1)}{\psi}{}'^2_{k,i} + \rho_{(i)}(k^2-1)(2\rho'_{(i)} + k^2\rho'_{(i)} - k^2\rho'_{(i+1)})\overset{(1)}{\psi}_{k,i}\overset{(1)}{\psi}{}'_{k,i}\Big]. \end{aligned}$$

At the line generated by $A_i(u_i,\rho_i)$, $i=1,2,\dots,n$, based on (2.8) we have

$$\begin{aligned} & u_i\overset{(2)}{M}_{2k,i} - u_i\overset{(2)}{M}_{2k,i+1} + \overset{(2)}{N}_{2k,i} - \overset{(2)}{N}_{2k,i+1} \\ &\quad = \frac{(k^2-1)}{2k^2\rho_i}\Bigg[\frac{(1+k_{i+1}^2)(k_{i+1}\overset{(1)}{N}_{k,i+1} - \overset{(1)}{M}_{k,i+1}n_{i+1})^2}{k_{i+1}^2} \\ &\qquad - \frac{(1+k_i^2)(k_i\overset{(1)}{N}_{k,i} - \overset{(1)}{M}_{k,i}n_i)^2}{k_i^2}\Bigg],\ (i=1,2,\dots,n). \end{aligned} \tag{2.15}$$

The equation (2.14) gives

$$\begin{aligned} & [-\rho_i-(4k^2-1)k_iu_i]\overset{(2)}{M}_{2k,i} + [\rho_i+(4k^2-1)k_{i+1}u_i]\overset{(2)}{M}_{2k,i+1} \\ &\quad - (4k^2-1)k_i\overset{(2)}{N}_{2k,i} + (4k^2-1)k_{i+1}\overset{(2)}{N}_{2k,i+1} \\ &\quad = \frac{(k^2-1)^2}{2k^2\rho_i}\Bigg[\frac{(1+k_{i+1}^2)(k_{i+1}\overset{(1)}{N}_{k,i+1} - \overset{(1)}{M}_{k,i+1}n_{i+1})^2}{k_{i+1}} \\ &\qquad - \frac{(1+k_i^2)(k_i\overset{(1)}{N}_{k,i} - \overset{(1)}{M}_{k,i}n_i)^2}{k_i}\Bigg] \\ &\quad + \frac{(4k^2-1)(k^2-1)^2}{2k^2\rho_i}\Big[-(1+k_{i+1}^2)(k_{i+1}\overset{(1)}{N}_{k,i+1} - \overset{(1)}{M}_{k,i+1}n_{i+1})^2 \\ &\qquad + (1+k_i^2)(k_i\overset{(1)}{N}_{k,i} - \overset{(1)}{M}_{k,i}n_i)^2\Big] \\ &\quad + \frac{k_i-k_{i+1}}{\rho_ik^2}\Big[(k^2-1)^2(k_i^2-k^2-k_ik_{i+1}k^2)(\overset{(1)}{M}_{k,i}u_i + \overset{(1)}{N}_{k,i})^2 \\ &\quad + (1+k^2)\rho^2\overset{(1)}{M}{}_{k,i}^2 + \rho_i(k^2-1)(2k_i+k^2k_i-k^2k_{i+1})(\overset{(1)}{M}_{k,i}u_i + \overset{(1)}{N}_{k,i})\overset{(1)}{M}_{k,i}\Big]. \end{aligned} \tag{2.16}$$

Solving the system (1.14′) we have

$$\begin{aligned} \overset{(2)}{\varphi}_0(u) &= \int(\overset{(2)}{A}_0(u) - \rho'(u)\overset{(2)}{B}{}'_0(u))\,du + \overset{(2)}{M}_0, \\ \overset{(2)}{\psi}_0(u) &= \overset{(2)}{B}_0(u), \\ \overset{(2)}{\chi}_0(u) &= \rho(u)\Bigg[\int\frac{\overset{(2)}{C}_0(u)}{\rho^2(u)}\,du + \overset{(2)}{N}_0\Bigg]. \end{aligned} \tag{2.17}$$

where $\overset{(2)}{A}_0(u)$, $\overset{(2)}{B}_0(u)$, $\overset{(2)}{C}_0(u)$ are given by (1.14). According to (2.9) we have

$$\overset{(2)}{M}_{0,i} - \overset{(2)}{M}_{0,i+1} = \Big[\overset{(2)}{A}_{0,i+1}(\sigma) - \overset{(2)}{A}_{0,i}(\sigma)\Big]\sigma + (k_i-k_{i+1})\overset{(2)}{C}_{0,i}(\sigma), \tag{2.18}$$

$(i = 1, 2, \ldots, n)$. From (2.9) we also have

$$\overset{(2)}{N}_{0,i} = \overset{(2)}{N}_{0,i+1}, \ (i = 1, 2, \ldots, n). \tag{2.19}$$

From above exposed, the following theorem holds:

THEOREM. *Neccesary and sufficient condition for the toroid rotational surface generated by polygonal meridian to have a field of infinitesimal bendings of the second order as an extension of the field* $\overset{(1)}{z}(u, v)$, *is the systems of linear equations* (2.15, 16) *and* (2.18, 19) *to be compatible.*

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